(Ω, 𝒻, (ℱₜ)ₜ≥₀, ℙ) is a filtered probability space, with complete and right continuous filtration. 

B := (Bₜ)ₜ≥₀ is a d-dimensional Brownian motion issued from the origin, d ≥ 1.

If Z is a semi-martingale, we denote by <Z> the increasing process of its local martingale part.

If Z = Z₀ + M + V, do not confuse <Z> = ⟨M⟩ with the finite variation part V of Z.

**Exercise 1** (Nature of an integral). Set d = 1. Let us consider the following integral, for t ≥ 0,

\[ Iₜ := \int₀ᵗ Bₛds. \]

2. Show that d(tBₜ) = Bₜdt + tdBₜ;
3. Deduce from the preceding question that \(Iₜ = \int₀ᵗ (t - s)dBₛ\) for all t ≥ 0;
4. Deduce from the preceding question that \(Iₜ \sim \mathcal{N}(0, \frac{1}{3}t^3)\) for all t ≥ 0;
5. For all t ≥ 0, n ≥ 1, 0 ≤ k ≤ n, let us define \(tₖ := \frac{k}{n}t\). Show that

\[ \sum_{k=0}^{n-1} B_{tₖ}(tₖ₊₁ - tₖ) = \frac{t}{n} \sum_{j=0}^{n-2} (n - j - 1)(B_{t_{j+1}} - B_{t_{j}}). \]
6. Deduce from the preceding question another proof that \(Iₜ \sim \mathcal{N}(0, \frac{1}{3}t^3)\) for all t ≥ 0;
7. Is the process \((Iₜ)ₜ≥₀\) a martingale?

**Elements of solution for Exercise 1.**

1. Since the integrator is of finite variation and the integrand is bounded and measurable (actually continuous), it is a Lebesgue–Stieltjes integral, and in particular an Itô integral with respect to a semi-martingale without martingale part. However it is not a Wiener integral.
2. The Itô formula for \(f(x, y) = xy\) and \(Xₜ = (t, Bₜ)\) gives

\[ tBₜ = 0 + \int₀ᵗ Bₛds + \int₀ᵗ sdBₛ, \]
3. From the preceding question (actually, it is an integration by parts)

\[ \int₀ᵗ Bₛds = tBₜ - \int₀ᵗ sdBₛ = \int₀ᵗ (t - s)dBₛ. \]
4. The integral in the right hand side is a Wiener integral. Thus it is Gaussian with mean zero and variance equal to the squared \(L^2\) norm of the integrand:

\[ \mathbb{E} \left[ \int₀ᵗ Bₛds \right] = 0 \quad \text{and} \quad \mathbb{E} \left[ \left( \int₀ᵗ Bₛds \right)^2 \right] = \int₀ᵗ (t - s)^2ds = \frac{t^3}{3}. \]
5. With \(tₖ = \frac{k}{n}t\) for all 0 ≤ k ≤ n, we have

\[ Sₙ := \sum_{k=0}^{n-1} B_{tₖ}(tₖ₊₁ - tₖ) = \frac{t}{n} \sum_{k=0}^{n-1} Bₖ = \frac{t}{n} \sum_{k=0}^{n-1} \sum_{j=0}^{k-1} (B_{t_{j+1}} - B_{tₖ}) = \frac{t}{n} \sum_{j=0}^{n-2} (n - j - 1)(B_{t_{j+1}} - B_{tₖ}). \]
6. Fix \( t \geq 0 \). Since \( I_t \) is a Lebesgue–Stieltjes integral with continuous integrand, we have \( \lim_{n \to \infty} S_n = I_t \) almost surely and thus in law. For all \( n \) and \( j \), since \( B_{t_{j+1}} - B_{t_j} \) are independent and Gaussian, we get that
\[
S_n \sim \mathcal{N}(E(S_n), \mathbb{E}(S_n^2) - \mathbb{E}(S_n)^2).
\]
But the convergence in law of Gaussians is equivalent to the convergence of the first two moments. Now it remains to note that we have \( \mathbb{E}(S_n) = 0 \) and
\[
\mathbb{E}(S_n^2) = \frac{t^2}{n^2} \sum_{j=0}^{n-2} (n - j - 1)^2 \mathbb{E}((B_{t_{j+1}} - B_{t_j})^2) = \frac{t^3}{n^3} \sum_{j=0}^{n-1} j^2 = \frac{t^3}{n} \sum_{j=1}^{n-1} j^2 \frac{1}{n} \xrightarrow{n \to \infty} \int_0^1 x^2 \, dx = \frac{t^3}{3}.
\]

7. Beware that the integrand in \( f_0^t (t - s) \, dB_s \) depends on \( t \). The process \( (-f_0^t \, s \, dB_s)_{t \geq 0} \) is a martingale, however the process \( (f_0^t \, t \, dB_s)_{t \geq 0} = (tB_t)_{t \geq 0} \) is not a martingale: for all \( 0 \leq s \leq t \),
\[
\mathbb{E}((t + s)B_{t+s} \mid \mathcal{F}_t) = (t + s)B_s \neq sB_s.
\]

**Exercise 2** (Study of a special process). Set \( d = 2 \). For all \( t \geq 0 \), we write \( B_t = (X_t, Y_t) \) and
\[
A_t := \int_0^t X_s \, dY_s - \int_0^t Y_s \, dX_s.
\]
1. Show that \( \langle A \rangle = \int_0^t (X_s^2 + Y_s^2) \, ds \) and that the process \( A \) is a square integrable martingale;
2. From now on let \( \lambda > 0 \). Show that for all \( t \geq 0 \),
\[
\mathbb{E}e^{\lambda A_t} = \mathbb{E}\cos(\lambda A_t).
\]
3. From now on, let \( f : \mathbb{R}_+ \to \mathbb{R} \) be \( \mathcal{C}^2 \), and let us define the continuous semi-martingales
\[
(Z_t)_{t \geq 0} := (\cos(\lambda A_t))_{t \geq 0} \quad \text{and} \quad (W_t)_{t \geq 0} := \left(-\frac{f'(t)}{2} (X_t^2 + Y_t^2) + f(t)\right)_{t \geq 0}.
\]
Show that for all \( t \geq 0 \),
\[
Z_t = 1 - \lambda \int_0^t \sin(\lambda A_s) \, dA_s - \frac{\lambda^2}{2} \int_0^t (X_s^2 + Y_s^2) \, dZ_s,
\]
and
\[
W_t = f(0) - \int_0^t f'(s) X_s \, dX_s - \int_0^t f'(s) Y_s \, dY_s - \frac{1}{2} \int_0^t f''(s) (X_s^2 + Y_s^2) \, ds,
\]
and deduce that
\[
\langle Z, W \rangle = 0.
\]
4. Show that if \( f \) solves \( f'' = f'^2 - \lambda^2 \) then \( Ze^W \) is a continuous local martingale and
\[
Ze^W_t = e^{f(0)} - \lambda \int_0^t \sin(\lambda A_s) e^{W_s} \, dA_s - \int_0^t f'(s) Ze^{W_s} X_s \, dX_s - \int_0^t f'(s) Ze^{W_s} Y_s \, dY_s.
\]
5. Let \( r > 0 \). By using \( f(t) = -\log \cosh(\lambda(r - t)) \) deduce from the previous question that
\[
\mathbb{E}e^{\lambda A_t} = \frac{1}{\cosh(\lambda r)}.
\]

**Elements of solution for Exercise 2.** For all \( t \geq 0 \), \( A_t \) is the algebraic area between planar Brownian motion and its chord, and the process \( A \) is the Lévy area. This exercise is a slightly more detailed version of [1, Exercise 5.30 pages 144–145]. Its goal is to compute the characteristic function or Fourier transform of \( A_t \).
1. Since \( \langle X \rangle_t = \langle Y \rangle_t = t \) and \( \langle X, Y \rangle_t = 0 \), we get
   \[
   \langle A \rangle_t = \langle A, A \rangle_t = \left( \int_0^t X_s dY_s \right)_t + \left( \int_0^t Y_s dX_s \right)_t + 2 \left( \int_0^t X_s dY_s, \int_0^t Y_s dX_s \right)_t,
   \]
   \[
   = \int_0^t X_s^2 d\langle X \rangle_s + \int_0^t Y_s^2 d\langle Y \rangle_s + \int_0^t X_s Y_s d\langle X, Y \rangle_s
   \]
   \[
   = \int_0^t (X_s^2 + Y_s^2) ds.
   \]
   It follows by the Fubini–Tonelli theorem that
   \[
   \mathbb{E}(A)_t = \int_0^t \mathbb{E}(X_s^2 + Y_s^2) ds = \int_0^t 2sd = t^2 < \infty
   \]
   and thus, by a famous martingale criterion, the process \( A \) is a square integrable martingale.

   Alternatively, since for all \( t \geq 0 \), \( \mathbb{E} \int_0^t X_s^2 d\langle Y \rangle_s = \int_0^t \mathbb{E}(X_s^2) ds < \infty \) the process \( \int_0^t X_s dY_s \) and by symmetry the process \( \int_0^t Y_s dX_s \) are both square integrable martingales, and thus the process \( A \) is also a square integrable martingale as being the difference of two square integrable martingales.

2. For all \( \lambda \in \mathbb{R}, \ t \geq 0, \mathbb{E}(e^{\lambda A_t}) = E(\cos(\lambda A_t)) + i\mathbb{E}(\sin(\lambda A_t)). \) Since \( \langle X, Y \rangle_t = \langle X, Y \rangle_t \), we get, for all \( t \geq 0, \)
   \[
   -A_t = \int_0^t Y_s dX_s - \int_0^t X_s dY_s \frac{d}{t} - \int_0^t X_s dY_s = \int_0^t Y_s dX_s = A_t,
   \]
   and thus the characteristic function or Fourier transform of \( A_t \) is real.

3. The canonical decompositions are given by the Itô formula. Namely, for \( Z, \)
   \[
   Z_t = 1 - \lambda \int_0^t \sin(\lambda A_s) dA_s - \frac{\lambda^2}{2} \int_0^t \cos(\lambda A_s) d(\langle A \rangle)_s
   \]
   \[
   = 1 - \lambda \int_0^t \sin(\lambda A_s) dA_s - \frac{\lambda^2}{2} \int_0^t (X_s^2 + Y_s^2) Z_s ds.
   \]
   Similarly, for \( W, \) by the Itô formula for the function \( g(x, y, t) := -\frac{f'(t)}{2}(x^2 + y^2) + f(t) \) and the vector of semi-martingale \( S_t := (X_t, Y_t, t) \) with martingale part \( (X_t, Y_t), \)
   \[
   W_t = g(0, 0, 0) + \int_0^t \partial_1 g(S_s) dX_s + \int_0^t \partial_2 g(S_s) dY_s + \int_0^t \partial_3 g(S_s) ds + \frac{1}{2} \int_0^t (\partial_1^2 g + \partial_2^2 g)(S_s) ds
   \]
   \[
   = f(0) - \int_0^t f'(s) X_s dX_s - \int_0^t f'(s) Y_s dY_s + \int_0^t \left( -\frac{f''(s)}{2}(X_s^2 + Y_s^2) + f'(s) \right) ds - \int_0^t f'(s) ds
   \]
   \[
   = f(0) - \int_0^t f'(s) X_s dX_s - \int_0^t f'(s) Y_s dY_s - \frac{1}{2} \int_0^t f''(s)(x_s^2 + y_s^2) ds.
   \]
   The computation of \( \langle Z, W \rangle \) involves only the local martingale parts, namely
   \[
   \langle Z, W \rangle_t = \lambda \left( \int_0^t \sin(\lambda A_s) dA_s, \int_0^t f'(s) X_s dX_s + \int_0^t f'(s) Y_s dY_s \right)_t
   \]
   \[
   = \lambda \int_0^t f'(s) \sin(\lambda A_s) X_s d\langle A, X \rangle_s + \lambda \int_0^t f'(s) \sin(\lambda A_s) Y_s d\langle A, Y \rangle_s.
   \]
   Now since \( \langle A, X \rangle_t = -\int_0^t Y_s ds \) and \( \langle A, Y \rangle_t = \int_0^t X_s ds \), we get
   \[
   \langle Z, W \rangle_t = \lambda \int_0^t f'(s)(-X_s Y_s + X_s Y_s) \sin(\lambda A_s) ds = 0.
   \]

4. The Itô formula gives (we benefit from the fact that \( \langle Z, W \rangle = 0 \) from the previous question)
   \[
   Z_t e^{W_t} = e^{f(0)} + \int_0^t e^{W_s} dZ_s + \int_0^t Z_s e^{W_s} dW_s + \frac{1}{2} \int_0^t Z_s e^{W_s} d\langle W \rangle_s.
   \]
By collecting the finite variation parts from $dZ$ and $dW$ from a previous question we get
\[ -\frac{\lambda^2}{2} \int_0^t (X_s^2 + Y_s^2) Z_s e^{W_t} \, ds - \frac{1}{2} \int_0^t f''(s)(X_s^2 + Y_s^2) Z_s e^{W_t} \, ds + \frac{1}{2} \int_0^t Z_s e^{W_t} d(W)_s. \]

Now from a previous question
\[ \langle W \rangle_t = \left( \int_0^t f'(s) X_s \, dX_s + \int_0^t f'(s) Y_s \, dY_s \right)_t = \int_0^t f''(s)(X_s^2 + Y_s^2) \, ds. \]

It follows that the finite variation part of $Ze^W$ vanishes when $f'' = f'^2 - \lambda^2$.

5. With $f(t) = -\log \cosh(\lambda(r-t))$, we have
\[ f'(t) = \frac{\lambda \sinh(\lambda(r-t))}{\cosh(\lambda(r-t))} = \lambda \tanh(\lambda(r-t)) \]
and
\[ f''(t) = -\frac{\lambda^2}{\cosh(\lambda(r-t))^2} = -\lambda^2 (1 - \tanh(\lambda(r-t))^2) = -\lambda^2 + f'^2(t). \]

It follows from the previous question that $Ze^W$ is a continuous local martingale. Note that $f(r) = f'(r) = 0$ and $W_t = 0$, and by using previous questions,
\[ \mathbb{E}e^{i \lambda A_t} = \mathbb{E} \cos(\lambda A_t) = \mathbb{E} Z_r = \mathbb{E}(Z_r e^{W_t}). \]

On the other hand, since $f(0) = -\log \cosh(\lambda r)$, $Z_0 = 1$, $W_0 = f(0)$, we get
\[ \mathbb{E}(Z_0 e^{W_t}) = e^{f(0)} = \frac{1}{\cosh(\lambda r)}. \]

It remains to show that the local martingale $Ze^W$ is a martingale on the time interval $[0, r]$. From the previous question, since $f$, $\cos$, and $\sin$ are bounded, it suffices to show that
\[ \mathbb{E} \int_0^t e^{2W_s} d(\langle A \rangle)_s < \infty \quad \text{and} \quad \mathbb{E} \int_0^t e^{2W_s}(X_s^2 + Y_s^2) \, ds < \infty. \]

But the first condition follows from the second thanks to the formula for $\langle A \rangle$ provided by a previous question. On the other hand, if $t \in [0, r]$ then $f'(t) \geq 0$ and thus $W_s \leq f(t)$ for all $s \in [0, t]$, which implies that the second condition is satisfied by using $\mathbb{E}(X_s^2 + Y_s^2) = 2s$.

**Exercise 3** (Criterion for a stochastic differential equation). Set $d = 1$. Let $\sigma, b$ be two functions $\mathbb{R} \to \mathbb{R}$ such that for some finite constant $C < \infty$ and for all $x, y \in \mathbb{R}$,
\[ |\sigma(x) - \sigma(y)| \leq C \sqrt{|x-y|} \quad \text{and} \quad |b(x) - b(y)| \leq C|x-y| \]

The goal of this exercise is to prove pathwise uniqueness for the stochastic differential equation
\[ dX_t = \sigma(X_t) dB_t + b(X_t) dt. \] (SDE)

A solution $X$ is a continuous semi-martingale with canonical decomposition $X = X_0 + M + V$ with $X_0 \in L^2$, local martingale part $M := \int_0^t \sigma(X_s) \, dB_s$, and finite variation part $V := \int_0^t b(X_s) \, ds$. Note that the continuity of $\sigma, X, b$ gives that almost surely, for all $t \geq 0$, $s \mapsto \sigma(X_s) + b(X_s)$ is locally bounded.

1. Let $Z$ be a continuous semi-martingale such that $\langle Z \rangle = \int_0^t \varphi \, ds$ for a progressive process $\varphi$ such that $0 \leq \varphi \leq C|Z|$ for some constant $C < \infty$. Prove that for all $t \geq 0$ and all $\alpha > 0$,
\[ \mathbb{E} \int_0^t 1_{0 < |Z_s| \leq \alpha} \, d|Z_s| \leq Ct. \]
2. Deduce from the preceding question that for all \( t \geq 0 \),

\[
\lim_{n \to \infty} nE \int_0^t 1_{0 < |Z_s| \leq \frac{1}{n}} d\langle Z \rangle_s = 0.
\]

3. For all \( n \geq 1 \), \( x \in \mathbb{R} \), let us define \( g_n(x) := 2n(1 + nx)1_{x \in \left[ -\frac{1}{n}, 0 \right]} + 2n1_{x=0} + 2n(1 - nx)1_{x \in \left( 0, \frac{1}{n} \right]} \).

Let \( f_n : \mathbb{R} \to \mathbb{R} \) be the twice differentiable function such that \( f''_n = g_n \) and \( f_n(0) = f'_n(0) = 0 \).

Show that for all \( x \in \mathbb{R} \), the following properties hold true:

(a) \( f'_n(x) \in [-1, 1] \) and \( \lim_{n \to \infty} f'_n(x) = \text{sign}(x) := 1_{x > 0} - 1_{x < 0} \);

(b) \( |f_n(x)| \leq |x| \) and \( \lim_{n \to \infty} f_n(x) = |x| \).

4. By using Itô formula, prove that for all continuous semi-martingale \( Z = (Z_t)_{t \geq 0} \), all \( t \geq 0 \),

\[
\int_0^t 1_{Z_s = 0} d\langle Z \rangle_s = 0.
\]

5. From now on, let \( X \) and \( X' \) be two solutions of (SDE) on \( (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}) \) and with respect to the Brownian motion \( B \). Show that for all \( t \geq 0 \),

\[
\langle X - X' \rangle_t = \int_0^t (\sigma(X_s) - \sigma(X'_s))^2 ds.
\]

6. By using the assumption on \( \sigma \), deduce from the preceding questions that for all \( t \geq 0 \),

\[
\lim_{n \to \infty} \mathbb{E} \int_0^t g_n(X_s - X'_s) d\langle X - X' \rangle_s = 0.
\]

7. Set \( Z := X - X' \). From now on, let \( T \) be a stopping time such that the semi-martingale \( (Z_{t \wedge T})_{t \geq 0} \) is bounded. By using notably the assumption on \( \sigma \), prove that for all \( t \geq 0, n \geq 1 \),

\[
\mathbb{E}(f_n(Z_{t \wedge T})) = \mathbb{E}(f_n(Z_0)) + \mathbb{E} \int_0^{t \wedge T} f'_n(Z_s)(b(X_s) - b(X'_s)) ds + \frac{1}{2} \mathbb{E} \int_0^{t \wedge T} f''_n(Z_s) d\langle Z \rangle_s.
\]

8. Deduce from the preceding questions and the assumption on \( b \) that for all \( t \geq 0 \),

\[
\mathbb{E}(|X_{t \wedge T} - X'_{t \wedge T}|) = \mathbb{E}(|X_0 - X'_0|) + \mathbb{E} \int_0^{t \wedge T} (b(X_s) - b(X'_s)) \text{sign}(X_s - X'_s) ds.
\]

9. By using the Grönwall lemma, deduce that if \( X_0 = X'_0 \), then \( X_t = X'_t \) for all \( t \geq 0 \).

**Elements of solution for Exercise 3.** The result is known as the Yamada–Watanabe criterion. This is a slightly more detailed version of [1, Exercise 8.14 pages 231–232].

1. We have, using the properties of \( Z \) and \( \varphi \),

\[
\int_0^t 1_{0 < |Z_s| \leq \alpha} d\langle Z \rangle_s = \int_0^t \frac{1_{0 < |Z_s| \leq \alpha}}{|Z_s|} \varphi_s ds \leq \int_0^t C ds = Ct.
\]

2. For all \( n \geq 1 \), we have \( n1_{0 < |Z_s| \leq \frac{1}{n}} \leq \frac{1_{0 < |Z_s| \leq 1}}{|Z_s|} \leq \frac{1_{0 < |Z_s| \leq 1}}{|Z_s|} = 1 \), which is integrable on \( [0, t] \) by the preceding question used with \( \alpha = 1 \), and thus the desired result follows then by dominated convergence.

3. The function \( g_n \) is 0 on \( (-\infty, -\frac{1}{n}] \), then increases from 0 to 2 on \( [-\frac{1}{n}, 0] \), then decreases from 2 to 0 on \( [0, \frac{1}{n}] \), then stays at 0 on \( (\frac{1}{n}, +\infty) \). Since \( f^0_{-\infty} g_n(y) dy = 1 \), we have, for all \( x \in \mathbb{R} \),

\[
f'_n(x) = \int_0^x g_n(u) du,
\]
in such a way that \( f'_n(0) = 0 \) and \( f''_n = g_n \).
The function $f_n$ is $-1$ on $(-\infty, -\frac{1}{n}]$, $0$ at $0$, and $1$ on $[\frac{1}{n}, +\infty)$. Also for all $x \in \mathbb{R}$,

$$\lim_{n \to \infty} f_n'(x) = \mathbf{1}_{x>0} - \mathbf{1}_{x<0} =: \text{sign}(x).$$

Next, for all $x \in \mathbb{R}$, we have

$$f_n(x) = \int_0^x f_n'(u)\,d\mu \quad \text{in such a way that } f_n(0) = 0 \text{ and } f_n'' = g_n.$$ 

Since $g_n \geq 0$, we have that $f_n'$ is non-decreasing, and thus $f_n'$ takes actually its values in $[-1, 1]$, and is in particular bounded. It follows by dominated convergence that for all $x \in \mathbb{R}$,

$$\lim_{n \to \infty} f_n(x) = \int_0^x \lim_{n \to \infty} f_n'(u)\,d\mu = \int_0^x \text{sign}(u)\,d\mu = |x|.$$ 

Finally, for all $x \in \mathbb{R}$, $|f_n(x)| \leq f_0^1|\,d \mu = |x|$.

4. The Itô formula for function $f_n$ of question 3 and semi-martingale $Z$ gives, for all $t \geq 0$,

$$f_n(Z_t) = f_n(Z_0) + \int_0^t f_n'(Z_s)\,dZ_s + \frac{1}{n} \int_0^t f_n''(Z_s)\,d\langle Z \rangle_s.$$ 

Since $|f_n'/(2n)| \leq 1$ and $\lim_{n \to \infty} f_n'(x)/2n = 1_{x=0}$ for all $x \in \mathbb{R}$, by dominated convergence,

$$\lim_{n \to \infty} 2n \int_0^t f_n''(Z_s)\,d\langle Z \rangle_s = \int_0^t 1_{Z_s=0}\,d\langle Z \rangle_s \quad \text{a.s.}.$$ 

On the other hand, since by question 3, $\lim_{n \to \infty} f_n(Z_0)/2n = 0$ for all $x \in \mathbb{R}$, it follows that a.s.

$$\lim_{n \to \infty} f_n(Z_0)/2n = 0 \quad \text{and} \quad \lim_{n \to \infty} f_n(Z_0)/2n = 0.$$ 

Finally since by question 3, $\frac{1}{2n}|f_n'| \leq \frac{1}{2n} \leq 1$ and $\lim_{n \to \infty} \frac{1}{2n}|f_n'(x)| = 0$ for all $x \in \mathbb{R}$, dominated convergence for stochastic integrals gives

$$\frac{1}{2n} \int_0^t f_n'(Z_s)\,dZ_s \overset{p}{\longrightarrow} 0.$$ 

5. Since $X$ and $X'$ are both solutions on the same space and for the same Brownian motion, we have, for all $X_0$ and $X'_0$ and for all $t \geq 0$,

$$X_t - X'_t = X_0 - X'_0 + \int_0^t (\sigma(X_s) - \sigma(X'_s))\,dB_s + \int_0^t (b(X_s) - b(X'_s))\,ds.$$ 

The right hand side gives the canonical decomposition of the semi-martingale $X - X'$. In this decomposition, the first integral is the local martingale part, and

$$\langle X - X' \rangle = \int_0^t (\sigma(X_s) - \sigma(X'_s))^2\,ds.$$ 

6. By assumption on $\sigma$, the process $\varphi := (\sigma(X) - \sigma(X'))^2$ satisfies $0 \leq \varphi \leq C|X - X'|$. We can then use question 5 and question 2 with $Z := X - X'$ to get, for all $t \geq 0$,

$$\lim_{n \to \infty} nE\int_0^t 1_{0<|Z_s|\leq \frac{1}{n}}\,d\langle Z \rangle_s = 0.$$ 

If $g_n$ is as in question 3, then for all $x \in \mathbb{R}, 0 \leq g_n(x) \leq 2n1_{0<|x|\leq \frac{1}{n}} + 2n1_{x=0}$. Thus, for all $t \geq 0$,

$$0 \leq E\int_0^t g_n(Z_s)\,d\langle Z \rangle_s \leq 2nE\int_0^t 1_{0<|Z_s|\leq \frac{1}{n}}\,d\langle Z \rangle_s + 2nE\int_0^t 1_{Z_s=0}\,d\langle Z \rangle_s \overset{n \to \infty}{\longrightarrow} 0,$$

where we have used question 2 and question 4.
7. The Itô formula for the $\mathcal{C}^2$ function $f_n$ and the continuous semi-martingale $Z^T$ gives
\[
f_n(Z^T_t) = f_n(Z^T_0) + \int_0^{t \wedge T} f''_n(Z_s) b(X_s) - b(X'_s) \, ds + \frac{1}{2} \int_0^{t \wedge T} f'''_n(Z_s) \, dB_s + \frac{1}{2} \int_0^{t \wedge T} f'''_n(Z_s) \, d\langle Z \rangle_s,
\]
and since $dZ_s = (\sigma(X_s) - \sigma(X'_s))dB_s + (b(X_s) - b(X'_s)) \, ds$, we get
\[
\int_0^{t \wedge T} f''(Z_s) \, dB_s = \int_0^{t \wedge T} f''(Z'_s)(\sigma(X_s) - \sigma(X'_s))1_{s < T} \, dB_s + \int_0^{t \wedge T} f''(Z_s)(b(X_s) - b(X'_s)) \, ds.
\]

Now, by the assumptions on $\sigma$ and $T$, we get
\[
|\sigma(X_s) - \sigma(X'_s)| 1_{s < T} \leq C \sqrt{|Z^T_s|} \leq C'.
\]

This boundedness, together with the one of $f''_n$, imply that the first integral in the right hand side above (the $dB_s$ one) is a martingale. Since this martingale is issued from the origin, its expectation vanishes for all times. On the other hand, since $f_n$ is continuous and $Z^T$ is bounded, the random variables $f_n(Z^T_t)$ and $f_n(Z^T_0)$ are integrable. All in all, we obtain
\[
E(f_n(Z^T_t)) = E(f_n(Z^T_0)) + E \int_0^{t \wedge T} f''_n(Z_s)(b(X_s) - b(X'_s)) \, ds + \frac{1}{2} E \int_0^{t \wedge T} f'''_n(Z_s) \, dB_s + \frac{1}{2} E \int_0^{t \wedge T} f'''_n(Z_s) \, d\langle Z \rangle_s.
\]

8. Since $f''_n = g_n \geq 0$, we get, by using question 6, that
\[
0 \leq E \int_0^{t \wedge T} f''_n(Z_s) \, d\langle Z \rangle_s \leq E \int_0^{t \wedge T} g_n(Z_s) \, dB_s \xrightarrow{n \to \infty} 0.
\]

On the other hand, by the assumption on $b$ and the boundedness of $Z^T$, we have, on $\{s \leq T\}$,
\[
|b(X_s) - b(X'_s)|^2 \leq C^2 |X_s - X'_s|^2 = C^2 |Z^T_s| \leq C'.
\]

But since $f''_n$ is bounded (takes its values in $[-1, 1]$), we get, by dominated convergence
\[
\lim_{n \to \infty} \int_0^{t \wedge T} f''_n(Z_s)(b(X_s) - b(X'_s)) \, ds = \int_0^{t \wedge T} \text{sign}(Z_s)(b(X_s) - b(X'_s)) \, ds.
\]

Finally, since $Z^T_t$ is bounded, and since from question 3, for all $x \in \mathbb{R}$, $|f_n(x)| \leq |x|$ and $\lim_{n \to \infty} f_n(x) = |x|$, we get, by dominated convergence, $\lim_{n \to \infty} E(f_n(Z^T_t)) = E(|Z^T_t|)$. Finally
\[
E(|X_{t \wedge T} - X'_{t \wedge T}|) = E(|X_0 - X'_0|) + E \int_0^{t \wedge T} (b(X_s) - b(X'_s)) \text{sign}(X_s - X'_s) \, ds.
\]

9. From the preceding question, we get, by using the assumption on $b$,
\[
\alpha(t) := E(|X_{t \wedge T} - X'_{t \wedge T}|) \leq E(|X_0 - X'_0|) + C \int_0^t \text{sign}(X_s - X'_s) \, ds = \alpha(0) + C \int_0^t \alpha(s) \, ds.
\]

By the Grönwall lemma, we obtain $\alpha(t) \leq \alpha(0)e^{Ct}$ for all $t \geq 0$. It follows that if $\alpha(0) = 0$ then $\alpha(t) = 0$ for all $t \geq 0$. This means that if $X_0 = X'_0$ then $X_{t \wedge T} = X'_{t \wedge T}$ for all $t \geq 0$. By writing this for $t \in \mathbb{Q}_+$, and by taking $T = T_m$ such that $\lim_{m \to \infty} T_m = +\infty$ almost surely, we get that $X_t = X'_t$ for all $t \in \mathbb{Q}_+$, and thus for all $t \geq 0$ since $X$ and $X'$ are continuous.

References