Random positive maps

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Overview

Joint work in preparation with Patrick Hayden (Stanford) and Ion Nechita (CNRS, Toulouse)

Outline:
1. Positive maps: why do we care? (a primer of quantum information theory)
2. Random positive maps with random matrices (convergence of the largest eigenvalue).
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Quantum information: a primer

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- On the other hand, positive maps are still completely unclassified.
  (roughly speaking) The only final results available are: maps from $\mathcal{M}_{n_1}(\mathbb{C}) \rightarrow \mathcal{M}_{n_2}(\mathbb{C})$ with $(n_1, n_2) = \{(1, n); (n, 1); (2, 2); (2, 3); (3, 2)\}$ are positive iff they are CP.
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- If $\rho \in \text{Sep}(n_1, n_2)$ and $\Phi : \mathcal{M}_{n_1}(\mathbb{C}) \to \mathcal{M}_{n_3}(\mathbb{C})$ is positive then $\Phi \otimes I_{n_2}(\rho)$ is positive.
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So, trying to find positive but not completely positive maps is a strategy to witness entanglement.
Example: the PPT (positive partial transpose) test, with the transpose map.
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The following state is not separable in $\mathcal{M}_2(\mathbb{C}) \otimes \mathcal{M}_2(\mathbb{C})$

\[
\begin{pmatrix}
0.2 & 0 & 0 & 0 \\
0 & 0.3 & 0.3 & 0 \\
0 & 0.3 & 0.3 & 0 \\
0 & 0 & 0 & 0.2
\end{pmatrix}
\]
A corollary of Hahn Banach theorem is that, for every entangled state $\rho$ there exists a positive map $\Phi$ such that $\Phi \otimes ld(\rho)$ fails to be positive.
Quantum information: entanglement witness

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RMT can help.
Choi matrix

To a linear map $\Phi : \mathcal{M}_{n_1}(\mathbb{C}) \to \mathcal{M}_{n_2}(\mathbb{C})$ we associate its Choi matrix $C_\Phi \in \mathcal{M}_{n_1}(\mathbb{C}) \otimes \mathcal{M}_{n_2}(\mathbb{C})$ given by

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- Theorem (Choi, 70’s): $\Phi$ is completely positive iff $C_{\Phi}$ is positive.
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More recently: $\Phi$ is positive iff $p \otimes 1_{n_2} C_\Phi p \otimes 1_{n_2}$ is positive.
One toy random example

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- Let \( p \) be a rank 1 projection in \( \mathcal{M}_k(\mathbb{C}) \). Then, the non-trivial eigenvalues of
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- Let $p$ be a rank 1 projection in $\mathcal{M}_k(\mathbb{C})$. Then, the non-trivial eigenvalues of
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follow a GUE centered at 1 and of variance $a/k$.
That is, the eigenvalues are located in a semi-circle distribution on the interval $[1 - 2\sqrt{a/k}, 1 + 2\sqrt{a/k}]$. 
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- Fixing $k$, if we construct a (random) map $\Phi : \mathcal{M}_k(\mathbb{C}) \to \mathcal{M}_n(\mathbb{C})$ whose Choi matrix is $X$. 

And if $a$ is such that $1 - 2\sqrt{a/k} > 0$, with probability tending to 1 as $n$ becomes large, we obtain a random positive map. 

[largest eigenvalue convergence + $\varepsilon$-net + union bound argument]

In addition, if $1 - 2\sqrt{a/k} < 0$, $\Phi$ is not completely positive with probability tending to 1 as $n$ becomes large, therefore it 'detects' many entangled states.
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- Let $\mu$ be a compactly supported real probability measure of 1st moment $> 0$. 
- The free CLT and super convergence (Bercovici Voiculescu) imply that for $l$ large enough, $\mu \boxplus l$ has positive support. 
- Picking a random selfadjoint matrix $X \in M_k(C) \otimes M_n(C)$ with random eigenvectors ($UXU^* \sim X$) whose eigenvalue distribution converges strongly to $\mu$ yields a random map whose Choi map is $X$. 
- If $l > l$, this map is positive with probability one as $n \to \infty$. 
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Let $X$ be a $GUE$ as above: $X \in \mathcal{M}_k(\mathbb{C}) \otimes \mathcal{M}_n(\mathbb{C})$ its eigenvalues are approximately in $[1 - 2\sqrt{a}, 1 + 2\sqrt{a}]$. $k$ is fixed, $n$ tends to $\infty$.

Set $\alpha = 2\sqrt{a}$.

$X$ is positive with probability one as $n \to \infty$ as soon as $0 < \alpha < 1$.

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With probability one, as $n \to \infty$:

- With $1 > \alpha > \frac{4}{\sqrt{k}}$, $X$ is PPT but not separable.
- If $\alpha < \sqrt{\frac{k}{2(k-1) + \sqrt{k}}}$, $X$ is separable.

The criterion starts to become useful when $k > 16$.

In both cases, $\alpha$ is of order $C/\sqrt{k}$.

The order $C/\sqrt{k}$ is optimal.

We use the non-centered GUE random positive maps exhibited earlier to prove this result.

Conclusion: Random maps are much more efficient than PPT.
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Let $X$ be a $GUE$ as above: $X \in \mathcal{M}_k(\mathbb{C}) \otimes \mathcal{M}_n(\mathbb{C})$ its eigenvalues are approximately in $[1 - 2\sqrt{a}, 1 + 2\sqrt{a}]$. $k$ is fixed, $n$ tends to $\infty$. Set $\alpha = 2\sqrt{a}$.

With probability one, as $n \to \infty$:

- With $1 > \alpha > 4/\sqrt{k}$, $X$ is PPT but not separable.
- If $\alpha < \sqrt{k}/(2(k - 1) + \sqrt{k})$, $X$ is separable.
- The criterion starts to become useful when $k > 16$.
- In both cases, $\alpha$ is of order $C/\sqrt{k}$. The order $C/\sqrt{k}$ is optimal.
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Conclusion: Random maps are much more efficient than PPT.
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Thank you!