Exam 2018/2019

January 9, 2019, from 09:00 to 12:00
Documents allowed, Internet not allowed
Do what you can, and do not worry

\((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})\) is a filtered probability space, with complete and right continuous filtration. 
\(B := (B_t)_{t \geq 0}\) is a \(d\)-dimensional Brownian motion issued from the origin, \(d \geq 1\).
If \(Z\) is a semi-martingale, we denote by \(\langle Z \rangle\) the increasing process of its local martingale part.
If \(Z = Z_0 + M + V\), do not confuse \(\langle Z \rangle = \langle M \rangle\) with the finite variation part \(V\) of \(Z\).

**Exercise 1** (Nature of an integral). Set \(d = 1\). Let us consider the following integral, for \(t \geq 0\),

\[ I_t := \int_0^t B_s \, ds. \]

2. Show that \(d(tB_t) = B_t \, dt + tdB_t\);
3. Deduce from the preceding question that \(I_t = \int_0^t (t-s) \, dB_s\) for all \(t \geq 0\);
4. Deduce from the preceding question that \(I_t \sim \mathcal{N}(0, \frac{1}{3} t^3)\) for all \(t \geq 0\);
5. For all \(t \geq 0\), \(n \geq 1\), \(0 \leq k \leq n\), let us define \(t_k := \frac{k}{n} t\). Show that

\[ \sum_{k=0}^{n-1} B_{t_k} (t_{k+1} - t_k) = t \sum_{j=0}^{n-2} (n - j - 1)(B_{t_{j+1}} - B_{t_j}). \]

6. Deduce from the preceding question another proof that \(I_t \sim \mathcal{N}(0, \frac{1}{3} t^3)\) for all \(t \geq 0\);
7. Is the process \((I_t)_{t \geq 0}\) a martingale?

**Elements of solution for Exercise 1.**

1. Since the integrator is of finite variation and the integrand is bounded and measurable (actually continuous), it is a Lebesgue–Stieltjes integral, and in particular an Itô integral with respect to a semi-martingale without martingale part. However it is not a Wiener integral.
2. The Itô formula for \(f(x, y) = xy\) and \(X_t = (t, B_t)\) gives

\[ tB_t = 0 + \int_0^t B_s \, ds + \int_0^t s \, dB_s, \]

3. From the preceding question (actually, it is an integration by parts)

\[ \int_0^t B_s \, ds = tB_t - \int_0^t s \, dB_s = \int_0^t (t-s) \, dB_s. \]

4. The integral in the right hand side is a Wiener integral. Thus it is Gaussian with mean zero and variance equal to the squared \(L^2\) norm of the integrand:

\[ \mathbb{E} \left( \int_0^t B_s \, ds \right) = 0 \quad \text{and} \quad \mathbb{E} \left( \left( \int_0^t B_s \, ds \right)^2 \right) = \int_0^t (t-s)^2 \, ds = \frac{t^3}{3}. \]

5. With \(t_k = \frac{k}{n} t\) for all \(0 \leq k \leq n\), we have

\[ S_n := \sum_{k=0}^{n-1} B_{t_k} (t_{k+1} - t_k) = \frac{t}{n} \sum_{k=0}^{n-1} B_{t_k} = \frac{t}{n} \sum_{k=0}^{n-1} \sum_{j=0}^{k-1} (B_{t_{j+1}} - B_{t_j}) = \frac{t}{n} \sum_{j=0}^{n-2} (n - j - 1)(B_{t_{j+1}} - B_{t_j}). \]
6. Fix \( t \geq 0 \). Since \( I_t \) is a Lebesgue–Stieltjes integral with continuous integrand, we have \( \lim_{n \to \infty} S_n = I_t \) almost surely and thus in law. For all \( n \) and \( j \), since \( R_{t_j+} - R_{t_j} \) are independent and Gaussian, we get that \( S_n \sim \mathcal{N}(\mathbb{E}(S_n), \mathbb{E}(S_n^2) - \mathbb{E}(S_n)^2) \). But the convergence in law of Gaussians is equivalent to the convergence of the first two moments. Now it remains to note that we have \( \mathbb{E}(S_n) = 0 \) and

\[
\mathbb{E}(S_n^2) = \frac{t^2}{n^2} \sum_{j=0}^{n-1} (n-j-1)^2 \mathbb{E}((R_{t_j+} - R_{t_j})^2) = \frac{t^2}{n^2} \sum_{j=0}^{n-1} j^2 \left( \frac{j}{n} \right)^2 \frac{1}{n} \to \frac{t^2}{3} \int_0^1 x^2 \, dx = \frac{t^3}{3}.
\]

7. Beware that the integrand in \( \int_0^t (t-s) \, dB_s \) depends on \( t \). The process \( -\int_0^t s \, dB_s \) is a martingale, however the process \( (\int_0^t t \, dB_s)_{t \geq 0} = (t \, B_t)_{t \geq 0} \) is not a martingale: for all \( 0 \leq s \leq t \),

\[
\mathbb{E}((t+s) \, B_{t+s} | \mathcal{F}_t) = (t+s) \, B_s \neq s \, B_s.
\]

**Exercise 2** (Study of a special process). Set \( d = 2 \). For all \( t \geq 0 \), we write \( B_t = (X_t, Y_t) \) and

\[
A_t := \int_0^t X_s \, dY_s - \int_0^t Y_s \, dX_s.
\]

1. Show that \( \langle A \rangle = \int_0^t (X_s^2 + Y_s^2) \, ds \) and that the process \( A \) is a square integrable martingale;

2. From now on let \( \lambda > 0 \). Show that for all \( t \geq 0 \),

\[
\mathbb{E} e^{i \lambda A_t} = \mathbb{E} \cos(\lambda A_t).
\]

3. From now on, let \( f : \mathbb{R} \to \mathbb{R} \) be \( \mathcal{C}^2 \), and let us define the continuous semi-martingales

\[
(Z_t)_{t \geq 0} := (\cos(\lambda A_t))_{t \geq 0} \quad \text{and} \quad (W_t)_{t \geq 0} := \left( -\frac{f'(t)}{2} \, (X_t^2 + Y_t^2) + f(t) \right)_{t \geq 0}.
\]

Show that for all \( t \geq 0 \),

\[
Z_t = 1 - \lambda \int_0^t \sin(\lambda A_s) \, dA_s - \frac{\lambda^2}{2} \int_0^t (X_s^2 + Y_s^2) \, Z_s \, ds.
\]

and

\[
W_t = f(0) - \int_0^t f'(s) \, X_s \, dX_s - \int_0^t f'(s) \, Y_s \, dY_s - \frac{1}{2} \int_0^t f''(s) \, (X_s^2 + Y_s^2) \, ds,
\]

and deduce that

\[
\langle Z, W \rangle = 0.
\]

4. Show that if \( f \) solves \( f'' = f'^2 - \lambda^2 \) then \( Z_t e^{W_t} \) is a continuous local martingale and

\[
Z_t e^{W_t} = e^{f(0)} - \lambda \int_0^t \sin(\lambda A_s) e^{W_s} \, dA_s - \int_0^t f'(s) Z_s e^{W_s} X_s \, dX_s - \int_0^t f'(s) Z_s e^{W_s} Y_s \, dY_s.
\]

5. Let \( r > 0 \). By using \( f(t) = -\log \cosh(\lambda (r-t)) \) deduce from the previous question that

\[
\mathbb{E} e^{i \lambda A_t} = \frac{1}{\cosh(\lambda r)}.
\]

**Elements of solution for Exercise 2.** For all \( t \geq 0 \), \( A_t \) is the algebraic area between planar Brownian motion and its chord, and the process \( A \) is the **Lévy area**. This exercise is a slightly more detailed version of [1, Exercise 5.30 pages 144–145]. Its goal is to compute the characteristic function or Fourier transform of \( A_t \).
1. Since \( \langle X \rangle_t = \langle Y \rangle_t = t \) and \( \langle X, Y \rangle_t = 0 \), we get
\[
\langle A \rangle_t = \langle A, A \rangle_t = \left( \int_0^t X_s dX_s \right)_t + \left( \int_0^t Y_s dY_s \right)_t + 2 \left( \int_0^t X_s dY_s, \int_0^t Y_s dX_s \right)_t
\]
\[
= \int_0^t X_s^2 d\langle X \rangle_t + \int_0^t Y_s^2 d\langle Y \rangle_t + \int_0^t X_s Y_s d\langle X, Y \rangle_t
\]
\[
= \int_0^t (X_s^2 + Y_s^2) ds.
\]
It follows by the Fubini–Tonelli theorem that
\[
\mathbb{E} \langle A \rangle_t = \int_0^t \mathbb{E}(X_s^2 + Y_s^2) ds = \int_0^t 2s ds = t^2 < \infty
\]
and thus, by a famous martingale criterion, the process \( A \) is a square integrable martingale.

Alternatively, since for all \( t \geq 0 \), \( \mathbb{E} \int_0^t X_s^2 d\langle Y \rangle_s = \int_0^t \mathbb{E}(X_s^2) ds < \infty \) the process \( \int_0^t X_s dY_s \) and by symmetry the process \( \int_0^t Y_s dX_s \) are both square integrable martingales, and thus the process \( A \) is also a square integrable martingales as being the difference of two square integrable martingales.

2. For all \( \lambda \in \mathbb{R}, t \geq 0, \mathbb{E}(e^{\lambda A_t}) = \mathbb{E}(\cos(\lambda A_t)) + i \mathbb{E}(\sin(\lambda A_t)) \). Since \( \langle X, Y \rangle_t \equiv (X, Y) \), we get, for all \( t \geq 0, \)
\[
-A_t = \int_0^t Y_s dX_s - \int_0^t X_s dY_s \equiv \int_0^t X_s dY_s - \int_0^t Y_s dX_s = A_t,
\]
and thus the characteristic function or Fourier transform of \( A_t \) is real.

3. The canonical decompositions are given by the Itô formula. Namely, for \( Z \),
\[
Z_t = 1 - \lambda \int_0^t \sin(\lambda A_s) dA_s - \frac{\lambda^2}{2} \int_0^t \cos(\lambda A_s) d\langle A \rangle_s
\]
\[
= 1 - \lambda \int_0^t \sin(\lambda A_s) dA_s - \frac{\lambda^2}{2} \int_0^t (X_s^2 + Y_s^2) Z_s ds.
\]
Similarly, for \( W \), by the Itô formula for the function \( g(x, y) := \frac{1}{2}(x^2 + y^2) + f(t) \) and the vector of semi-martingale \( S_t := (X_t, Y_t, t) \) with martingale part \( (X_t, Y_t) \),
\[
W_t = g(0, 0, 0) + \int_0^t \partial_1 g(S_s) dX_s + \int_0^t \partial_2 g(S_s) dY_s + \int_0^t \partial_3 g(S_s) ds + \frac{1}{2} \int_0^t (\partial^2_{x^2} g + \partial^2_{y^2} g)(S_s) ds
\]
\[
= f(0) - \int_0^t f'(s) X_s dX_s - \int_0^t f'(s) Y_s dY_s + \int_0^t \left( -\frac{f''(s)}{2} (X_s^2 + Y_s^2) + f'(s) \right) ds - \int_0^t f'(s) ds
\]
\[
= f(0) - \int_0^t f'(s) X_s dX_s - \int_0^t f'(s) Y_s dY_s - \frac{1}{2} \int_0^t f''(s) (X_s^2 + Y_s^2) ds.
\]

The computation of \( \langle Z, W \rangle \) involves only the local martingale parts, namely
\[
\langle Z, W \rangle_t = \lambda \int_0^t \sin(\lambda A_s) dA_s, \int_0^t f'(s) X_s dX_s + \int_0^t f'(s) Y_s dY_s \rangle_t
\]
\[
= \lambda \int_0^t f'(s) \sin(\lambda A_s) X_s d(A, X)_s + \lambda \int_0^t f'(s) \sin(\lambda A_s) Y_s d(A, Y)_s.
\]
Now since \( \langle X, A \rangle_t = -\int_0^t X_s ds \) and \( \langle A, Y \rangle_t = \int_0^t X_s ds \), we get
\[
\langle Z, W \rangle_t = \lambda \int_0^t f'(s)(-X_s Y_s + X_s Y_s) \sin(\lambda A_s) ds = 0.
\]
4. The Itô formula gives (we benefit from the fact that \( \langle Z, W \rangle = 0 \) from the previous question)
\[
Z_t e^{W_t} = e^{f(0)} + \int_0^t e^{W_s} dZ_s + \int_0^t Z_s e^{W_s} dW_s + \frac{1}{2} \int_0^t Z_s e^{W_s} d\langle W \rangle_s.
\]
Exercise 3

A solution that for some finite constant $C$ the local martingale part $\sigma$ of $X^2$ deduce from the preceding question that for all $t$.

Let, $\sigma$, $X^2$. It follows that the finite variation part of $Z$.

Now from a previous question it follows that the finite variation part of $Z$ vanishes when $f'' = f'^2 - \lambda^2$.

5. With $f(t) = -\log \cosh(\lambda(r - t))$, we have

$$f'(t) = \lambda \frac{\sinh(\lambda(r - t))}{\cosh(\lambda(r - t))} = \lambda \tanh(\lambda(r - t))$$

and

$$f''(t) = -\lambda^2 \frac{\cosh(\lambda(r - t))^2}{\cosh(\lambda(r - t))^2} = -\lambda^2 (1 - \tanh(\lambda(r - t))^2) = -\lambda^2 + f'^2(t).$$

It follows from the previous question that $Ze^W$ is a continuous local martingale. Note that $f(r) = f'(r) = 0$ and $W_r = 0$, and by using previous questions,

$$\mathbb{E} e^{\lambda A_r} = \mathbb{E} \cos(\lambda A_r) = \mathbb{E} Z_r = \mathbb{E} (Z_r e^{W_r}).$$

On the other hand, since $f(0) = -\log \cosh(\lambda r)$, $Z_0 = 1$, $W_0 = f(0)$, we get

$$\mathbb{E} (Z_0 e^{W_0}) = e^{f(0)} = \frac{1}{\cosh(\lambda r)}.$$

It remains to show that the local martingale $Ze^W$ is a martingale on the time interval $[0, r]$. From the previous question, since $f$, $\cos$, and $\sin$ are bounded, it suffices to show that

$$\mathbb{E} \int_0^t e^{2W_s} d(A)_s < \infty \quad \text{and} \quad \mathbb{E} \int_0^t e^{2W_s} (X_s^2 + Y_s^2) ds < \infty.$$ 

But the first condition follows from the second thanks to the formula for $\langle A \rangle$ provided by a previous question. On the other hand, if $t \in [0, r]$ then $f'(t) \geq 0$ and thus $W_s \leq f(t)$ for all $s \in [0, t]$, which implies that the second condition is satisfied by using $\mathbb{E} (X_s^2 + Y_s^2) = 2s$.

Exercise 3 (Criterion for a stochastic differential equation). Set $d = 1$. Let $\sigma$, $b$ be two functions $\mathbb{R} \to \mathbb{R}$ such that for some finite constant $C < \infty$ and for all $x, y \in \mathbb{R}$,

$$|\sigma(x) - \sigma(y)| \leq C\sqrt{x - y} \quad \text{and} \quad |b(x) - b(y)| \leq C|x - y|$$

The goal of this exercise is to prove pathwise uniqueness for the stochastic differential equation

$$dX_t = \sigma(X_t) dB_t + b(X_t) dt. \quad \text{(SDE)}$$

A solution $X$ is a continuous semi-martingale with canonical decomposition $X = X_0 + M + V$ with $X_0 \in L^2$, local martingale part $M := \int_0^\cdot \sigma(X_s) dB_s$, and finite variation part $V := \int_0^\cdot b(X_s) ds$. Note that the continuity of $\sigma$, $X$, $b$ gives that almost surely, for all $t \geq 0$, $s \rightarrow \sigma(X_s) + b(X_s)$ is locally bounded.

1. Let $Z$ be a continuous semi-martingale such that $\langle Z \rangle = \int_0^\cdot \varphi_s ds$ for a progressive process $\varphi$ such that $0 \leq \varphi \leq C|Z|$ for some constant $C < \infty$. Prove that for all $t \geq 0$ and all $a > 0$,

$$\mathbb{E} \int_0^t \frac{1_{|Z_s| \leq a}}{|Z_s|} d\langle Z \rangle_s \leq Ct.$$

2. Deduce from the preceding question that for all $t \geq 0$,

$$\lim_{n \to \infty} n\mathbb{E} \int_0^t 1_{|Z_s| \leq \frac{a}{n}} d\langle Z \rangle_s = 0.$$
3. For all \( n \geq 1, x \in \mathbb{R} \), let us define \( g_n(x) := 2n(1 + nx)\mathbf{1}_{x \in (-\frac{1}{n}, 0)} + 2n\mathbf{1}_{x = 0} + 2n(1 - nx)\mathbf{1}_{x \in (0, \frac{1}{n})} \).
Let \( f_n : \mathbb{R} \to \mathbb{R} \) be the twice differentiable function such that \( f_n'' = g_n \) and \( f_n(0) = f_n'(0) = 0 \). Show that for all \( x \in \mathbb{R} \), the following properties hold true:
(a) \( f_n'(x) \in [-1, 1] \) and \( \lim_{n \to \infty} f_n'(x) = \text{sign}(x) := \mathbf{1}_{x > 0} - \mathbf{1}_{x < 0} \);
(b) \( |f_n(x)| \leq |x| \) and \( \lim_{n \to \infty} f_n(x) = |x| \).

4. By using Itô formula, prove that for all continuous semi-martingale \( Z = (Z_t)_{t \geq 0} \), all \( t \geq 0 \),
\[
\int_0^t 1_{Z_s = 0} d(Z)_s = 0.
\]

5. From now on, let \( X \) and \( X' \) be two solutions of (SDE) on \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})\) and with respect to the Brownian motion \( B \). Show that for all \( t \geq 0 \),
\[
<X - X'>_t = \int_0^t (\sigma(X_s) - \sigma(X'_s))^2 ds.
\]

6. By using the assumption on \( \sigma \), deduce from the preceding questions that for all \( t \geq 0 \),
\[
\lim_{n \to \infty} \mathbb{E} \int_0^t g_n(X_s - X'_s) d\langle X - X' \rangle_s = 0.
\]

7. Set \( Z := X - X' \). From now on, let \( T \) be a stopping time such that the semi-martingale \( (Z_{t \wedge T})_{t \geq 0} \) is bounded. By using notably the assumption on \( \sigma \), prove that for all \( t \geq 0, \ n \geq 1 \),
\[
\mathbb{E}(f_n(Z_{t \wedge T})) = \mathbb{E}(f_n(Z_0)) + \int_0^{t \wedge T} f_n'(Z_s)(b(X_s) - b(X'_s)) ds + \frac{1}{2} \int_0^{t \wedge T} f_n''(Z_s) d\langle Z \rangle_s.
\]

8. Deduce from the preceding questions and the assumption on \( b \) that for all \( t \geq 0 \),
\[
\mathbb{E}(|X_{t \wedge T} - X'_{t \wedge T}|) = \mathbb{E}(|X_0 - X'_0|) + \int_0^{t \wedge T} (b(X_s) - b(X'_s)) \text{sign}(X_s - X'_s) ds.
\]

9. By using the Grönwall lemma, deduce that if \( X_0 = X'_0 \) then \( X_t = X'_t \) for all \( t \geq 0 \).

**Elements of solution for Exercise 3.** The result is known as the Yamada–Watanabe criterion. This is a slightly more detailed version of [1, Exercise 8.14 pages 231–232].

1. We have, using the properties of \( Z \) and \( \varphi \),
\[
\int_0^t \frac{\mathbf{1}_{0 <|Z_s| \leq a}}{|Z_s|} d\langle Z \rangle_s = \int_0^t \frac{\mathbf{1}_{0 <|Z_s| \leq a}}{|Z_s|} \varphi_s ds \leq \int_0^t C ds = Ct.
\]

2. For all \( n \geq 1 \), have \( n \mathbf{1}_{0 <|Z_s| \leq \frac{1}{n}} \leq \frac{n \mathbf{1}_{0 <|Z_s| \leq \frac{1}{n}}}{|Z_s|} \leq \frac{1}{2} n \mathbf{1}_{0 <|Z_s| \leq \frac{1}{n}} \), which is integrable on \([0, t] \) by the preceding question used with \( a = 1 \), and thus the desired result follows then by dominated convergence.

3. The function \( g_n \) is \( 0 \) on \((-\infty, -\frac{1}{n}) \), then increases from \( 0 \) to \( 2 \) on \([-\frac{1}{n}, 0] \), then decreases from \( 2 \) to \( 0 \) on \([0, \frac{1}{n}] \), then stays at 0 on \( [\frac{1}{n}, +\infty) \). Since \( \int_{-\infty}^0 g_n(y) dy = 1 \), we have, for all \( x \in \mathbb{R} \),
\[
f_n'(x) = \int_0^x g_n(u) du \quad \text{in such a way that } f_n'(0) = 0 \text{ and } f_n'' = g_n.
\]
The function \( f_n' \) is \(-1 \) on \((-\infty, -\frac{1}{n}) \), \( 0 \) at 0, and \( 1 \) on \([\frac{1}{n}, +\infty) \). Also for all \( x \in \mathbb{R} \),
\[
\lim_{n \to \infty} f_n'(x) = \mathbf{1}_{x > 0} - \mathbf{1}_{x < 0} =: \text{sign}(x).
\]

Next, for all \( x \in \mathbb{R} \), we have
\[
f_n(x) = \int_0^x f_n'(u) du \quad \text{in such a way that } f_n(0) = 0 \text{ and } f_n'' = g_n.
\]

Since \( g_n \geq 0 \), we have that \( f_n' \) is non-decreasing, and thus \( f_n' \) takes actually its values in \([-1, 1] \), and is in particular bounded. It follows by dominated convergence that for all \( x \in \mathbb{R} \),
\[
\lim_{n \to \infty} f_n(x) = \int_0^x \lim_{n \to \infty} f_n'(u) du = \int_0^x \text{sign}(u) du = |x|.
\]

Finally, for all \( x \in \mathbb{R} \), \( |f_n(x)| \leq \int_0^{|x|} du = |x| \).
4. The Itô formula for function $f_n$ of question 3 and semi-martingale $Z$ gives, for all $t \geq 0$,

$$f_n(Z_t) = f_n(Z_0) + \int_0^t f'_n(Z_s)\,dZ_s + \frac{1}{2} \int_0^t f''_n(Z_s)\,d\langle Z \rangle_s.$$ 

Since $\frac{f''_n(x)}{2n} \leq 1$ and $\lim_{n \to \infty} \frac{f''_n(x)}{2n} = 1_{x=0}$ for all $x \in \mathbb{R}$, by dominated convergence,

$$\lim_{n \to \infty} \frac{1}{2n} \int_0^t f''_n(Z_s)\,d\langle Z \rangle_s = \int_0^t 1_{Z_s=0} d\langle Z \rangle_s \quad \text{a.s.}$$

On the other hand, since by question 3, $\lim_{n \to \infty} \frac{f'(x)}{2n} = 0$ for all $x \in \mathbb{R}$, it follows that a.s.

$$\lim_{n \to \infty} \frac{f_n(Z_t)}{2n} = 0 \quad \text{and} \quad \lim_{n \to \infty} \frac{f_n(Z_0)}{2n} = 0.$$ 

Finally since by question 3, $\frac{1}{2n} f'(x) \leq 1$ and $\lim_{n \to \infty} \frac{f'(x)}{2n} = 0$ for all $x \in \mathbb{R}$, dominated convergence for stochastic integrals gives

$$\frac{1}{2n} \int_0^t f''_n(Z_s)\,dZ_s \xrightarrow{n \to \infty} 0.$$ 

5. Since $X$ and $X'$ are both solutions on the same space and for the same Brownian motion, we have, for all $X_0$ and $X'_0$ and for all $t \geq 0$,

$$X_t - X'_t = X_0 - X'_0 + \int_0^t (\sigma(X_s) - \sigma(X'_s))\,dB_s + \int_0^t (b(X_s) - b(X'_s))\,ds.$$ 

The right hand side gives the canonical decomposition of the semi-martingale $X - X'$. In this decomposition, the first integral is the local martingale part, and

$$\langle X - X' \rangle = \int_0^t (\sigma(X_s) - \sigma(X'_s))^2\,ds.$$ 

6. By assumption on $\sigma$, the process $\varphi := (\sigma(X) - \sigma(X'))^2$ satisfies $0 \leq \varphi \leq C|X - X'|$. We can then use question 5 and question 2 with $Z := X - X'$ to get, for all $t \geq 0$,

$$\lim_{n \to \infty} n\mathbb{E} \int_0^t 1_{0 < |Z_s| \leq \frac{1}{n}} d\langle Z \rangle_s = 0.$$ 

If $g_n$ is as in question 3, then for all $x \in \mathbb{R}$, $0 \leq g_n(x) \leq 2n 1_{0 < |x| \leq \frac{1}{n}} + 2n 1_{x=0}$. Thus, for all $t \geq 0$,

$$0 \leq \mathbb{E} \int_0^t g_n(Z_s)\,d\langle Z \rangle_s \leq 2n \mathbb{E} \int_0^t 1_{0 < |Z_s| \leq \frac{1}{n}} d\langle Z \rangle_s + 2n \mathbb{E} \int_0^t 1_{Z_s=0} d\langle Z \rangle_s \xrightarrow{n \to \infty} 0,$$

where we have used question 2 and question 4.

7. The Itô formula for the $t^2$ function $f_n$ and the continuous semi-martingale $Z^T$ gives

$$f_n(Z^T_t) = f_n(Z^T_0) + \int_0^{t \wedge T} f'_n(Z_s)\,dZ_s + \frac{1}{2} \int_0^{t \wedge T} f''_n(Z_s)\,d\langle Z \rangle_s,$$

and since $dZ_s = (\sigma(X_s) - \sigma(X'_s)) dB_s + (b(X_s) - b(X'_s)) \,ds$, we get

$$\int_0^{t \wedge T} f'(Z_s)\,dZ_s = \int_0^t f'(Z^T_s)(\sigma(X_s) - \sigma(X'_s)) 1_{s \leq T} dB_s + \int_0^{t \wedge T} f'(Z_s)(b(X_s) - b(X'_s))\,ds.$$ 

Now, by the assumptions on $\sigma$ and $T$, we get

$$|\sigma(X_s) - \sigma(X'_s)| 1_{s \leq T} \leq C \sqrt{|Z^T_s|} \leq C'.$$

This boundedness, together with the one of $f''_n$, imply that the first integral in the right hand side above (the dB integral) is a martingale. Since this martingale is issued from the origin, its expectation vanishes for all times. On the other hand, since $f_n$ is continuous and $Z^T$ is bounded, the random variables $f_n(Z^T_t)$ and $f_n(Z^T_0)$ are integrable. All in all, we obtain

$$\mathbb{E}(f_n(Z^T_T)) = \mathbb{E}(f_n(Z^T_0)) + \mathbb{E} \int_0^{t \wedge T} f'_n(Z_s)(b(X_s) - b(X'_s))\,ds + \frac{1}{2} \mathbb{E} \int_0^{t \wedge T} f''_n(Z_s)\,d\langle Z \rangle_s.$$ 

8. Since \( f''_n = g_n \geq 0 \), we get, by using question 6, that

\[
0 \leq \mathbb{E} \int_0^{t \wedge T} f''_n(Z_s) d\langle Z \rangle_s \leq \mathbb{E} \int_0^t g_n(Z_s) d\langle Z \rangle_s \overset{n \to \infty}{\to} 0.
\]

On the other hand, by the assumption on \( b \) and the boundedness of \( Z^T \), we have, on \( \{s \leq T\} \),

\[
|b(X_s) - b(X'_s)|^2 \leq C^2 |X_s - X'_s|^2 = C^2 |Z_s^T| \leq C'.
\]

But since \( f'_n \) is bounded (takes its values in \([-1, 1]\)), we get, by dominated convergence

\[
\lim_{n \to \infty} \int_0^{t \wedge T} f'_n(Z_s)(b(X_s) - b(X'_s)) ds = \int_0^{t \wedge T} \text{sign}(Z_s)(b(X_s) - b(X'_s)) ds.
\]

Finally, since \( Z^T \) is bounded, and since from question 3, for all \( x \in \mathbb{R} \), \( |f_n(x)| \leq |x| \) and \( \lim_{n \to \infty} f_n(x) = x \), we get, by dominated convergence, \( \lim_{n \to \infty} \mathbb{E}(f_n(Z^T_s)) = \mathbb{E}(|Z^T_s|) \). Finally

\[
\mathbb{E}(|X_{t \wedge T} - X'_{t \wedge T}|) = \mathbb{E}(|X_0 - X'_0|) + \mathbb{E} \int_0^{t \wedge T} (b(X_s) - b(X'_s)) \text{sign}(X_s - X'_s) ds.
\]

9. From the preceding question, we get, by using the assumption on \( b \),

\[
\alpha(t) := \mathbb{E}(|X_{t \wedge T} - X'_{t \wedge T}|) \leq \mathbb{E}(|X_0 - X'_0|) + C \mathbb{E} \int_0^t |X_{s \wedge T} - X'_{s \wedge T}| ds = \alpha(0) + C \int_0^t \alpha(s) ds.
\]

By the Grönwall lemma, we obtain \( \alpha(t) \leq \alpha(0)e^{Ct} \) for all \( t \geq 0 \). It follows that if \( \alpha(0) = 0 \) then \( \alpha(t) = 0 \) for all \( t \geq 0 \). This means that if \( X_0 = X'_0 \) then \( X_{t \wedge T} = X'_{t \wedge T} \) for all \( t \geq 0 \). By writing this for \( t \in \mathbb{Q}_+ \), and by taking \( T = T_m \) such that \( \lim_{m \to \infty} T_m = +\infty \) almost surely, we get that \( X_t = X'_t \) for all \( t \in \mathbb{Q}_+ \), and thus for all \( t \geq 0 \) since \( X \) and \( X' \) are continuous.

References