Exercise 1 (Nature of an integral). Set $d = 1$. Let us consider the following integral, for $t \geq 0$,

$$ I_t = \int_0^t B_s \, ds. $$

2. Show that $d(t B_t) = B_t \, dt + t \, dB_t$;
3. Deduce from the preceding question that $I_t = \int_0^t (t-s) \, dB_s$ for all $t \geq 0$;
4. Deduce from the preceding question that $I_t \sim \mathcal{N}(0, \frac{1}{3} t^3)$ for all $t \geq 0$;
5. For all $t \geq 0$, $n \geq 1$, $0 \leq k \leq n$, let us define $t_k = \frac{k}{n} t$. Show that

$$ \sum_{k=0}^{n-1} B_{t_k} (t_{k+1} - t_k) = \frac{t}{n} \sum_{j=0}^{n-2} (n-j-1)(B_{t_{j+1}} - B_{t_j}). $$

6. Deduce from the preceding question another proof that $I_t \sim \mathcal{N}(0, \frac{1}{3} t^3)$ for all $t \geq 0$;
7. Is the process $(I_t)_{t \geq 0}$ a martingale?

Elements of solution for Exercise 1.

1. Since the integrator is of finite variation and the integrand is bounded and measurable (actually continuous), it is a Lebesgue–Stieltjes integral, and in particular an Itô integral with respect to a semi-martingale without martingale part. However it is not a Wiener integral.

2. The Itô formula for $f(x, y) = xy$ and $X_t = (t, B_t)$ gives

$$ t B_t = 0 + \int_0^t B_s \, ds + \int_0^t s \, dB_s, $$

3. From the preceding question (actually, it is an integration by parts)

$$ \int_0^t B_s \, ds = t B_t - \int_0^t s \, dB_s = \int_0^t (t-s) \, dB_s. $$

4. The integral in the right hand side is a Wiener integral. Thus it is Gaussian with mean zero and variance equal to the squared $L^2$ norm of the integrand:

$$ \mathbb{E} \left[ \int_0^t B_s \, ds \right] = 0 \quad \text{and} \quad \mathbb{E} \left( \left( \int_0^t B_s \, ds \right)^2 \right) = \int_0^t (t-s)^2 \, ds = \frac{t^3}{3}. $$

5. With $t_k = \frac{k}{n} t$ for all $0 \leq k \leq n$, we have

$$ S_n = \sum_{k=0}^{n-1} B_{t_k} (t_{k+1} - t_k) = \frac{t}{n} \sum_{k=0}^{n-1} B_{t_k} = \frac{t}{n} \sum_{k=0}^{n-1} \sum_{j=0}^{k-1} (B_{t_{j+1}} - B_{t_j}) = \frac{t}{n} \sum_{j=0}^{n-2} (n-j-1)(B_{t_{j+1}} - B_{t_j}). $$
Exercise 2 (Study of a special process). Set $d = 2$. For all $t \geq 0$, we write $B_t = (X_t, Y_t)$ and

$$A_t = \int_0^t X_s dY_s - \int_0^t Y_s dX_s.$$ 

1. Show that $(A) = \int_0^t (X_s^2 + Y_s^2) ds$ and that the process $A$ is a square integrable martingale;

2. From now on let $\lambda > 0$. Show that for all $t \geq 0$,

$$\mathbb{E}e^{i \lambda A_t} = \mathbb{E}\cos(\lambda A_t).$$

3. From now on, let $f : \mathbb{R}_+ \to \mathbb{R}$ be $\mathcal{C}^2$, and let us define the continuous semi-martingales

$$(Z_t)_{t \geq 0} = (\cos(\lambda A_t))_{t \geq 0} \quad \text{and} \quad (W_t)_{t \geq 0} = \left( -\frac{f''(t)}{2} (X_t^2 + Y_t^2) + f(t) \right)_{t \geq 0}.$$ 

Show that for all $t \geq 0$,

$$Z_t = 1 - \lambda \int_0^t \sin(\lambda A_s) dA_s - \frac{\lambda^2}{2} \int_0^t (X_s^2 + Y_s^2) Z_s ds,$$

and

$$W_t = f(0) - \int_0^t f'(s) X_s dY_s - \int_0^t f'(s) Y_s dX_s - \frac{1}{2} \int_0^t f''(s) (X_s^2 + Y_s^2) ds,$$

and deduce that

$$\langle Z, W \rangle = 0.$$ 

4. Show that if $f$ solves $f'' = f'^2 - \lambda^2$ then $Ze^W$ is a continuous local martingale and

$$Ze^{W_t} = e^{f(0)} - \lambda \int_0^t \sin(\lambda A_s) e^{W_s} dA_s - \int_0^t f'(s) Ze^{W_s} X_s dX_s - \int_0^t f'(s) Ze^{W_s} Y_s dY_s.$$ 

5. Let $r > 0$. By using $f(t) = -\log \cosh(\lambda (r - t))$ deduce from the previous question that

$$\mathbb{E}e^{i \lambda A_t} = \frac{1}{\cosh(\lambda r)}.$$ 

Elements of solution for Exercise 2. For all $t \geq 0$, $A_t$ is the algebraic area between planar Brownian motion and its chord, and the process $A$ is the Lévy area. This exercise is a slightly more detailed version of [1, Exercise 5.30 pages 144–145]. Its goal is to compute the characteristic function or Fourier transform of $A_t$. 

6. Fix $t \geq 0$. Since $I_t$ is a Lebesgue–Stieltjes integral with continuous integrand, we have $\lim_{n \to \infty} S_n = I_t$ almost surely and thus in law. For all $n$ and $j$, since $B_{t_{j+1}} - B_{t_j}$ are independent and Gaussian, we get that $S_n \sim \mathcal{N}(\mathbb{E}(S_n), \mathbb{E}(S_n^2) - \mathbb{E}(S_n)^2)$. But the convergence in law of Gaussians is equivalent to the convergence of the first two moments. Now it remains to note that we have $\mathbb{E}(S_n) = 0$ and

$$\mathbb{E}(S_n^2) = \frac{t^2}{n^2} \sum_{j=0}^{n-2} (n - j - 1)^2 \mathbb{E}((B_{t_{j+1}} - B_{t_j})^2) = \frac{t^3 n^3}{n^3} \sum_{j=0}^{n-1} j^2 = - \frac{1}{n} \mathbb{E}(\frac{t^3}{n} \sum_{j=0}^{n-1} j^2) \Rightarrow t^3 \int_0^1 x^2 dx = \frac{t^3}{3}.$$ 

7. Beware that the integrand in $\int_0^t (t - s) dB_s$ depends on $t$. The process $(-\int_0^t s dB_s)_{t \geq 0}$ is a martingale, however the process $(\int_0^t t dB_s)_{t \geq 0} = (t B_t)_{t \geq 0}$ is not a martingale: for all $0 \leq s \leq t$,

$$\mathbb{E}(t + s) | \mathcal{F}_t = (t + s) B_s \neq s B_s.$$ 

Elements of solution for Exercise 2. For all $t \geq 0$, $A_t$ is the algebraic area between planar Brownian motion and its chord, and the process $A$ is the Lévy area. This exercise is a slightly more detailed version of [1, Exercise 5.30 pages 144–145]. Its goal is to compute the characteristic function or Fourier transform of $A_t$. 

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1. Since \( \langle X \rangle_t = \langle Y \rangle_t = t \) and \( \langle X, Y \rangle_t = 0 \), we get
\[
\langle A \rangle_t = \langle A, A \rangle_t = \left( \int_0^t X_s dY_s \right)_t + \left( \int_0^t Y_s dX_s \right)_t + 2 \left( \int_0^t X_s dY_s, \int_0^t Y_s dX_s \right)_t
\]
\[
= \int_0^t X_s^2 d\langle X \rangle_s + \int_0^t Y_s^2 d\langle Y \rangle_s + \int_0^t X_s Y_s d\langle X, Y \rangle_s
\]
\[
= \int_0^t (X_s^2 + Y_s^2) ds.
\]
It follows by the Fubini–Tonelli theorem that
\[
E\langle A \rangle_t = \int_0^t E(X_s^2 + Y_s^2) ds = \int_0^t 2sds = t^2 < \infty
\]
and thus, by a famous martingale criterion, the process \( A \) is a square integrable martingale.

Alternatively, since for all \( t \geq 0 \), \( E \int_0^t X_s^2 d\langle Y \rangle_s = \int_0^t E(X_s^2) ds < \infty \) the process \( \int_0^t X_s dY_s \) and by symmetry the process \( \int_0^t Y_s dX_s \) are both square integrable martingales, and thus the process \( A \) is also a square integrable martingales as being the difference of two square integrable martingales.

2. For all \( \lambda \in \mathbb{R} \), \( t \geq 0 \), \( E(e^{tA}) = E(\cos(\lambda A_t)) + iE(\sin(\lambda A_t)) \). Since \( \langle X, Y \rangle \equiv \langle Y, X \rangle \), we get, for all \( t \geq 0 \),
\[
-A_t = \int_0^t Y_s dX_s - \int_0^t X_s dY_s = \int_0^t X_s dY_s - \int_0^t Y_s dX_s = A_t,
\]
and thus the characteristic function or Fourier transform of \( A_t \) is real.

3. The canonical decompositions are given by the Itô formula. Namely, for \( Z \),
\[
Z_t = 1 - \lambda \int_0^t \sin(\lambda A_s) dA_s - \frac{\lambda^2}{2} \int_0^t \cos(\lambda A_s) d\langle A \rangle_s
\]
\[
= 1 - \lambda \int_0^t \sin(\lambda A_s) dA_s - \frac{\lambda^2}{2} \int_0^t (X_s^2 + Y_s^2) Z_s ds.
\]
Similarly, for \( W \), by the Itô formula for the function \( g(x, y, t) = -\frac{E(f)}{2} (x^2 + y^2) + f(t) \) and the vector of semi-martingale \( S_t = (X_t, Y_t, t) \),
\[
W_t = g(0, 0, 0) + \int_0^t \partial_1 g(S_s) dX_s + \int_0^t \partial_2 g(S_s) dY_s + \int_0^t \partial_3 g(S_s) ds + \frac{1}{2} \int_0^t (\partial_1^2 g + \partial_2^2 g)(S_s) ds
\]
\[
= f(0) - \int_0^t f'(s) X_s dX_s - \int_0^t f'(s) Y_s dY_s + \int_0^t \left( - \frac{f''(s)}{2} (X_s^2 + Y_s^2) + f'(s) \right) ds - \int_0^t f'(s) ds
\]
\[
= f(0) - \int_0^t f'(s) X_s dX_s - \int_0^t f'(s) Y_s dY_s - \frac{1}{2} \int_0^t f''(s) (X_s^2 + Y_s^2) ds.
\]
The computation of \( \langle Z, W \rangle \) involves only the local martingale parts, namely
\[
\langle Z, W \rangle_t = \lambda \left( \int_0^t \sin(\lambda A_s) dA_s, \int_0^t f'(s) X_s dX_s + \int_0^t f'(s) Y_s dY_s \right)_t
\]
\[
= \lambda \int_0^t f'(s) \sin(\lambda A_s) X_s d\langle A, X \rangle_s + \lambda \int_0^t f'(s) \sin(\lambda A_s) Y_s d\langle A, Y \rangle_s.
\]
Now since \( \langle A, X \rangle_t = -\int_0^t X_s ds \) and \( \langle A, Y \rangle_t = \int_0^t Y_s ds \), we get
\[
\langle Z, W \rangle_t = \lambda \int_0^t f'(s)(-X_s Y_s + X_s Y_s) \sin(\lambda A_s) ds = 0.
\]
4. The Itô formula gives (we benefit from the fact that \( \langle Z, W \rangle = 0 \) from the previous question)
\[
Z_t e^{W_t} = e^{f(0)} + \int_0^t e^{W_s} dZ_s + \int_0^t Z_s e^{W_s} dW_s + \frac{1}{2} \int_0^t Z_s e^{W_s} d\langle W \rangle_s.
\]
By collecting the finite variation parts from $dZ$ and $dW$ from a previous question we get
\[
-\frac{\lambda^2}{2} \int_0^t (X^2 + Y^2) Z_s e^W dW + \frac{1}{2} \int_0^t f''(s)(X^2 + Y^2) Z_s e^W ds + \frac{1}{2} \int_0^t Z_s e^W d(W)_s.
\]

Now from a previous question
\[
\langle W \rangle_t = \left( \int_0^t f'(s) X_s dX_s + \int_0^t f'(s) Y_s dY_s \right)_t = \int_0^t f''(s)(X^2 + Y^2) ds.
\]

It follows that the finite variation part of $Ze^W$ vanishes when $f'' = f^2 - \lambda^2$.

5. With $f(t) = -\log \cosh(\lambda(r-t))$, we have
\[
f'(t) = \lambda \frac{\sinh(\lambda(r-t))}{\cosh(\lambda(r-t))} = \lambda \tanh(\lambda(r-t))
\]
and
\[
f''(t) = -\frac{\lambda^2}{\cosh(\lambda(r-t))^2} = -\lambda^2 (1 - \tanh(\lambda(r-t))^2) = -\lambda^2 + f'^2(t).
\]

It follows from the previous question that $Ze^W$ is a continuous local martingale. Note that $f(r) = f'(r) = 0$ and $W_t = 0$, and by using previous questions,
\[
\mathbb{E} e^{\lambda A_t} = \mathbb{E} \cos(\lambda A_t) = \mathbb{E} Z_t = \mathbb{E}(Z_t e^{W_t}).
\]

On the other hand, since $f(0) = -\log \cosh(\lambda r)$, $Z_0 = 1$, $W_0 = f(0)$, we get
\[
\mathbb{E}(Z_0 e^{W_0}) = e^{f(0)} = \frac{1}{\cosh(\lambda r)}.
\]

It remains to show that the local martingale $Ze^W$ is a martingale on the time interval $[0, r]$. From the previous question, since $f$, cos, and sin are bounded, it suffices to show that
\[
\mathbb{E} \int_0^t e^{2W_s} d(A)_s < \infty \quad \text{and} \quad \mathbb{E} \int_0^t e^{2W_s}(X^2_s + Y^2_s) ds < \infty.
\]

But the first condition follows from the second thanks to the formula for $\langle A \rangle$ provided by a previous question. On the other hand, if $t \in [0, r]$ then $f'(t) \geq 0$ and thus $W_s \leq f(t)$ for all $s \in [0, t]$, which implies that the second condition is satisfied by using $\mathbb{E}(X^2_s + Y^2_s) = 2s$.

**Exercise 3** (Criterion for a stochastic differential equation). Set $d = 1$. Let $\sigma, b$ be two functions $\mathbb{R} \to \mathbb{R}$ such that for some finite constant $C < \infty$ and for all $x, y \in \mathbb{R}$,
\[
|\sigma(x) - \sigma(y)| \leq C|x - y| \quad \text{and} \quad |b(x) - b(y)| \leq C|x - y|.
\]

The goal of this exercise is to prove pathwise uniqueness for the stochastic differential equation
\[
dX_t = \sigma(X_t) dB_t + b(X_t) dt. \tag{SDE}
\]

A solution $X$ is a continuous semi-martingale with canonical decomposition $X = X_0 + M + V$ with $X_0 \in L^2$, local martingale part $M = \int_0^\cdot \sigma(X_s) dB_s$, and finite variation part $V = \int_0^\cdot b(X_s) ds$. Note that the continuity of $\sigma, X, b$ gives that almost surely, for all $t \geq 0$, $s \mapsto \sigma(X_s) + b(X_s)$ is locally bounded.

1. Let $Z$ be a continuous semi-martingale such that $\langle Z \rangle = \int_0^\cdot \varphi_s ds$ for a progressive process $\varphi$ such that $0 \leq \varphi \leq C|Z|$ for some constant $C < \infty$. Prove that for all $t \geq 0$ and all $a > 0$,
\[
\mathbb{E} \int_0^t 1_{0 < |Z_s| \leq a} \frac{d|Z_s|}{|Z_s|} \leq Ct.
\]
2. Deduce from the preceding question that for all \( t \geq 0 \),
\[
\lim_{n \to \infty} nE \int_0^t 1_{|Z_s| \leq \frac{1}{n}} d\langle Z \rangle_s = 0.
\]

3. For all \( n \geq 1, x \in \mathbb{R} \), let us define \( g_n(x) = 2n(1 + nx)1_{x \in [-\frac{1}{n}, 0]} + 2n1_{x=0} + 2n(1 - nx)1_{x \in (0, \frac{1}{n})} \).
Let \( f_n : \mathbb{R} \to \mathbb{R} \) be the twice differentiable function such that \( f''_n = g_n \) and \( f_n(0) = f'_n(0) = 0 \).
Show that for all \( x \in \mathbb{R} \), the following properties hold true:
(a) \( f'_n(x) \in [-1, 1] \) and \( \lim_{n \to \infty} f'_n(x) = \text{sign}(x) = 1_{x>0} - 1_{x<0} \);
(b) \( |f_n(x)| \leq |x| \) and \( \lim_{n \to \infty} f_n(x) = |x| \).

4. By using Itô formula, prove that for all continuous semi-martingale \( Z = (Z_t)_{t \geq 0} \), all \( t \geq 0 \),
\[
\int_0^t 1_{Z_s=0} d\langle Z \rangle_s = 0.
\]

5. From now on, let \( X \) and \( X' \) be two solutions of (SDE) on \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}) \) and with respect to the Brownian motion \( B \). Show that for all \( t \geq 0 \),
\[
\langle X - X' \rangle_t = \int_0^t (\sigma(X_s) - \sigma(X'_s))^2 ds.
\]

6. By using the assumption on \( \sigma \), deduce from the preceding questions that for all \( t \geq 0 \),
\[
\lim_{n \to \infty} E \int_0^t g_n(X_s - X'_s) d\langle X - X' \rangle_s = 0.
\]

7. Set \( Z = X - X' \). From now on, let \( T \) be a stopping time such that the semi-martingale \((Z_{t \land T})_{t \geq 0}\) is bounded. By using notably the assumption on \( \sigma \), prove that for all \( t \geq 0, n \geq 1 \),
\[
E(f_n(Z_{t \land T})) = E(f_n(0)) + E \int_0^{t \land T} f'_n(Z_s)(b(X_s) - b(X'_s))ds + \frac{1}{2} E \int_0^{t \land T} f''_n(Z_s) d\langle Z \rangle_s.
\]

8. Deduce from the preceding questions and the assumption on \( b \) that for all \( t \geq 0 \),
\[
E(|X_{t \land T} - X'_{t \land T}|) = E(|X_0 - X'_0|) + E \int_0^{t \land T} (b(X_s) - b(X'_s)) \text{sign}(X_s - X'_s) ds.
\]

9. By using the Grönwall lemma, deduce that if \( X_0 = X'_0 \) then \( X_t = X'_t \) for all \( t \geq 0 \).

**Elements of solution for Exercise 3.** The result is known as the Yamada–Watanabe criterion. This is a slightly more detailed version of [1, Exercise 8.14 pages 231–232].

1. We have, using the properties of \( Z \) and \( \phi \),
\[
\int_0^t 1_{0 < |Z_s| \leq a} |Z_s|^{-\alpha} d\langle Z \rangle_s = \int_0^t 1_{0 < |Z_s| \leq a} |Z_s|^{-\alpha} ds \leq \int_0^t C ds = Ct.
\]

2. For all \( n \geq 1, \) we have \( n1_{0 < |Z_s| \leq \frac{1}{n}} \leq 1_{0 < |Z_s| \leq \frac{1}{n}} \leq 1_{0 < |Z_s| \leq a} \), which is integrable on \([0, t]\) by the preceding question used with \( \alpha = 1 \), and thus the desired result follows then by dominated convergence.

3. The function \( g_n \) is \( 0 \) on \((-\infty, -\frac{1}{n})\), then increases from \( 0 \) to \( 2 \) on \([-\frac{1}{n}, 0] \), then decreases from \( 2 \) to \( 0 \) on \([0, \frac{1}{n}]\), then stays at \( 0 \) on \([\frac{1}{n}, +\infty)\). Since \( \int_{-\infty}^0 g_n(y)dy = 1 \), we have, for all \( x \in \mathbb{R} \),
\[
f'_n(x) = \int_0^x g_n(u)du, \quad \text{in such a way that } f'_n(0) = 0 \text{ and } f''_n = g_n.
\]
The function $f_n$ is $-1$ on $(-\infty, -\frac{1}{n}]$, $0$ at $0$, and $1$ on $[\frac{1}{n}, +\infty)$. Also for all $x \in \mathbb{R}$,
\[
\lim_{n \to \infty} f_n'(x) = 1_{x>0} - 1_{x<0} =: \text{sign}(x).
\]

Next, for all $x \in \mathbb{R}$, we have
\[
f_n(x) = \int_0^x f_n'(u) \, du \quad \text{in such a way that } f_n(0) = 0 \text{ and } f_n'' = g_n.
\]

Since $g_n \geq 0$, we have that $f_n'$ is non-decreasing, and thus $f_n'$ takes actually its values in $[-1, 1]$, and is in particular bounded. It follows by dominated convergence that for all $x \in \mathbb{R}$,
\[
\lim_{n \to \infty} f_n(x) = \int_0^x \lim_{n \to \infty} f_n'(u) \, du = \int_0^x \text{sign}(u) \, du = |x|.
\]

Finally, for all $x \in \mathbb{R}$, $|f_n(x)| \leq f_0(x)$ implies $|f_n(x)| = |x|$.

4. The Itô formula for function $f_n$ of question 3 and semi-martingale $Z$ gives, for all $t \geq 0$,
\[
f_n(Z_t) = f_n(Z_0) + \int_0^t f_n'(Z_s) \, dZ_s + \frac{1}{2} \int_0^t f_n''(Z_s) \, d\langle Z \rangle_s.
\]

Since $|f_n''| \leq 1$ and $\lim_{n \to \infty} f_n''(x) = 1_{x=0}$ for all $x \in \mathbb{R}$, by dominated convergence,
\[
\lim_{n \to \infty} f_n(Z_t) = \int_0^t 1_{Z_s=0} \, d\langle Z \rangle_s \quad \text{a.s.}
\]

On the other hand, since by question 3, $\lim_{n \to \infty} f_n(Z_0) = 0$ for all $x \in \mathbb{R}$, it follows that a.s.
\[
\lim_{n \to \infty} \frac{f_n(Z_t)}{n} = 0 \quad \text{and} \quad \lim_{n \to \infty} \frac{f_n(Z_0)}{n} = 0.
\]

Finally since by question 3, $\frac{1}{2n} |f_n'(x)| \leq \frac{1}{2n} \leq 1$ and $\lim_{n \to \infty} \frac{1}{2n} |f_n'(x)| = 0$ for all $x \in \mathbb{R}$, dominated convergence for stochastic integrals gives
\[
\frac{1}{2n} \int_0^t f_n'(Z_s) \, dZ_s \xrightarrow{n \to \infty} 0.
\]

5. Since $X$ and $X'$ are both solutions on the same space and for the same Brownian motion, we have, for all $X_0$ and $X'_0$ and for all $t \geq 0$,
\[
X_t - X'_t = X_0 - X'_0 + \int_0^t (\sigma(X_s) - \sigma(X'_s)) \, dB_s + \int_0^t (b(X_s) - b(X'_s)) \, ds.
\]

The right hand side gives the canonical decomposition of the semi-martingale $X - X'$. In this decomposition, the first integral is the local martingale part, and
\[
\langle X - X' \rangle = \int_0^t (\sigma(X_s) - \sigma(X'_s))^2 \, ds.
\]

6. By assumption on $\sigma$, the process $\varphi = (\sigma(X) - \sigma(X'))^2$ satisfies $0 \leq \varphi \leq C |X - X'|$. We can then use question 5 and question 2 with $Z = X - X'$ to get, for all $t \geq 0$,
\[
\lim_{n \to \infty} n \mathbb{E} \int_0^t 1_{0<|Z_s|<\frac{1}{n}} \, d\langle Z \rangle_s = 0.
\]

If $g_n$ is as in question 3, then for all $x \in \mathbb{R}$, $0 \leq g_n(x) \leq 2n 1_{0<|x|<\frac{1}{n}} + 2n 1_{x=0}$. Thus, for all $t \geq 0$,
\[
0 \leq \mathbb{E} \int_0^t g_n(Z_s) \, d\langle Z \rangle_s \leq 2n \mathbb{E} \int_0^t 1_{0<|Z_s|<\frac{1}{n}} \, d\langle Z \rangle_s + 2n \mathbb{E} \int_0^t 1_{Z_s=0} \, d\langle Z \rangle_s \xrightarrow{n \to \infty} 0,
\]
where we have used question 2 and question 4.
7. The Itô formula for the \( \mathcal{C}^2 \) function \( f_n \) and the continuous semi-martingale \( Z_t^T \) gives

\[
f_n(Z_t^T) = f_n(Z_0^T) + \int_0^{t \wedge T} f_n'(Z_s) \, dB_s + \frac{1}{2} \int_0^{t \wedge T} f_n''(Z_s) \, dB^2_s,
\]

and since \( dB_s = (\sigma(X_s) - \sigma(X_t)) \, dB_s + (b(X_s) - b(X_t)) \, ds \), we get

\[
\int_0^{t \wedge T} f'(Z_s) \, dB_s = \int_0^t f'(Z_s) (\sigma(X_s) - \sigma(X_t)) \, 1_{s \leq T} \, dB_s + \int_0^{t \wedge T} f'(Z_s) (b(X_s) - b(X_t)) \, ds.
\]

Now, by the assumptions on \( \sigma \) and \( T \), we get

\[
|\sigma(X_s) - \sigma(X_t)| \leq C |Z_s| \leq C'.
\]

This boundedness, together with the one of \( f_n' \), imply that the first integral in the right hand side above (the \( dB_s \) one) is a martingale. Since this martingale is issued from the origin, its expectation vanishes for all times. On the other hand, since \( f_n \) is continuous and \( Z_t^T \) is bounded, the random variables \( f_n(Z_t^T) \) and \( f_n(Z_0^T) \) are integrable. All in all, we obtain

\[
\mathbb{E}(f_n(Z_t^T)) = \mathbb{E}(f_n(Z_0^T)) + \mathbb{E} \int_0^{t \wedge T} f_n'(Z_s) (b(X_s) - b(X_t)) \, ds + \frac{1}{2} \mathbb{E} \int_0^{t \wedge T} f_n''(Z_s) \, dB^2_s.
\]

8. Since \( f_n'' = g_n \geq 0 \), we get, by using question 6, that

\[
0 \leq \mathbb{E} \int_0^{t \wedge T} f_n''(Z_s) \, dB^2_s \leq \mathbb{E} \int_0^t g_n(Z_s) \, dB_s \xrightarrow{n \to \infty} 0.
\]

On the other hand, by the assumption on \( b \) and the boundedness of \( Z_t^T \), we have, on \( \{s \leq T\} \),

\[
|b(X_s) - b(X_t)|^2 \leq C^2 |X_s - X_t|^2 = C^2 |Z_s| \leq C'.
\]

But since \( f_n' \) is bounded (takes its values in \([-1, 1]\)), we get, by dominated convergence

\[
\lim_{n \to \infty} \int_0^{t \wedge T} f_n'(Z_s) (b(X_s) - b(X_t)) \, ds = \int_0^{t \wedge T} \text{sign}(Z_s) (b(X_s) - b(X_t)) \, ds.
\]

Finally, since \( Z_t^T \) is bounded, and since from question 3, for all \( x \in \mathbb{R} \), \( |f_n(x)| \leq |x| \) and \( \lim_{n \to \infty} f_n(x) = |x| \), we get, by dominated convergence, \( \lim_{n \to \infty} \mathbb{E}(f_n(Z_t^T)) = \mathbb{E}(|Z_t^T|) \). Finally

\[
\mathbb{E}(|X_{t \wedge T} - X_{t \wedge T}^T|) = \mathbb{E}(|X_0 - X_0^T|) + \mathbb{E} \int_0^{t \wedge T} (b(X_s) - b(X_t^T)) \text{sign}(X_s - X_t^T) \, ds.
\]

9. From the preceding question, we get, by using the assumption on \( b \),

\[
\alpha(t) = \mathbb{E}(|X_{t \wedge T} - X_{t \wedge T}^T|) \leq \mathbb{E}(|X_0 - X_0^T|) + C \mathbb{E} \int_0^t |X_{s \wedge T} - X_{s \wedge T}^T| \, ds = \alpha(0) + C \int_0^t \alpha(s) \, ds.
\]

By the Grönwall lemma, we obtain \( \alpha(t) \leq \alpha(0)e^{Ct} \) for all \( t \geq 0 \). It follows that if \( \alpha(0) = 0 \) then \( \alpha(t) = 0 \) for all \( t \geq 0 \). This means that if \( X_0 = X_0^T \) then \( X_{t \wedge T} = X_{t \wedge T}^T \) for all \( t \geq 0 \). By writing this for \( t \in \mathbb{Q}_+ \), and by taking \( T = T_m \) such that \( \lim_{m \to \infty} T_m = +\infty \) almost surely, we get that \( X_t = X_t^T \) for all \( t \in \mathbb{Q}_+ \), and thus for all \( t \geq 0 \) since \( X \) and \( X' \) are continuous.

References