Random positive maps

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Joint work in preparation with Patrick Hayden (Stanford) and Ion Nechita (CNRS, Toulouse)

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2. Random positive maps with random matrices (convergence of the largest eigenvalue).
3. Application: almost optimal entanglement witnesses.

Quantum information: a primer

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- *Entangled states* $Ent(n_1, n_2) := D(n_1 n_2) - Sep(n_1, n_2)$. A very important set (resource for quantum computing, etc).

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- Φ_1 is k-positive for all k whereas Φ_2 is 'only' 1-positive.
- A map that is positive for all k is called *completely positive*.

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- On the other hand, positive maps are still completely unclassified.
  (roughly speaking) The only final results available are: maps from $\mathcal{M}_{n_1}(\mathbb{C}) \to \mathcal{M}_{n_2}(\mathbb{C})$ with $(n_1, n_2) = \{(1,n); (n,1); (2,2); (2,3); (3,2)\}$ are positive iff they are CP.

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- However, if $\rho \in Ent(n_1, n_2)$ then $\Phi \otimes I_{n_2}(\rho)$ could in principle fail to be positive.
- But a failure to be positive can't happen if Φ is CP by definition.
  So, trying to find *positive but not completely positive maps* is a strategy to *witness* entanglement.

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- The following state is not separable in $\mathcal{M}_2(\mathbb{C}) \otimes \mathcal{M}_2(\mathbb{C})$

$$\begin{pmatrix} 0.2 & 0 & 0 & 0 \\ 0 & 0.3 & 0.3 & 0 \\ 0 & 0.3 & 0.3 & 0 \\ 0 & 0 & 0 & 0.2 \end{pmatrix}$$

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- RMT can help.

Choi matrix

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- More recently: Φ is *positive* iff $p \otimes 1_{n_2} C_\Phi p \otimes 1_{n_2}$ is positive.

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  That is, the eigenvalues are located in a semi-circle distribution on the interval $[1-2\sqrt{a/k}, 1+2\sqrt{a/k}]$.

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  with probability tending to 1 as n becomes large, we obtain a *random positive map*.
  [largest eigenvalue convergence + ε-net + union bound argument]
- In addition, if $1 - 2\sqrt{a} < 0$, Φ is not completely positive with probability tending to 1 as n becomes large, therefore it 'detects' many entangled states.

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- X is PPT with probability one as $n \to \infty$ as soon as $0 < \alpha < 1$.
- PPT states and general states have typical size – i.e. PPT is not so efficient in large dimension to detect entanglement.

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- Conclusion: Random maps are much more efficient than PPT.

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