Exercise 1 (Nature of an integral). Set $d = 1$. Let us consider the following integral, for $t \geq 0$,

$$I_t = \int_0^t B_s \, ds.$$ 


2. Show that $d(tB_t) = B_t \, dt + t \, dB_t$;

3. Deduce from the preceding question that $I_t = \int_0^t (t - s) \, dB_s$ for all $t \geq 0$;

4. Deduce from the preceding question that $I_t \sim \mathcal{N}(0, \frac{1}{3} t^3)$ for all $t \geq 0$;

5. For all $t \geq 0$, $n \geq 1$, $0 \leq k \leq n$, let us define $t_k = \frac{k}{n} t$. Show that

$$\sum_{k=0}^{n-1} B_{t_k}(t_{k+1} - t_k) = \frac{t}{n} \sum_{j=0}^{n-2} (n - j - 1)(B_{t_{j+1}} - B_{t_j}).$$

6. Deduce from the preceding question another proof that $I_t \sim \mathcal{N}(0, \frac{1}{3} t^3)$ for all $t \geq 0$;

7. Is the process $(I_t)_{t \geq 0}$ a martingale?

Elements of solution for Exercise 1.

1. Since the integrator is of finite variation and the integrand is bounded and measurable (actually continuous), it is a Lebesgue–Stieltjes integral, and in particular an Itô integral with respect to a semi-martingale without martingale part. However it is not a Wiener integral.

2. The Itô formula for $f(x, y) = xy$ and $X_t = (t, B_t)$ gives

$$tB_t = 0 + \int_0^t B_s \, ds + \int_0^t s \, dB_s.$$

3. From the preceding question (actually, it is an integration by parts)

$$\int_0^t B_s \, ds = tB_t - \int_0^t s \, dB_s = \int_0^t (t - s) \, dB_s.$$

4. The integral in the right hand side is a Wiener integral. Thus it is Gaussian with mean zero and variance equal to the squared $L^2$ norm of the integrand:

$$\mathbb{E} \left[ \int_0^t B_s \, ds \right] = 0 \quad \text{and} \quad \mathbb{E} \left( \left( \int_0^t B_s \, ds \right)^2 \right) = \int_0^t (t - s)^2 \, ds = \frac{t^3}{3}.$$

5. With $t_k = \frac{k}{n} t$ for all $0 \leq k \leq n$, we have

$$S_n = \sum_{k=0}^{n-1} B_{t_k}(t_{k+1} - t_k) = \frac{t}{n} \sum_{k=0}^{n-1} B_{t_k} = \frac{t}{n} \sum_{k=0}^{n-1} \sum_{j=0}^{k-1} (B_{t_{j+1}} - B_{t_j}) = \frac{t}{n} \sum_{j=0}^{n-2} (n - j - 1)(B_{t_{j+1}} - B_{t_j}).$$
6. Fix \( t \geq 0 \). Since \( I_t \) is a Lebesgue–Stieltjes integral with continuous integrand, we have \( \lim_{n \to \infty} S_n = I_t \) almost surely and thus in law. For all \( n \) and \( j \), since \( B_{t+j} - B_t \) are independent and Gaussian, we get that \( S_n \sim \mathcal{N}(E(S_n),E(S_n^2) - E(S_n)^2) \). But the convergence in law of Gaussians is equivalent to the convergence of the first two moments. Now it remains to note that we have \( E(S_n) = 0 \) and

\[
\mathbb{E}(S_n^2) = \frac{t^2}{n^2} \sum_{j=0}^{n-2} (n-j-1)^2 \mathbb{E}((B_{t+j} - B_t)^2) = \frac{t^3}{n^3} \sum_{j=0}^{n-1} j^2 = t^3 \sum_{j=1}^{n} \left( \frac{1}{n} \right)^2 = \frac{t^3}{n} \to \frac{t^3}{t^3} = \frac{t^2}{3}.
\]

7. Beware that the integrand in \( f_0^t (t-s)dB_s \) depends on \( t \). The process \((- f_0^t s dB_s)_{t \geq 0} \) is a martingale, however the process \((f_0^t t dB_s)_{t \geq 0} = (t B_t)_{t \geq 0} \) is not a martingale: for all \( 0 \leq s \leq t \),

\[
\mathbb{E}((t + s)B_{t+s} \mid \mathcal{F}_t) = (t + s)B_s \neq sB_s.
\]

Exercise 2 (Study of a special process). Set \( d = 2 \). For all \( t \geq 0 \), we write \( B_t = (X_t, Y_t) \) and

\[
A_t = \int_0^t X_s dY_s - \int_0^t Y_s dX_s.
\]

1. Show that \( \langle A \rangle = \int_0^t (X_s^2 + Y_s^2)ds \) and that the process \( A \) is a square integrable martingale;

2. From now on let \( \lambda > 0 \). Show that for all \( t \geq 0 \),

\[
\mathbb{E}e^{\lambda A_t} = \mathbb{E}\cos(\lambda A_t).
\]

3. From now on, let \( f : \mathbb{R}_+ \to \mathbb{R} \) be \('e^2'\), and let us define the continuous semi-martingales

\[
(Z_t)_{t \geq 0} = (\cos(\lambda A_t))_{t \geq 0} \quad \text{and} \quad (W_t)_{t \geq 0} = \left( -\frac{f'(t)}{2} (X_t^2 + Y_t^2) + f(t) \right)_{t \geq 0}.
\]

Show that for all \( t \geq 0 \),

\[
Z_t = 1 - \Lambda \int_0^t \sin(\lambda A_s) dA_s - \frac{\Lambda^2}{2} \int_0^t (X_s^2 + Y_s^2) Z_s ds.
\]

and

\[
W_t = f(0) - \int_0^t f'(s) X_s dX_s - \int_0^t f'(s) Y_s dY_s - \frac{1}{2} \int_0^t f''(s) (X_s^2 + Y_s^2) ds,
\]

and deduce that

\[
\langle Z, W \rangle = 0.
\]

4. Show that if \( f \) solves \( f'' = f'^2 - \lambda^2 \) then \( Z e^{W} \) is a continuous local martingale and

\[
Z e^{W} = e^{f(0)} - \Lambda \int_0^t \sin(\lambda A_s) e^{W}_s dA_s - \int_0^t f'(s) Z_s e^{W_s} X_s dX_s - \int_0^t f'(s) Z_s e^{W_s} Y_s dY_s.
\]

5. Let \( r > 0 \). By using \( f(t) = -\cosh(\lambda(r-t)) \) deduce from the previous question that

\[
\mathbb{E}e^{\lambda A_t} = \frac{1}{\cosh(\lambda r)}.
\]

Elements of solution for Exercise 2. For all \( t \geq 0 \), \( A_t \) is the algebraic area between planar Brownian motion and its chord, and the process \( A \) is the \( \text{Lévy area} \). This exercise is a slightly more detailed version of \([1, \text{Exercise 5.30 pages 144–145}]. Its goal is to compute the characteristic function or Fourier transform of \( A_t \).
1. Since \(\langle X \rangle_t = \langle Y \rangle_t = t\) and \(\langle X, Y \rangle_t = 0\), we get
\[
\langle A \rangle_t = \langle A, A \rangle_t = \left( \int_0^t X_s \, dY_s \right)^2 + \left( \int_0^t Y_s \, dX_s \right)^2 + 2 \left( \int_0^t X_s \, dY_s \int_0^t Y_s \, dX_s \right)_t
= \int_0^t X_s^2 \, d\langle X \rangle_s + \int_0^t Y_s^2 \, d\langle Y \rangle_s + \int_0^t X_sY_s \, d\langle X, Y \rangle_s
= \int_0^t (X_s^2 + Y_s^2) \, ds.
\]
It follows by the Fubini–Tonelli theorem that
\[
\mathbb{E}(A)_t = \int_0^t \mathbb{E}(X_s^2 + Y_s^2) \, ds = \int_0^t 2s \, ds = t^2 < \infty
\]
and thus, by a famous martingale criterion, the process \(A\) is a square integrable martingale.

Alternatively, since for all \(t \geq 0\), \(\int_0^t X_s^2 \, d\langle Y \rangle_s = \int_0^t Y_s^2 \, d\langle X \rangle_s < \infty\) the process \(\int_0^t X_s \, dY_s\) and by symmetry the process \(\int_0^t Y_s \, dX_s\) are both square integrable martingales, and thus the process \(A\) is also a square integrable martingales as being the difference of two square integrable martingales.

2. For all \(\lambda \in \mathbb{R}, t \geq 0\), \(\mathbb{E}(e^{i\lambda A_t}) = \mathbb{E}(\cos(\lambda A_t)) + i\mathbb{E}(\sin(\lambda A_t))\). Since \(\langle X, Y \rangle_t = (X_t, Y_t)\), we get, for all \(t \geq 0\)
\[
-A_t = \int_0^t Y_s^2 \, ds - \int_0^t X_s^2 \, ds - \int_0^t X_s \, dY_s - \int_0^t Y_s \, dX_s = A_t,
\]
and thus the characteristic function or Fourier transform of \(A_t\) is real.

3. The canonical decompositions are given by the Itô formula. Namely, for \(Z\),
\[
Z_t = 1 - \lambda \int_0^t \sin(\lambda A_s) \, dA_s - \frac{\lambda^2}{2} \int_0^t \cos(\lambda A_s) \, d\langle A \rangle_s
= 1 - \lambda \int_0^t \sin(\lambda A_s) \, dA_s - \frac{\lambda^2}{2} \int_0^t (X_s^2 + Y_s^2) \, Z_s \, ds.
\]
Similarly, for \(W\), by the Itô formula for the function \(g(x, y, t) = -\frac{1}{2}t(x^2 + y^2) + f(t)\) and the vector of semi-martingale \(S_t = (X_t, Y_t, t)\) with martingale part \((X_t, Y_t)\),
\[
W_t = g(0, 0, 0) + \int_0^t \partial_1 g(S_s) \, dX_s + \int_0^t \partial_2 g(S_s) \, dY_s + \int_0^t \partial_3 g(S_s) \, ds + \frac{1}{2} \int_0^t (\partial_2^2 g + \partial_3^2 g)(S_s) \, ds
= f(0) - \int_0^t f'(s)X_s \, ds - \int_0^t f'(s)Y_s \, ds + \int_0^t \left( -\frac{f''(s)}{2}X_s^2 + Y_s^2 \right) \, ds - \int_0^t f'(s) \, ds
= f(0) - \int_0^t f'(s)X_s \, ds - \int_0^t f'(s)Y_s \, ds - \frac{1}{2} \int_0^t f''(s)(X_s^2 + Y_s^2) \, ds.
\]
The computation of \(\langle Z, W \rangle_t\) involves only the local martingale parts, namely
\[
\langle Z, W \rangle_t = \lambda \left( \int_0^t \sin(\lambda A_s) \, dA_s \int_0^t f'(s)X_s \, dX_s + \int_0^t f'(s)Y_s \, dY_s \right)_t
= \lambda \int_0^t f'(s) \sin(\lambda A_s) X_s \, d(A, X)_s + \lambda \int_0^t f'(s) \sin(\lambda A_s) Y_s \, d(A, Y)_s.
\]
Now since \(\langle X, Y \rangle_t = -\int_0^t Y_s \, ds\) and \(\langle A, Y \rangle_t = \int_0^t X_s \, ds\), we get
\[
\langle Z, W \rangle_t = \lambda \int_0^t f'(s)(-X_sY_s + X_sY_s) \, ds = 0.
\]

4. The Itô formula gives (we benefit from the fact that \(\langle Z, W \rangle = 0\) from the previous question)
\[
Z_t e^{W_t} = e^{f(0)} + \int_0^t e^{W_s} \, dZ_s + \int_0^t Z_s e^{W_s} \, dW_s + \frac{1}{2} \int_0^t Z_s e^{W_s} \, d\langle W \rangle_s.
\]
The goal of this exercise is to prove pathwise uniqueness for the stochastic differential equation

\[
M_t = \int_0^t \sigma(s) X_s dB_s + \int_0^t X_s dB_t + \int_0^t \int_0^t Z_s e^W_t d(W)_s.
\]

Now from a previous question

\[
\langle W \rangle_t = \left( \int_0^t f'(s) X_s dX_s + \int_0^t f'(s) Y_s dY_s \right)_t = \int_0^t f''(s) (X_s^2 + Y_s^2) ds.
\]

It follows that the finite variation part of \( Z e^W \) vanishes when \( f'' = f'^2 - \lambda^2 \).

5. With \( f(t) = -\log \cosh(\lambda (r-t)) \), we have

\[
f'(t) = \lambda \frac{\sinh(\lambda (r-t))}{\cosh(\lambda (r-t))} = \lambda \tanh(\lambda (r-t))
\]

and

\[
f''(t) = -\frac{\lambda^2}{\cosh(\lambda (r-t))^2} = -\lambda^2 (1 - \tanh(\lambda (r-t))^2) = -\lambda^2 + f'^2(t).
\]

It follows from the previous question that \( Z e^W \) is a continuous local martingale. Note that \( f(r) = f'(r) = 0 \) and \( W_r = 0 \), and by using previous questions,

\[
E e^{jA_r} = E \cos(\lambda A_r) = E Z_r = E (Z_r e^{W_r}).
\]

On the other hand, since \( f(0) = -\log \cosh(\lambda r) \), \( Z_0 = 1 \), \( W_0 = f(0) \), we get

\[
E (Z_0 e^{W_0}) = e^{f(0)} = \frac{1}{\cosh(\lambda r)}.
\]

It remains to show that the local martingale \( Z e^W \) is a martingale on the time interval \([0, r]\). From the previous question, since \( f \), \( \cos \), and \( \sin \) are bounded, it suffices to show that

\[
E \int_0^t e^{2W_s} d(A)_s < \infty \quad \text{and} \quad E \int_0^t e^{2W_s} (X_s^2 + Y_s^2) ds < \infty.
\]

But the first condition follows from the second thanks to the formula for \( \langle A \rangle \) provided by a previous question. On the other hand, if \( t \in [0, r] \) then \( f'(t) \geq 0 \) and thus \( W_s \leq f(t) \) for all \( s \in [0, t] \), which implies that the second condition is satisfied by using \( E(X_s^2 + Y_s^2) = 2s \).

**Exercise 3** (Criterion for a stochastic differential equation). Set \( d = 1 \). Let \( \sigma, b \) be two functions \( \mathbb{R} \to \mathbb{R} \) such that for some finite constant \( C < \infty \) and for all \( x, y \in \mathbb{R}, \)

\[
|\sigma(x) - \sigma(y)| \leq C |x - y| \quad \text{and} \quad |b(x) - b(y)| \leq C |x - y|
\]

The goal of this exercise is to prove pathwise uniqueness for the stochastic differential equation

\[
dX_t = \sigma(X_t) dB_t + b(X_t) dt.
\]

A solution \( X \) is a continuous semi-martingale with canonical decomposition \( X = X_0 + M + V \) with \( X_0 \in L^2 \), local martingale part \( M = \int_0^t \sigma(X_s) dB_s \), and finite variation part \( V = \int_0^t b(X_s) ds \). Note that the continuity of \( \sigma, X, b \) gives that almost surely, for all \( t \geq 0 \), \( s \mapsto \sigma(X_s) + b(X_s) \) is locally bounded.

1. Let \( Z \) be a continuous semi-martingale such that \( \langle Z \rangle = \int_0^t \varphi ds \) for a progressive process \( \varphi \) such that \( 0 \leq \varphi \leq C |Z| \) for some constant \( C < \infty \). Prove that for all \( t \geq 0 \) and all \( a > 0 \),

\[
E \int_0^t \frac{1_{0 < |Z_s| \leq a}}{|Z_s|} d\langle Z \rangle_s \leq Ct.
\]
2. Deduce from the preceding question that for all \( t \geq 0 \),
\[
\lim_{n \to \infty} nE \int_0^t 1_{0 < |Z_s| \leq \frac{1}{n}} d\langle Z \rangle_s = 0.
\]

3. For all \( n \geq 1 \), \( x \in \mathbb{R} \), let us define \( g_n(x) = 2n(1 + nx)1_{x \in (-\frac{1}{n}, 0)} + 2n1_{x = 0} + 2n(1 - nx)1_{x \in (0, \frac{1}{n})} \).
Let \( f_n : \mathbb{R} \to \mathbb{R} \) be the twice differentiable function such that \( f_n'' = g_n \) and \( f_n(0) = f_n'(0) = 0 \).
Show that for all \( x \in \mathbb{R} \), the following properties hold true:

   (a) \( f_n'(x) \in [-1, 1] \) and \( \lim_{n \to \infty} f_n'(x) = \text{sign}(x) = 1_{x > 0} - 1_{x < 0} \);
   
   (b) \(|f_n(x)| \leq |x|\) and \( \lim_{n \to \infty} f_n(x) = |x| \).

4. By using Itô formula, prove that for all continuous semi-martingale \( Z = (Z_t)_{t \geq 0} \), all \( t \geq 0 \),
\[
\int_0^t 1_{Z_s = 0} d\langle Z \rangle_s = 0.
\]

5. From now on, let \( X \) and \( X' \) be two solutions of (SDE) on \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})\) and with respect to the Brownian motion \( B \). Show that for all \( t \geq 0 \),
\[
\langle X - X' \rangle_t = \int_0^t (\sigma(X_s) - \sigma(X'_s))^2 \, ds.
\]

6. By using the assumption on \( \sigma \), deduce from the preceding questions that for all \( t \geq 0 \),
\[
\lim_{n \to \infty} \mathbb{E} \int_0^t g_n(X_s - X'_s) d\langle X - X' \rangle_s = 0.
\]

7. Set \( Z = X - X' \). From now on, let \( T \) be a stopping time such that the semi-martingale \((Z_{t \wedge T})_{t \geq 0}\) is bounded. By using notably the assumption on \( \sigma \), prove that for all \( t \geq 0, n \geq 1 \),
\[
\mathbb{E}(f_n(Z_{t \wedge T})) = \mathbb{E}(f_n(Z_0)) + \mathbb{E} \int_0^{t \wedge T} f_n'(Z_s)(b(X_s) - b(X'_s)) \, ds + \frac{1}{2} \mathbb{E} \int_0^{t \wedge T} f_n''(Z_s) d\langle Z \rangle_s.
\]

8. Deduce from the preceding questions and the assumption on \( b \) that for all \( t \geq 0 \),
\[
\mathbb{E}(|X_{t \wedge T} - X'_{t \wedge T}|) = \mathbb{E}(|X_0 - X'_0|) + \mathbb{E} \int_0^{t \wedge T} (b(X_s) - b(X'_s)) \text{sign}(X_s - X'_s) \, ds.
\]

9. By using the Grönwall lemma, deduce that if \( X_0 = X'_0 \) then \( X_T = X'_T \) for all \( T \geq 0 \).

**Elements of solution for Exercise 3.** The result is known as the Yamada–Watanabe criterion. This is a slightly more detailed version of [1, Exercise 8.14 pages 231–232].

1. We have, using the properties of \( Z \) and \( \varphi \),
\[
\int_0^t 1_{0 < |Z_s| \leq \varphi_s} \, d\langle Z \rangle_s = \int_0^t \frac{1_{0 < |Z_s| \leq \varphi_s}}{|Z_s|} \varphi_s \, ds \leq \int_0^t C \, ds = Ct.
\]

2. For all \( n \geq 1 \), we have \( n1_{0 < |Z_s| \leq \frac{1}{n}} \leq \frac{1_{0 < |Z_s| \leq \varphi_s}}{|Z_s|} \leq 1_{0 < |Z_s| \leq 1} \), which is integrable on \([0, t]\) by the preceding question used with \( a = 1 \), and thus the desired result follows then by dominated convergence.

3. The function \( g_n \) is 0 on \((\infty, -\frac{1}{n})\), then increases from 0 to 2 on \([-\frac{1}{n}, 0)\), then decreases from 2 to 0 on \([0, \frac{1}{n})\), then stays at 0 on \([\frac{1}{n}, +\infty)\). Since \( \int_{-\infty}^0 g_n(y) \, dy = 1 \), we have, for all \( x \in \mathbb{R} \),
\[
f'_n(x) = \int_0^x g_n(u) \, du, \quad \text{in such a way that } f'_n(0) = 0 \text{ and } f''_n = g_n.
\]

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The function \( f_n' \) is \(-1\) on \((\infty, -\frac{1}{n}]\), \(0\) at \(0\), and \(1\) on \([\frac{1}{n}, +\infty)\). Also for all \(x \in \mathbb{R}\),

\[
\lim_{n \to \infty} f_n'(x) = 1_{x>0} - 1_{x<0} =: \text{sign}(x).
\]

Next, for all \(x \in \mathbb{R}\), we have

\[
f_n(x) = \int_0^x f_n'(u)\,du \quad \text{in such a way that } f_n(0) = 0 \text{ and } f_n'' = g_n.
\]

Since \(g_n \geq 0\), we have that \(f_n'\) is non-decreasing, and thus \(f_n'\) takes actually its values in \([-1,1]\), and is in particular bounded. It follows by dominated convergence that for all \(x \in \mathbb{R}\),

\[
\lim_{n \to \infty} f_n(x) = \int_0^x \lim_{n \to \infty} f_n'(u)\,du = \int_0^x \text{sign}(u)\,du = |x|.
\]

Finally, for all \(x \in \mathbb{R}\), \(|f_n(x)| \leq f_0(|x|)\,du = |x|\).

4. The Itô formula for function \(f_n\) of question 3 and semi-martingale \(Z\) gives, for all \(t \geq 0\),

\[
f_n(Z_t) = f_n(Z_0) + \int_0^t f_n'(Z_s)\,dZ_s + \frac{1}{2} \int_0^t f_n''(Z_s)\,d\langle Z \rangle_s.
\]

Since \(|f_n''(x)| \leq 1\) and \(\lim_{n \to \infty} f_n''(x) = 1_{x=0}\) for all \(x \in \mathbb{R}\), by dominated convergence,

\[
\lim_{n \to \infty} \int_0^t f_n''(Z_s)\,d\langle Z \rangle_s = \int_0^t 1_{Z_s=0}\,d\langle Z \rangle_s \quad \text{a.s.}
\]

On the other hand, since by question 3, \(\lim_{n \to \infty} \int_0^t |f_n'(Z_s)|\,d\langle Z \rangle_s = 0\) for all \(x \in \mathbb{R}\), it follows that a.s.

\[
\lim_{n \to \infty} \frac{f_n(Z_t)}{2n} = 0 \quad \text{and} \quad \lim_{n \to \infty} \frac{f_n(Z_0)}{2n} = 0.
\]

Finally since by question 3, \(\frac{1}{2n} |f_n'(x)| \leq \frac{1}{2n} \leq 1\) and \(\lim_{n \to \infty} \frac{1}{2n} |f_n'(x)| = 0\) for all \(x \in \mathbb{R}\), dominated convergence for stochastic integrals gives

\[
\frac{1}{2n} \int_0^t f_n'(Z_s)\,dZ_s \xrightarrow{n \to \infty} 0.
\]

5. Since \(X\) and \(X'\) are both solutions on the same space and for the same Brownian motion, we have, for all \(X_0\) and \(X'_0\) and for all \(t \geq 0\),

\[
X_t - X'_t = X_0 - X'_0 + \int_0^t (\sigma(X_s) - \sigma(X'_s))\,dB_s + \int_0^t (b(X_s) - b(X'_s))\,ds.
\]

The right hand side gives the canonical decomposition of the semi-martingale \(X - X'\). In this decomposition, the first integral is the local martingale part, and

\[
\langle X - X' \rangle = \int_0^t (\sigma(X_s) - \sigma(X'_s))^2\,ds.
\]

6. By assumption on \(\sigma\), the process \(\varphi = (\sigma(X) - \sigma(X'))^2\) satisfies \(0 \leq \varphi \leq C|X - X'|\). We can then use question 5 and question 2 with \(Z = X - X'\) to get, for all \(t \geq 0\),

\[
\lim_{n \to \infty} \mathbb{E} \int_0^t 1_{0<|Z_s| \leq \frac{1}{n}} \,d\langle Z \rangle_s = 0.
\]

If \(g_n\) is as in question 3, then for all \(x \in \mathbb{R}\), \(0 \leq g_n(x) \leq 2n 1_{0<|x| \leq \frac{1}{n}} + 2n 1_{x=0}\). Thus, for all \(t \geq 0\),

\[
0 \leq \mathbb{E} \int_0^t g_n(Z_s)\,d\langle Z \rangle_s \leq 2n \mathbb{E} \int_0^t 1_{0<|Z_s| \leq \frac{1}{n}} \,d\langle Z \rangle_s + 2n \mathbb{E} \int_0^t 1_{Z_s=0}\,d\langle Z \rangle_s \xrightarrow{n \to \infty} 0,
\]

where we have used question 2 and question 4.
7. The Itô formula for the \(e^2\) function \(f_n\) and the continuous semi-martingale \(Z^T\) gives
\[
f_n(Z^T_t) = f_n(Z^T_0) + \int_0^{t \wedge T} f'_n(Z_s) \, dB_s + \frac{1}{2} \int_0^{t \wedge T} f''_n(Z_s) \, dB_s^2,
\]
and since \(dB_s = (\sigma(X_s) - \sigma(X'_s)) \, dB_s + (b(X_s) - b(X'_s)) \, ds\), we get
\[
\int_0^{t \wedge T} f'_n(Z_s) \, dB_s = \int_0^t f'_n(Z_s) \, ds + \int_0^{t \wedge T} f'_n(Z_s) \, dB_s.
\]
Now, by the assumptions on \(\sigma\) and \(T\), we get
\[
|\sigma(X_s) - \sigma(X'_s)| \leq C|Z^T_s| \leq C'.
\]
This boundedness, together with the one of \(f'_n\), imply that the first integral in the right hand side above (the \(dB_s\) one) is a martingale. Since this martingale is issued from the origin, its expectation vanishes for all times. On the other hand, since \(f_n\) is continuous and \(Z^T\) is bounded, the random variables \(f_n(Z^T_t)\) and \(f_n(Z^T_0)\) are integrable. All in all, we obtain
\[
E(f_n(Z^T_t)) = E(f_n(Z^T_0)) + \frac{1}{2} \int_0^{t \wedge T} f''_n(Z_s) \, dB_s^2 + \frac{1}{2} \int_0^{t \wedge T} f''_n(Z_s) \, dB_s^2.
\]

8. Since \(f''_n \equiv g_n \geq 0\), we get, by using question 6, that
\[
0 \leq \int_0^{t \wedge T} f''_n(Z_s) \, dB_s^2 \leq \int_0^{t \wedge T} g_n(Z_s) \, dB_s^2 \xrightarrow{n \to \infty} 0.
\]
On the other hand, by the assumption on \(b\) and the boundedness of \(Z^T\), we have, on \(s \leq T\),
\[
|b(X_s) - b(X'_s)|^2 \leq C^2 |X_s - X'_s|^2 = C^2 |Z^T_s| \leq C'.
\]
But since \(f'_n\) bounded (takes its values in \([-1, 1]\)), we get, by dominated convergence
\[
\lim_{n \to \infty} \int_0^{t \wedge T} f'_n(Z_s) (b(X_s) - b(X'_s)) \, ds = \int_0^{t \wedge T} \text{sign}(Z_s) (b(X_s) - b(X'_s)) \, ds.
\]
Finally, since \(Z^T_t\) is bounded, and since from question 3, for all \(x \in \mathbb{R}, |f_n(x)| \leq |x|\) and \(\lim_{n \to \infty} f_n(x) = |x|\), we get, by dominated convergence, \(\lim_{n \to \infty} E(f_n(Z^T_t)) = E(|Z^T_t|)\). Finally
\[
E(|X_{s \wedge T} - X'_{s \wedge T}|) = E(|X_0 - X'_0|) + \int_0^{t \wedge T} (b(X_s) - b(X'_s)) \text{sign}(X_s - X'_s) \, ds.
\]

9. From the preceding question, we get, by using the assumption on \(b\),
\[
\alpha(t) = E(|X_{s \wedge T} - X'_{s \wedge T}|) \leq E(|X_0 - X'_0|) + CE \int_0^t |X_{s \wedge T} - X'_{s \wedge T}| \, ds = \alpha(0) + C \int_0^t \alpha(s) \, ds.
\]
By the Grönwall lemma, we obtain \(\alpha(t) \leq \alpha(0)e^{Ct}\) for all \(t \geq 0\). It follows that if \(\alpha(0) = 0\) then \(\alpha(t) = 0\) for all \(t \geq 0\). This means that if \(X_0 = X'_0\), then \(X_{s \wedge T} = X'_{s \wedge T}\) for all \(t \geq 0\). By writing this for \(t \in \mathbb{Q}_+\), and by taking \(T = T_m\) such that \(\lim_{m \to \infty} T_m = +\infty\) almost surely, we get that \(X_t = X'_t\) for all \(t \in \mathbb{Q}_+\), and thus for all \(t \geq 0\) since \(X\) and \(X'\) are continuous.

References