Exercise 1 (Nature of an integral). Set \( d = 1 \). Let us consider the following integral, for \( t \geq 0 \),

\[
I_t := \int_0^t B_s \, ds.
\]

2. Show that \( d(tB_t) = B_t \, dt + t \, dB_t \);
3. Deduce from the preceding question that \( I_t = \int_0^t (t - s) \, dB_s \) for all \( t \geq 0 \);
4. Deduce from the preceding question that \( I_t \sim \mathcal{N}(0, \frac{1}{3} t^3) \) for all \( t \geq 0 \);
5. For all \( t \geq 0, n \geq 1, 0 \leq k \leq n \), let us define \( t_k := \frac{k}{n} t \). Show that

\[
\sum_{k=0}^{n-1} B_{t_k} (t_{k+1} - t_k) = \frac{t}{n} \sum_{j=0}^{n-2} (n - j - 1)(B_{t_{j+1}} - B_{t_j}).
\]

6. Deduce from the preceding question another proof that \( I_t \sim \mathcal{N}(0, \frac{1}{3} t^3) \) for all \( t \geq 0 \);
7. Is the process \((I_t)_{t \geq 0}\) a martingale?

Elements of solution for Exercise 1.

1. Since the integrator is of finite variation and the integrand is bounded and measurable (actually continuous), it is a Lebesgue–Stieltjes integral, and in particular an Itô integral with respect to a semi-martingale without martingale part. However it is not a Wiener integral.
2. The Itô formula for \( f(x, y) = xy \) and \( X_t = (t, B_t) \) gives

\[
tB_t = 0 + \int_0^t B_s \, ds + \int_0^t s \, dB_s.
\]

3. From the preceding question (actually, it is an integration by parts)

\[
\int_0^t B_s \, ds = tB_t - \int_0^t s \, dB_s = \int_0^t (t - s) \, dB_s.
\]

4. The integral in the right hand side is a Wiener integral. Thus it is Gaussian with mean zero and variance equal to the squared \( L^2 \) norm of the integrand:

\[
E \int_0^t B_s \, ds = 0 \quad \text{and} \quad E \left( \left( \int_0^t B_s \, ds \right)^2 \right) = \int_0^t (t - s)^2 \, ds = \frac{t^3}{3}.
\]

5. With \( t_k = \frac{k}{n} t \) for all \( 0 \leq k \leq n \), we have

\[
S_n := \sum_{k=0}^{n-1} B_{t_k} (t_{k+1} - t_k) = \frac{t}{n} \sum_{k=0}^{n-1} B_{t_k} = \frac{t}{n} \sum_{k=0}^{n-1} \sum_{j=0}^{k-1} (B_{t_{j+1}} - B_{t_j}) = \frac{t}{n} \sum_{j=0}^{n-2} (n - j - 1)(B_{t_{j+1}} - B_{t_j}).
\]
6. Fix \( t \geq 0 \). Since \( I_t \) is a Lebesgue–Stieltjes integral with continuous integrand, we have \( \lim_{n \to \infty} S_n = I_t \) almost surely and thus in law. For all \( n \) and \( j \), since \( R_{t_{j+1}} - R_{t_j} \) are independent and Gaussian, we get that \( S_n \sim \mathcal{N}(\mathbb{E}(S_n), \mathbb{E}(S_n^2) - \mathbb{E}(S_n)^2) \). But the convergence in law of Gaussians is equivalent to the convergence of the first two moments. Now it remains to note that we have \( \mathbb{E}(S_n) = 0 \) and

\[
\mathbb{E}(S_n^2) = \frac{t^2}{n^2} \sum_{j=0}^{n-1} (n - j - 1)^2 \mathbb{E}((B_{t_{j+1}} - B_{t_j})^2) = \frac{t^2}{n^3} \sum_{j=0}^{n-1} j^2 = t^2 \sum_{j=1}^{n-1} \frac{1}{n} \to t^2 \int_0^1 x^2 \, dx = \frac{t^3}{3}.
\]

7. Beware that the integrand in \( \int_0^t (t - s) \, dB_s \) depends on \( t \). The process \( (- \int_0^t s \, dB_s)_{t \geq 0} \) is a martingale, however the process \( (\int_0^t t \, dB_s)_{t \geq 0} = (t B_t)_{t \geq 0} \) is not a martingale: for all \( 0 \leq s \leq t \),

\[
\mathbb{E}(((t + s)B_{t+s} | \mathcal{F}_t)) = (t + s)B_s \neq sB_s.
\]

**Exercise 2** (Study of a special process). Set \( d = 2 \). For all \( t \geq 0 \), we write \( B_t = (X_t, Y_t) \) and

\[
A_t := \int_0^t X_s \, dY_s - \int_0^t Y_s \, dX_s.
\]

1. Show that \( \langle A \rangle = \int_0^t (X_s^2 + Y_s^2) \, ds \) and that the process \( A \) is a square integrable martingale;

2. From now on let \( \lambda > 0 \). Show that for all \( t \geq 0 \),

\[
\mathbb{E} e^{i\lambda A_t} = \mathbb{E} \cos(\lambda A_t).
\]

3. From now on, let \( f : \mathbb{R}_+ \to \mathbb{R} \) be \( \mathcal{C}^2 \), and let us define the continuous semi-martingales

\[
(Z_t)_{t \geq 0} := (\cos(\lambda A_t))_{t \geq 0} \quad \text{and} \quad (W_t)_{t \geq 0} := \left( -\frac{f'(t)}{2}(X_t^2 + Y_t^2) + f(t) \right)_{t \geq 0}.
\]

Show that for all \( t \geq 0 \),

\[
Z_t = 1 - \lambda \int_0^t \sin(\lambda A_s) \, dA_s - \frac{\lambda^2}{2} \int_0^t (X_s^2 + Y_s^2) \, dZ_s,
\]

and

\[
W_t = f(0) - \int_0^t f'(s) X_s \, dX_s - \int_0^t f'(s) Y_s \, dY_s - \frac{1}{2} \int_0^t f''(s)(X_s^2 + Y_s^2) \, ds,
\]

and deduce that

\[
\langle Z, W \rangle = 0.
\]

4. Show that if \( f \) solves \( f'' = f'^2 - \lambda^2 \) then \( Z e^{W_t} \) is a continuous local martingale and

\[
Z_t e^{W_t} = e^{f(0)} - \lambda \int_0^t \sin(\lambda A_s) e^{W_s} \, dA_s - \int_0^t f'(s) Z_s e^{W_s} X_s \, dX_s - \int_0^t f'(s) Z_s e^{W_s} Y_s \, dY_s.
\]

5. Let \( r > 0 \). By using \( f(t) = -\log \cosh(\lambda(r - t)) \) deduce from the previous question that

\[
\mathbb{E} e^{i\lambda A_t} = \frac{1}{\cosh(\lambda r)}.
\]

**Elements of solution for Exercise 2.** For all \( t \geq 0 \), \( A_t \) is the algebraic area between planar Brownian motion and its chord, and the process \( A \) is the *Lévy area*. This exercise is a slightly more detailed version of [1, Exercise 5.30 pages 144–145]. Its goal is to compute the characteristic function or Fourier transform of \( A_t \).
1. Since \( \langle X \rangle_t = \langle Y \rangle_t = t \) and \( \langle X, Y \rangle_t = 0 \), we get

\[
\langle A \rangle_t = \langle A, A \rangle_t = \left( \int_0^t X_s dY_s \right) + \left( \int_0^t Y_s dX_s \right) + 2 \left( \int_0^t X_s dY_s, \int_0^t Y_s dX_s \right) = \int_0^t X_s^2 d\langle X \rangle_s + \int_0^t Y_s^2 d\langle Y \rangle_s + \int_0^t X_s Y_s d\langle X, Y \rangle_s
\]

\[
= \int_0^t (X_s^2 + Y_s^2) ds.
\]

It follows by the Fubini–Tonelli theorem that

\[
E \langle A \rangle_t = \int_0^t E(X_s^2 + Y_s^2) ds = \int_0^t 2 ds = t < \infty
\]

and thus, by a famous martingale criterion, the process \( A \) is a square integrable martingale.

Alternatively, since for all \( t \geq 0 \), \( E \int_0^t X_s^2 d\langle Y \rangle_s = \int_0^t E(X_s^2) ds < \infty \) the process \( \int_0^t X_s dY_s \) and by symmetry the process \( \int_0^t Y_s dX_s \) are both square integrable martingales, and thus the process \( A \) is also a square integrable martingales as the difference of two square integrable martingales.

2. For all \( \lambda \in \mathbb{R}, t \geq 0 \), \( E(e^{i \lambda A_t}) = E(\cos(\lambda A_t)) + i E(\sin(\lambda A_t)) \). Since \( \langle X, Y \rangle_t = \langle X, Y \rangle_0 \), we get, for all \( t \geq 0 \),

\[
-A_t = \int_0^t Y_s dX_s - \int_0^t X_s dY_s = \int_0^t X_s dY_s - \int_0^t Y_s dX_s = A_t,
\]

and thus the characteristic function or Fourier transform of \( A_t \) is real.

3. The canonical decompositions are given by the Itô formula. Namely, for \( Z \),

\[
Z_t = 1 - \lambda \int_0^t \sin(\lambda A_s) dA_s - \frac{\lambda^2}{2} \int_0^t \cos(\lambda A_s) d\langle A \rangle_s
\]

\[
= 1 - \lambda \int_0^t \sin(\lambda A_s) dA_s - \frac{\lambda^2}{2} \int_0^t (X_s^2 + Y_s^2) Z_s ds.
\]

Similarly, for \( W \), by the Itô formula for the function \( g(x, y, t) := -\frac{f(t)}{2}(x^2 + y^2) + f(t) \) and the vector of semi-martingale \( S_t := (X_t, Y_t, t) \) with martingale part \( (X_t, Y_t) \),

\[
W_t = g(0, 0, 0) + \int_0^t \partial_1 g(S_s) dX_s + \int_0^t \partial_2 g(S_s) dY_s + \int_0^t \partial_3 g(S_s) ds + \frac{1}{2} \int_0^t (\partial_1^2 g + \partial_2^2 g)(S_s) ds
\]

\[
= f(0) - \int_0^t f'(s) X_s dX_s - \int_0^t f'(s) Y_s dY_s + \int_0^t \left( -\frac{f''(s)}{2}(X_s^2 + Y_s^2) + f'(s) \right) ds - \int_0^t f'(s) ds
\]

The computation of \( \langle Z, W \rangle \) involves only the local martingale parts, namely

\[
\langle Z, W \rangle_t = \lambda \left( \int_0^t \sin(\lambda A_s) dA_s, \int_0^t f'(s) X_s dX_s, \int_0^t f'(s) Y_s dY_s \right)_t
\]

\[
= \lambda \int_0^t f'(s) \sin(\lambda A_s) X_s d(A, X)_s + \lambda \int_0^t f'(s) \sin(\lambda A_s) Y_s d(A, Y)_s.
\]

Now since \( \langle A, X \rangle_t = -\int_0^t Y_s ds \) and \( \langle A, Y \rangle_t = \int_0^t X_s ds \), we get

\[
\langle Z, W \rangle_t = \lambda \int_0^t f'(s)(-X_s Y_s + X_s Y_s) \sin(\lambda A_s) ds = 0.
\]

4. The Itô formula gives (we benefit from the fact that \( \langle Z, W \rangle = 0 \) from the previous question)

\[
Z_t e^{W_t} = e^{f(0)} + \int_0^t e^{W_s} dZ_s + \int_0^t Z_s e^{W_s} dW_s + \frac{1}{2} \int_0^t Z_s e^{W_s} d\langle W \rangle_s.
\]
By collecting the finite variation parts from \(dZ\) and \(dW\) from a previous question we get
\[
\frac{-\lambda^2}{2} \int_0^t (X_s^2 + Y_s^2)Z_s e^{W_s} ds - \frac{1}{2} \int_0^t f''(s)(X_s^2 + Y_s^2)Z_s e^{W_s} ds + \frac{1}{2} \int_0^t Z_s e^{W_s} d(W)_s.
\]

Now from a previous question
\[
\langle W \rangle_t = \left( \int_0^t f'(s)X_s ds + \int_0^t f'(s)Y_s ds \right)_t = \int_0^t f''(s)(X_s^2 + Y_s^2) ds.
\]

It follows that the finite variation part of \(Ze^W\) vanishes when \(f'' = f'^2 - \lambda^2\).

5. With \(f(t) = -\log \cosh(\lambda(r - t))\), we have
\[
f'(t) = \lambda \frac{\sinh(\lambda(r - t))}{\cosh(\lambda(r - t))} = \lambda \tanh(\lambda(r - t))
\]
and
\[
f''(t) = -\frac{\lambda^2}{\cosh(\lambda(r - t))^2} = -\lambda^2(1 - \tanh(\lambda(r - t))^2) = -\lambda^2 + f'^2(t).
\]
It follows from the previous question that \(Ze^W\) is a continuous local martingale. Note that \(f(r) = f'(r) = 0\) and \(W_0 = 0\), and by using previous questions,
\[
\mathbb{E}e^{\lambda A_r} = \mathbb{E} \cos(\lambda A_r) = \mathbb{E}Z_r = \mathbb{E}(Z_0 e^{W_0}).
\]

On the other hand, since \(f(0) = -\log \cosh(\lambda r)\), \(Z_0 = 1\), \(W_0 = f(0)\), we get
\[
\mathbb{E}(Z_0 e^{W_0}) = e^{f(0)} = \frac{1}{\cosh(\lambda r)}.
\]

It remains to show that the local martingale \(Ze^W\) is a martingale on the time interval \([0, r]\). From the previous question, since \(f\), \(c\), \(s\), and \(s\) are bounded, it suffices to show that
\[
\mathbb{E} \int_0^t e^{2W_s} d(A)_s < \infty \quad \text{and} \quad \mathbb{E} \int_0^t e^{2W_s}(X_s^2 + Y_s^2) ds < \infty.
\]

But the first condition follows from the second thanks to the formula for \(\langle A \rangle\) provided by a previous question. On the other hand, if \(t \in [0, r]\) then \(f'(t) \geq 0\) and thus \(W_s \leq f(t)\) for all \(s \in [0, t]\), which implies that the second condition is satisfied by using \(\mathbb{E}(X_s^2 + Y_s^2) = 2s\).

**Exercise 3** (Criterion for a stochastic differential equation). Set \(d = 1\). Let \(\sigma, b\) be two functions \(\mathbb{R} \to \mathbb{R}\) such that for some finite constant \(C < \infty\) and for all \(x, y \in \mathbb{R}\),
\[
|\sigma(x) - \sigma(y)| \leq C|x - y| \quad \text{and} \quad |b(x) - b(y)| \leq C|x - y|.
\]

The goal of this exercise is to prove pathwise uniqueness for the stochastic differential equation
\[
dX_t = \sigma(X_t)dB_t + b(X_t)dt. \tag{SDE}
\]
A solution \(X\) is a continuous semi-martingale with canonical decomposition \(X = X_0 + M + V\) with \(X_0 \in L^2\), local martingale part \(M := \int_0^t \sigma(X_s)dB_s\), and finite variation part \(V := \int_0^t b(X_s)ds\). Note that the continuity of \(\sigma, X, b\) gives that almost surely, for all \(t \geq 0\), \(s \to \sigma(X_s) + b(X_s)\) is locally bounded.

1. Let \(Z\) be a continuous semi-martingale such that \(\langle Z \rangle = \int_0^t \varphi_s ds\) for a progressive process \(\varphi\) such that \(0 \leq \varphi \leq C|Z|\) for some constant \(C < \infty\). Prove that for all \(t \geq 0\) and all \(a > 0\),
\[
\mathbb{E} \int_0^t \frac{1_{|Z_s| \leq a}}{|Z_t|} d(Z)_s \leq Ct.
\]
2. Deduce from the preceding question that for all \(t \geq 0\),
\[
\lim_{n \to \infty} n \mathbb{E} \int_0^t 1_{|Z_s| \geq n} d(Z)_s = 0.
\]
3. For all \( n \geq 1, x \in \mathbb{R} \), let us define \( g_n(x) := 2n(1 + nx)1_{x \leq -\frac{1}{n}} + 2n1_{x = 0} + 2n(1 - nx)1_{x \in \mathbb{R} \setminus \{0\}} \).
Let \( f_n : \mathbb{R} \to \mathbb{R} \) be the twice differentiable function such that \( f''_n = g_n \) and \( f_n(0) = f'_n(0) = 0 \).

4. By using Itô formula, prove that for all continuous semi-martingale \( Z = (Z_t)_{t \geq 0} \), all \( t \geq 0 \),
\[
\int_0^t 1_{Z_s = 0} d(Z)_s = 0.
\]

5. From now on, let \( X \) and \( X' \) be two solutions of (SDE) on \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})\) and with respect to the Brownian motion \( B \). Show that for all \( t \geq 0 \),
\[
\langle X - X' \rangle_t = \int_0^t (\sigma(X_s) - \sigma(X'_s))^2 ds.
\]

6. By using the assumption on \( \sigma \), deduce from the preceding questions that for all \( t \geq 0 \),
\[
\lim_{n \to \infty} \mathbb{E} \int_0^t g_n(X_s - X'_s) d\langle X - X' \rangle_s = 0.
\]

7. Set \( Z := X - X' \). From now on, let \( T \) be a stopping time such that the semi-martingale \((Z_t)_{t \geq 0}\) is bounded. By using notably the assumption on \( \sigma \), prove that for all \( t \geq 0, n \geq 1 \),
\[
\mathbb{E}(f_n(Z_T)) = \mathbb{E}(f_n(Z_0)) + \int_0^T f'_n(Z_s)(b(X_s) - b(X'_s)) ds + \frac{1}{2} \int_0^T f''_n(Z_s) \mathbb{E}d\langle Z \rangle_s.
\]

8. Deduce from the preceding questions and the assumption on \( b \) that for all \( t \geq 0 \),
\[
\mathbb{E}(|X_{t \wedge T} - X'_{t \wedge T}|) = \mathbb{E}(|X_0 - X'_0|) + \int_0^{t \wedge T} (b(X_s) - b(X'_s)) \mathbb{E}d\langle X - X' \rangle_s.
\]

9. By using the Grönwall lemma, deduce that if \( X_0 = X'_0 \) then \( X_t = X'_t \) for all \( t \geq 0 \).

**Elements of solution for Exercise 3.** The result is known as the Yamada–Watanabe criterion. This is a slightly more detailed version of [1, Exercise 8.14 pages 231–232].

1. We have, using the properties of \( Z \) and \( \varphi \),
\[
\int_0^t \frac{1_{0 < \langle Z \rangle_s < a}}{\langle Z \rangle_s} d\langle Z \rangle_s = \int_0^t \frac{1_{0 < \langle Z \rangle_s < a}}{\langle Z \rangle_s} \varphi_s ds \leq \int_0^t \varphi ds = Ct.
\]

2. For all \( n \geq 1 \), we have \( n1_{0 < \langle Z \rangle_s < a} \leq \frac{1_{|Z_s| < \frac{1}{n}}}{\langle Z \rangle_s} \leq \frac{1_{|Z_s| < a}}{\langle Z \rangle_s} \), which is integrable on \([0, t]\) by the preceding question used with \( a = 1 \), and thus the desired result follows then by dominated convergence.

3. The function \( g_n \) is \( = 0 \) on \((-\infty, -\frac{1}{n})\), then increases from \( 0 \) to \( 2 \) on \([-\frac{1}{n}, 0]\), then decreases from \( 2 \) to \( 0 \) on \([0, \frac{1}{n}]\), then stays at \( 0 \) on \([\frac{1}{n}, +\infty)\). Since \( \int_{-\infty}^0 g_n(y) dy = 1 \), we have, for all \( x \in \mathbb{R} \),
\[
f'_n(x) = \int_0^x g_n(u) du \quad \text{in such a way that} \quad f'_n(0) = 0 \quad \text{and} \quad f''_n = g_n.
\]

The function \( f'_n \) is \( = -1 \) on \((-\infty, -\frac{1}{n})\), \( = 0 \) at \( 0 \), and \( = 1 \) on \([\frac{1}{n}, +\infty)\). Also for all \( x \in \mathbb{R} \),
\[
\lim_{n \to \infty} f'_n(x) = 1_{x > 0} - 1_{x < 0} = : \text{sign}(x).
\]

Next, for all \( x \in \mathbb{R} \), we have
\[
f_n(x) = \int_0^x f'_n(u) du \quad \text{in such a way that} \quad f_n(0) = 0 \quad \text{and} \quad f''_n = g_n.
\]

Since \( g_n \geq 0 \), we have that \( f'_n \) is non-decreasing, and thus \( f'_n \) takes actually its values in \([-1, 1]\), and is in particular bounded. It follows by dominated convergence that for all \( x \in \mathbb{R} \),
\[
\lim_{n \to \infty} f_n(x) = \int_0^x \lim_{n \to \infty} f'_n(u) du = \int_0^x \text{sign}(u) du = |x|.
\]

Finally, for all \( x \in \mathbb{R} \), \( |f_n(x)| \leq \int_0^{|x|} du = |x| \).
4. The Itô formula for function \( f_n \) of question 3 and semi-martingale \( Z \) gives, for all \( t \geq 0 \),
\[
    f_n(Z_t) = f_n(Z_0) + \int_0^t f_n'(Z_s) \, dB_s + \frac{1}{2} \int_0^t f_n''(Z_s) \, dB_s^2.
\]
Since \( \frac{f_n}{2n} \leq 1 \) and \( \lim_{n \to \infty} \frac{f_n(x)}{2n} = 1 \) for all \( x \in \mathbb{R} \), by dominated convergence,
\[
    \lim_{n \to \infty} \frac{1}{2n} \int_0^t f_n''(Z_s) \, dB_s^2 = \int_0^t 1_{Z_s = 0} \, dB_s^2 \quad \text{a.s.}
\]
On the other hand, since by question 3, \( \lim_{n \to \infty} \frac{f_n(x)}{2n} = 0 \) for all \( x \in \mathbb{R} \), it follows that a.s.
\[
    \lim_{n \to \infty} \frac{f_n(Z_t)}{2n} = 0 \quad \text{and} \quad \lim_{n \to \infty} \frac{f_n(Z_0)}{2n} = 0.
\]
Finally since by question 3, \( \frac{1}{2n} |f_n'| \leq \frac{1}{2n} \leq 1 \) and \( \lim_{n \to \infty} \frac{1}{2n} |f_n(x)| = 0 \) for all \( x \in \mathbb{R} \), dominated convergence for stochastic integrals gives
\[
    \frac{1}{2n} \int_0^t f_n'(Z_s) \, dB_s \xrightarrow{p} 0, \quad n \to \infty.
\]
5. Since \( X \) and \( X' \) are both solutions on the same space and for the same Brownian motion, we have, for all \( X_0 \) and \( X'_0 \) and for all \( t \geq 0 \),
\[
    X_t - X'_t = X_0 - X'_0 + \int_0^t (\sigma(X_s) - \sigma(X'_s)) \, dB_s + \int_0^t (b(X_s) - b(X'_s)) \, ds.
\]
The right hand side gives the canonical decomposition of the semi-martingale \( X - X' \). In this decomposition, the first integral is the local martingale part, and
\[
    \langle X - X' \rangle = \int_0^t \langle \sigma(X_s) - \sigma(X'_s) \rangle \, ds.
\]
6. By assumption on \( \sigma \), the process \( \varphi := \langle \sigma(X) - \sigma(X') \rangle \) satisfies \( 0 \leq \varphi \leq C |X - X'| \). We can then use question 5 and question 2 with \( Z := X - X' \) to get, for all \( t \geq 0 \),
\[
    \lim_{n \to \infty} n \mathbb{E} \int_0^t 1_{0 < |Z_s| \leq \frac{1}{n}} \, d(Z)_s = 0.
\]
If \( g_n \) is as in question 3, then for all \( x \in \mathbb{R} \), \( 0 \leq g_n(x) \leq 2n 1_{0 < |x| \leq \frac{1}{n}} + 2n 1_{x = 0} \). Thus, for all \( t \geq 0 \),
\[
    0 \leq \mathbb{E} \int_0^t g_n(Z_s) \, dB_s \leq 2n \mathbb{E} \int_0^t 1_{0 < |Z_s| \leq \frac{1}{n}} \, d(Z)_s + 2n \mathbb{E} \int_0^t 1_{Z_s = 0} \, dB_s \xrightarrow{n \to \infty} 0,
\]
where we have used question 2 and question 4.
7. The Itô formula for the \( \mathcal{C}^2 \) function \( f_n \) and the continuous semi-martingale \( Z^T \) gives
\[
    f_n(Z^T_t) = f_n(Z^T_0) + \int_0^{t \wedge T} f_n'(Z_s) \, dZ_s + \frac{1}{2} \int_0^{t \wedge T} f_n''(Z_s) \, dB_s^2,
\]
and since \( dZ_s = (\sigma(X_s) - \sigma(X'_s)) \, dB_s + (b(X_s) - b(X'_s)) \, ds \), we get
\[
    \int_0^{t \wedge T} f'(Z_s) \, dB_s = \int_0^{t \wedge T} f'(Z^T_s) (\sigma(X_s) - \sigma(X'_s)) 1_{s \leq T} \, dB_s + \int_0^{t \wedge T} f'(Z_s) (b(X_s) - b(X'_s)) \, ds.
\]
Now, by the assumptions on \( \sigma \) and \( T \), we get
\[
    |\sigma(X_s) - \sigma(X'_s)| 1_{s \leq T} \leq C \sqrt{|Z^T_s|} \leq C'.
\]
This boundedness, together with the one of \( f_n'' \), imply that the first integral in the right hand side above (the \( dB_s \) one) is a martingale. Since this martingale is issued from the origin, its expectation vanishes for all times. On the other hand, since \( f_n \) is continuous and \( Z^T \) is bounded, the random variables \( f_n(Z^T_t) \) and \( f_n(Z^T_0) \) are integrable. All in all, we obtain
\[
    \mathbb{E}(f_n(Z^T_t)) = \mathbb{E}(f_n(Z^T_0)) + \mathbb{E} \int_0^{t \wedge T} f_n'(Z_s) (b(X_s) - b(X'_s)) \, ds + \frac{1}{2} \mathbb{E} \int_0^{t \wedge T} f_n''(Z_s) \, dB_s^2.
\]
8. Since $f_n'' = g_n \geq 0$, we get, by using question 6, that

\[ 0 \leq \mathbb{E} \int_0^{t \wedge T} f_n''(Z_s) d\langle Z \rangle_s \leq \mathbb{E} \int_0^t g_n(Z_s) d\langle Z \rangle_s \xrightarrow{n \to \infty} 0. \]

On the other hand, by the assumption on $b$ and the boundedness of $Z^T$, we have, on $\{s \leq T\}$,

\[ |b(X_s) - b(X'_s)|^2 \leq C^2|X_s - X'_s|^2 = C^2|Z^T_s| \leq C'. \]

But since $f_n'$ is bounded (takes its values in $[-1, 1]$), we get, by dominated convergence

\[ \lim_{n \to \infty} \int_0^{t \wedge T} f_n'(Z_s)(b(X_s) - b(X'_s)) ds = \int_0^{t \wedge T} \text{sign}(Z_s)(b(X_s) - b(X'_s)) ds. \]

Finally, since $Z^T_T$ is bounded, and since from question 3, for all $x \in \mathbb{R}$, $|f_n(x)| \leq |x|$ and $\lim_{n \to \infty} f_n(x) = |x|$, we get, by dominated convergence, $\lim_{n \to \infty} \mathbb{E}(f_n(Z^T_T)) = \mathbb{E}(|Z^T_T|)$. Finally

\[ \mathbb{E}(|X_{t \wedge T} - X'_{t \wedge T}|) = \mathbb{E}(|X_0 - X'_0|) + \mathbb{E} \int_0^{t \wedge T} (b(X_s) - b(X'_s))\text{sign}(X_s - X'_s) ds. \]

9. From the preceding question, we get, by using the assumption on $b$,

\[ \alpha(t) := \mathbb{E}(|X_{t \wedge T} - X'_{t \wedge T}|) \leq \mathbb{E}(|X_0 - X'_0|) + C \mathbb{E} \int_0^t |X_{s \wedge T} - X'_{s \wedge T}| ds = \alpha(0) + C \int_0^t \alpha(s) ds. \]

By the Grönwall lemma, we obtain $\alpha(t) \leq \alpha(0)e^{Ct}$ for all $t \geq 0$. It follows that if $\alpha(0) = 0$ then $\alpha(t) = 0$ for all $t \geq 0$. This means that if $X_0 = X'_0$ then $X_{t \wedge T} = X'_{t \wedge T}$ for all $t \geq 0$. By writing this for $t \in \mathbb{Q}_+$, and by taking $T = T_m$ such that $\lim_{m \to \infty} T_m = +\infty$ almost surely, we get that $X_t = X'_t$ for all $t \in \mathbb{Q}_+$, and thus for all $t \geq 0$ since $X$ and $X'$ are continuous.

References