Exercise 1 (Nature of an integral). Set $d = 1$. Let us consider the following integral, for $t \geq 0$,

$$I_t = \int_0^t B_s \, ds.$$  


2. Show that $d(tB_t) = B_t \, dt + t \, dB_t$.

3. Deduce from the preceding question that $I_t = \int_0^t (t - s) \, dB_s$ for all $t \geq 0$.

4. Deduce from the preceding question that $I_t \sim \mathcal{N}(0, \frac{1}{3} t^3)$ for all $t \geq 0$.

5. For all $t \geq 0$, $n \geq 1$, $0 \leq k \leq n$, let us define $t_k = \frac{k}{n} t$. Show that

$$\sum_{k=0}^{n-1} B_{t_k} (t_{k+1} - t_k) = \frac{t}{n} \sum_{j=0}^{n-2} (n - j - 1)(B_{t_{j+1}} - B_{t_j}).$$

6. Deduce from the preceding question another proof that $I_t \sim \mathcal{N}(0, \frac{1}{3} t^3)$ for all $t \geq 0$.

7. Is the process $(I_t)_{t \geq 0}$ a martingale?

Elements of solution for Exercise 1.

1. Since the integrator is of finite variation and the integrand is bounded and measurable (actually continuous), it is a Lebesgue–Stieltjes integral, and in particular an Itô integral with respect to a semi-martingale without martingale part. However it is not a Wiener integral.

2. The Itô formula for $f(x, y) = xy$ and $X_t = (t, B_t)$ gives

$$tB_t = 0 + \int_0^t B_s \, ds + \int_0^t s \, dB_s.$$  

3. From the preceding question (actually, it is an integration by parts)

$$\int_0^t B_s \, ds = tB_t - \int_0^t s \, dB_s = \int_0^t (t - s) \, dB_s.$$  

4. The integral in the right hand side is a Wiener integral. Thus it is Gaussian with mean zero and variance equal to the squared $L^2$ norm of the integrand:

$$\mathbb{E} \left( \int_0^t B_s \, ds \right)^2 = \int_0^t (t - s)^2 \, ds = \frac{t^3}{3}.$$  

5. With $t_k = \frac{k}{n} t$ for all $0 \leq k \leq n$, we have

$$S_n = \sum_{k=0}^{n-1} B_{t_k} (t_{k+1} - t_k) = \frac{t}{n} \sum_{k=0}^{n-1} B_{t_k} = \frac{t}{n} \sum_{k=0}^{n-1} \sum_{j=0}^{k-1} (B_{t_{j+1}} - B_{t_j}) = \frac{t}{n} \sum_{j=0}^{n-2} (n - j - 1)(B_{t_{j+1}} - B_{t_j}).$$
6. Fix $t \geq 0$. Since $I_t$ is a Lebesgue–Stieltjes integral with continuous integrand, we have $\lim_{n \to \infty} S_n = I_t$ almost surely and thus in law. For all $n$ and $j$, since $B_{t_{j+1}} - B_{t_j}$ are independent and Gaussian, we get that $S_n \sim \mathcal{N}(\mathbb{E}(S_n), \mathbb{E}(S_n^2) - \mathbb{E}(S_n)^2)$. But the convergence in law of Gaussians is equivalent to the convergence of the first two moments. Now it remains to note that we have $\mathbb{E}(S_n) = 0$ and

$$
\mathbb{E}(S_n^2) = \frac{t^2}{n^2} \sum_{j=0}^{n-2} (n-j-1)^2 \mathbb{E}((B_{t_{j+1}} - B_{t_j})^2) = \frac{t^3}{n^3} \sum_{j=0}^{n-2} j^2 = t^3 \left( \frac{1}{n} \sum_{j=1}^{n-1} \frac{j^2}{n} \right) \to t^3 \int_0^1 x^2 \, dx = \frac{t^3}{3}.
$$

7. Beware that the integrand in $\int_0^t (t-s) \, dB_s$ depends on $t$. The process $(-\int_0^t s \, dB_s)_{t \geq 0}$ is a martingale, however the process $(\int_0^t t \, dB_s)_{t \geq 0} = (tB_t)_{t \geq 0}$ is not a martingale: for all $0 \leq s \leq t$,

$$
\mathbb{E}((t+s)B_{t+s} \mid \mathcal{F}_t) = (t+s)B_s \neq sB_s.
$$

**Exercise 2** (Study of a special process). Set $d = 2$. For all $t \geq 0$, we write $B_t = (X_t, Y_t)$ and

$$
A_t = \int_0^t X_s \, dY_s - \int_0^t Y_s \, dX_s.
$$

1. Show that $(A) = \int_0^t (X_s^2 + Y_s^2) \, ds$ and that the process $A$ is a square integrable martingale.

2. From now on let $\lambda > 0$. Show that for all $t \geq 0$,

$$
\mathbb{E}e^{i\lambda A_t} = \mathbb{E}\cos(\lambda A_t).
$$

3. From now on, let $f : \mathbb{R}_+ \to \mathbb{R}$ be $\mathcal{C}^2$, and let us define the continuous semi-martingales

$$
(Z_t)_{t \geq 0} = (\cos(\lambda A_t))_{t \geq 0} \quad \text{and} \quad (W_t)_{t \geq 0} = \left( -\frac{f'(t)}{2} (X_t^2 + Y_t^2) + f(t) \right)_{t \geq 0}.
$$

Show that for all $t \geq 0$,

$$
Z_t = 1 - \lambda \int_0^t \sin(\lambda A_s) \, dA_s - \frac{\lambda^2}{2} \int_0^t (X_s^2 + Y_s^2) \, Z_s \, ds.
$$

and

$$
W_t = f(0) - \int_0^t f'(s) X_s \, dY_s - \int_0^t f'(s) Y_s \, dX_s - \frac{1}{2} \int_0^t f''(s) (X_s^2 + Y_s^2) \, ds,
$$

and deduce that

$$
\langle Z, W \rangle = 0.
$$

4. Show that if $f$ solves $f'' = f' - \lambda^2$ then $Ze^W$ is a continuous local martingale and

$$
Ze^{W_t} = e^{f(0)} - \lambda \int_0^t \sin(\lambda A_s) e^{W_s} \, dA_s - \int_0^t f'(s) Ze^{W_s} X_s \, dX_s - \int_0^t f'(s) Ze^{W_s} Y_s \, dY_s.
$$

5. Let $r > 0$. By using $f(t) = -\cosh(\lambda (r-t))$ deduce from the previous question that

$$
\mathbb{E}e^{i\lambda A_t} = \frac{1}{\cosh(\lambda r)}.
$$

**Elements of solution for Exercise 2.** For all $t \geq 0$, $A_t$ is the algebraic area between planar Brownian motion and its chord, and the process $A$ is the **Lévy area**. This exercise is a slightly more detailed version of [1, Exercise 5.30 pages 144–145]. Its goal is to compute the characteristic function or Fourier transform of $A_t$. 

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1. Since \( \langle X \rangle_t = \langle Y \rangle_t = t \) and \( \langle X, Y \rangle_t = 0 \), we get
\[
\langle A \rangle_t = \langle A, A \rangle_t = \left( \int_0^t X_s \, dY_s \right)_t + \left( \int_0^t Y_s \, dX_s \right)_t + 2 \left( \int_0^t X_s \, dY_s \right)_t = \int_0^t X_s^2 \, dY_s + \int_0^t Y_s^2 \, dX_s + \int_0^t X_s \, dY_s = \int_0^t \left( X_s^2 + Y_s^2 \right) ds.
\]
It follows by the Fubini–Tonelli theorem that
\[
\mathbb{E}(A)_t = \int_0^t \mathbb{E}(X_s^2 + Y_s^2) \, ds = \int_0^t 2s \, ds = t^2 < \infty
\]
and thus, by a famous martingale criterion, the process \( A \) is a square integrable martingale.

Alternatively, since for all \( t \geq 0 \), \( \int_0^t X_s^2 \, dY_s + \int_0^t Y_s^2 \, dX_s < \infty \) the process \( \int_0^t X_s \, dY_s \) and by symmetry the process \( \int_0^t Y_s \, dX_s \) are both square integrable martingales, and thus the process \( A \) is also a square integrable martingales as being the difference of two square integrable martingales.

2. For all \( \lambda \in \mathbb{R} \), \( t \geq 0 \), \( E(e^{i\lambda A_t}) = E(\cos(\lambda A_t)) + iE(\sin(\lambda A_t)) \). Since \( \langle X, Y \rangle = (X, Y) \), we get, for all \( t \geq 0 \),
\[
-A_t = \int_0^t Y_s \, dX_s - \int_0^t X_s \, dY_s = \int_0^t X_s \, dY_s - \int_0^t Y_s \, dX_s = A_t,
\]
and thus the characteristic function or Fourier transform of \( A_t \) is real.

3. The canonical decompositions are given by the Itô formula. Namely, for \( Z \),
\[
Z_t = 1 - \lambda \int_0^t \sin(\lambda A_s) \, dA_s - \frac{\lambda^2}{2} \int_0^t \cos(\lambda A_s) \, d\langle A \rangle_s - \frac{\lambda^2}{2} \int_0^t (X_s^2 + Y_s^2) \, dZ_s.
\]
Similarly, for \( W \), by the Itô formula for the function \( g(x, y, t) = -\frac{E(t)}{2} (x^2 + y^2) + f(t) \) and the vector of semi-martingale \( S_t = (X_t, Y_t, t) \) with the martingale part \( (X_t, Y_t) \),
\[
W_t = g(0, 0, 0) + \int_0^t \partial_1 g(S_s) \, dX_s + \int_0^t \partial_2 g(S_s) \, dY_s + \int_0^t \partial_3 g(S_s) \, ds + \frac{1}{2} \int_0^t (\partial_1^2 g + \partial_2^2 g)(S_s) \, ds
\]
\[
= f(0) - \int_0^t \left( f'(s) X_s \, dX_s - \int_0^t f'(s) Y_s \, dY_s + \frac{1}{2} \int_0^t \left( -f''(s) \right)(X_s^2 + Y_s^2) + f'(s) \right) \, ds - \int_0^t f'(s) \, ds.
\]
The computation of \( \langle Z, W \rangle \) involves only the local martingale parts, namely
\[
\langle Z, W \rangle_t = \lambda \left( \int_0^t \sin(\lambda A_s) \, dA_s + \int_0^t f'(s) X_s \, dX_s + \int_0^t f'(s) Y_s \, dY_s \right)_t
\]
\[
= \lambda \int_0^t f'(s) \sin(\lambda A_s) X_s \, dA(s, X_s) + \lambda \int_0^t f'(s) \sin(\lambda A_s) Y_s \, d\langle A, Y \rangle_s.
\]
Now since \( \langle A, X \rangle = -\int_0^t Y_s \, ds \) and \( \langle A, Y \rangle_t = \int_0^t X_s \, ds \), we get
\[
\langle Z, W \rangle_t = \lambda \int_0^t f'(s) (-X_s Y_s + X_s Y_s) \, ds = 0.
\]

4. The Itô formula gives (we benefit from the fact that \( \langle Z, W \rangle = 0 \) from the previous question)
\[
Z_t e^{W_t} = e^{f(0)} + \int_0^t e^{W_s} \, dZ_s + \int_0^t e^{W_s} Z_s \, dW_s + \frac{1}{2} \int_0^t e^{W_s} \, d\langle W \rangle_s.
\]
By collecting the finite variation parts from \(dZ\) and \(dW\) from a previous question we get
\[
-\frac{\lambda^2}{2} \int_0^t (X_s^2 + Y_s^2) Z_s e^{W_s} ds - \frac{1}{2} \int_0^t f''(s) (X_s^2 + Y_s^2) Z_s e^{W_s} ds + \frac{1}{2} \int_0^t Z_s e^{W_s} d(W)_s.
\]

Now from a previous question
\[
\langle W \rangle_t = \left( \int_0^t f'(s) X_s dX_s + \int_0^t f'(s) Y_s dY_s \right)_t = \int_0^t f''(s) (X_s^2 + Y_s^2) ds.
\]

It follows that the finite variation part of \(Ze^W\) vanishes when \(f'' = f'^2 - \lambda^2\).

5. With \(f(t) = -\log \cosh(\lambda(r-t))\), we have
\[
f'(t) = \lambda \frac{\sinh(\lambda(r-t))}{\cosh(\lambda(r-t))} = \lambda \tanh(\lambda(r-t))
\]
and
\[
f''(t) = -\frac{\lambda^2}{\cosh(\lambda(r-t))^2} = -\lambda^2 (1 - \tanh(\lambda(r-t))^2) = -\lambda^2 + f'^2(t).
\]

It follows from the previous question that \(Ze^W\) is a continuous local martingale. Note that \(f(r) = f'(r) = 0\) and \(W_r = 0\), and by using previous questions,
\[
\mathbb{E}e^{iA_r} = \mathbb{E} \cos(\lambda A_r) = \mathbb{E} Z_r = \mathbb{E} (Ze^{W_r}).
\]

On the other hand, since \(f(0) = -\log \cosh(\lambda r), Z_0 = 1, W_0 = f(0)\), we get
\[
\mathbb{E}(Z_0 e^{W_0}) = e^{f(0)} = \frac{1}{\cosh(\lambda r)}.
\]

It remains to show that the local martingale \(Ze^W\) is a martingale on the time interval \([0, r]\). From the previous question, since \(f, \cos, \sin\), and \(\sin\) are bounded, it suffices to show that
\[
\mathbb{E} \int_0^t e^{2W_s} d\langle A \rangle_s < \infty \quad \text{and} \quad \mathbb{E} \int_0^t e^{2W_s} (X_s^2 + Y_s^2) ds < \infty.
\]

But the first condition follows from the second thanks to the formula for \(\langle A \rangle\) provided by a previous question. On the other hand, if \(t \in [0, r]\) then \(f'(t) \geq 0\) and thus \(W_s \leq f(t)\) for all \(s \in [0, t]\), which implies that the second condition is satisfied by using \(\mathbb{E}(X_s^2 + Y_s^2) = 2s\).

**Exercise 3** (Criterion for a stochastic differential equation). Set \(d = 1\). Let \(\sigma, b\) be two functions \(\mathbb{R} \to \mathbb{R}\) such that for some finite constant \(C < \infty\) and for all \(x, y \in \mathbb{R}\),
\[
|\sigma(x) - \sigma(y)| \leq C \sqrt{x-y} \quad \text{and} \quad |b(x) - b(y)| \leq C|x-y|
\]
The goal of this exercise is to prove pathwise uniqueness for the stochastic differential equation
\[
dX_t = \sigma(X_t) dB_t + b(X_t) dt. \tag{SDE}
\]
A solution \(X\) is a continuous semi-martingale with canonical decomposition \(X = X_0 + M + V\) with \(X_0 \in L^2\), local martingale part \(M = \int_0^t \sigma(X_s) dB_s\), and finite variation part \(V = \int_0^t b(X_s) ds\). Note that the continuity of \(\sigma, X, b\) gives that almost surely, for all \(t \geq 0\), \(s \mapsto \sigma(X_s) + b(X_s)\) is locally bounded.

1. Let \(Z\) be a continuous semi-martingale such that \(\langle Z \rangle = \int_0^t \varphi(s) ds\) for a progressive process \(\varphi\) such that \(0 \leq \varphi \leq C|Z|\) for some constant \(C < \infty\). Prove that for all \(t \geq 0\) and all \(a > 0\),
\[
\mathbb{E} \int_0^t 1_{0 < \frac{\varphi(s)}{|Z_s|} \leq a} d\langle Z \rangle_s \leq Ct.
\]
2. Deduce from the preceding question that for all $t \geq 0$,
\[
\lim_{n \to \infty} n E \int_0^t 1_{|Z_s| \leq \frac{1}{n}} d\langle Z \rangle_s = 0.
\]

3. For all $n \geq 1$, $x \in \mathbb{R}$, let us define $g_n(x) = 2n(1 + nx)1_{x \in [-\frac{1}{n}, 0)} + 2n1_{x=0} + 2n(1 - nx)1_{x \in (0, \frac{1}{n})}$.
Let $f_n : \mathbb{R} \to \mathbb{R}$ be the twice differentiable function such that $f''_n = g_n$ and $f_n(0) = f'_n(0) = 0$. Show that for all $x \in \mathbb{R}$, the following properties hold true:

(a) $f'_n(x) \in [-1, 1]$ and $\lim_{n \to \infty} f'_n(x) = \text{sign}(x) = 1_{x>0} - 1_{x<0}$

(b) $|f_n(x)| \leq |x|$ and $\lim_{n \to \infty} f_n(x) = |x|$.

4. By using Itô formula, prove that for all continuous semi-martingale $Z = (Z_t)_{t \geq 0}$, all $t \geq 0$,
\[
\int_0^t 1_{Z_s = 0} d\langle Z \rangle_s = 0.
\]

5. From now on, let $X$ and $X'$ be two solutions of (SDE) on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ and with respect to the Brownian motion $B$. Show that for all $t \geq 0$,
\[
\langle X - X' \rangle_t = \int_0^t (\sigma(X_s) - \sigma(X'_s))^2 ds.
\]

6. By using the assumption on $\sigma$, deduce from the preceding questions that for all $t \geq 0$,
\[
\lim_{n \to \infty} E \int_0^t g_n(X_s - X'_s) d\langle X - X' \rangle_s = 0.
\]

7. Set $Z = X - X'$. From now on, let $T$ be a stopping time such that the semi-martingale $(Z_{t \wedge T})_{t \geq 0}$ is bounded. By using notably the assumption on $\sigma$, prove that for all $t \geq 0$, $n \geq 1$,
\[
E(f_n(Z_{t \wedge T})) = E(f_n(Z_0)) + E \int_0^{t \wedge T} f'_n(Z_s)(b(X_s) - b(X'_s)) ds + \frac{1}{2} E \int_0^{t \wedge T} f''_n(Z_s) d\langle Z_s \rangle.
\]

8. Deduce from the preceding questions and the assumption on $b$ that for all $t \geq 0$,
\[
E(|X_{t \wedge T} - X'_{t \wedge T}|) = E(|X_0 - X'_0|) + E \int_0^{t \wedge T} (b(X_s) - b(X'_s)) \text{sign}(X_s - X'_s) ds.
\]

9. By using the Grönwall lemma, deduce that if $X_0 = X'_0$ then $X_t = X'_t$ for all $t \geq 0$.

**Elements of solution for Exercise 3.** The result is known as the Yamada–Watanabe criterion. This is a slightly more detailed version of [1, Exercise 8.14 pages 231–232].

1. We have, using the properties of $Z$ and $\varphi$,
\[
\int_0^t 1_{0 < |Z_s| \leq a} |Z_s| d\langle Z \rangle_s = \int_0^t 1_{0 < |Z_s| \leq a} |Z_s| \varphi_s ds \leq \int_0^t C ds = C t.
\]

2. For all $n \geq 1$, we have $n 1_{0 < |Z_s| \leq \frac{1}{n}} = \frac{1_{0 < |Z_s| \leq \frac{1}{n}}}{|Z_s|} \leq \frac{1_{0 < |Z_s| \leq 1}}{|Z_s|}$, which is integrable on $[0, t]$ by the preceding question used with $a = 1$, and thus the desired result follows then by dominated convergence.

3. The function $g_n$ is $0$ on $(-\infty, -\frac{1}{n})$, then increases from $0$ to $2$ on $[-\frac{1}{n}, 0]$, then decreases from $2$ to $0$ on $[0, \frac{1}{n}]$, then stays at $0$ on $[\frac{1}{n}, +\infty)$. Since $\int_{-\infty}^0 g_n(y) dy = 1$, we have, for all $x \in \mathbb{R}$,
\[
f'_n(x) = \int_0^x g_n(u) du, \quad \text{in such a way that } f'_n(0) = 0 \text{ and } f''_n = g_n.$
The function $f_n$ is $-1$ on $(-\infty, -\frac{1}{n}]$, $0$ at $0$, and $1$ on $[\frac{1}{n}, +\infty)$. Also for all $x \in \mathbb{R}$,

$$\lim_{n \to \infty} f_n'(x) = 1_{x > 0} - 1_{x < 0} =: \text{sign}(x).$$

Next, for all $x \in \mathbb{R}$, we have

$$f_n(x) = \int_0^x f_n'(u)\,du \quad \text{in such a way that } f_n(0) = 0 \text{ and } f_n'' = g_n.$$

Since $g_n \geq 0$, we have that $f_n'$ is non-decreasing, and thus $f_n'$ takes actually its values in $[-1, 1]$, and is in particular bounded. It follows by dominated convergence that for all $x \in \mathbb{R}$,

$$\lim_{n \to \infty} f_n(x) = \int_0^x \lim_{n \to \infty} f_n'(u)\,du = \int_0^x \text{sign}(u)\,du = |x|.$$

Finally, for all $x \in \mathbb{R}$, $|f_n(x)| \leq f_0^{|x|} \, du = |x|$.

4. The Itô formula for function $f_n$ of question 3 and semi-martingale $Z$ gives, for all $t \geq 0$,

$$f_n(Z_t) = f_n(Z_0) + \int_0^t f_n'(Z_s)\,dZ_s + \frac{1}{2} \int_0^t f_n''(Z_s)\,d\langle Z \rangle_s.$$

Since $|f_n''| \leq 1$ and $\lim_{n \to \infty} \frac{f_n''(x)}{2n} = 1_{x=0}$ for all $x \in \mathbb{R}$, by dominated convergence,

$$\lim_{n \to \infty} \frac{1}{2n} \int_0^t f_n''(Z_s)\,d\langle Z \rangle_s = \int_0^t 1_{Z_s=0}\,d\langle Z \rangle_s \quad \text{a.s.}$$

On the other hand, since by question 3, $\lim_{n \to \infty} \frac{f_n(Z_t)}{2n} = 0$ for all $x \in \mathbb{R}$, it follows that a.s.

$$\lim_{n \to \infty} \frac{f_n(Z_t)}{2n} = 0 \quad \text{and} \quad \lim_{n \to \infty} \frac{f_n(Z_0)}{2n} = 0.$$

Finally since by question 3, $\frac{1}{2n} |f_n'| \leq \frac{1}{2n} \leq 1$ and $\lim_{n \to \infty} \frac{1}{2n} |f_n'(x)| = 0$ for all $x \in \mathbb{R}$, dominated convergence for stochastic integrals gives

$$\frac{1}{2n} \int_0^t f_n'(Z_s)\,dZ_s \xrightarrow{n \to \infty} 0.$$

5. Since $X$ and $X'$ are both solutions on the same space and for the same Brownian motion, we have, for all $X_0$ and $X'_0$ and for all $t \geq 0$,

$$X_t - X'_t = X_0 - X'_0 + \int_0^t (\sigma(X_s) - \sigma(X'_s))\,dB_s + \int_0^t (b(X_s) - b(X'_s))\,ds.$$

The right hand side gives the canonical decomposition of the semi-martingale $X - X'$. In this decomposition, the first integral is the local martingale part, and

$$\langle X - X' \rangle = \int_0^t (\sigma(X_s) - \sigma(X'_s))^2\,ds.$$

6. By assumption on $\sigma$, the process $\varphi = (\sigma(X) - \sigma(X'))^2$ satisfies $0 \leq \varphi \leq C|X - X'|$. We can then use question 5 and question 2 with $Z = X - X'$ to get, for all $t \geq 0$,

$$\lim_{n \to \infty} n\mathbb{E} \int_0^t 1_{0 < |Z_s| \leq \frac{1}{n}}\,d\langle Z \rangle_s = 0.$$

If $g_n$ is as in question 3, then for all $x \in \mathbb{R}$, $0 \leq g_n(x) \leq 2n 1_{0 < |x| \leq \frac{1}{n}} + 2n 1_{x=0}$. Thus, for all $t \geq 0$,

$$0 \leq \mathbb{E} \int_0^t g_n(Z_s)\,d\langle Z \rangle_s \leq 2n\mathbb{E} \int_0^t 1_{0 < |Z_s| \leq \frac{1}{n}}\,d\langle Z \rangle_s + 2n\mathbb{E} \int_0^t 1_{Z_s=0}\,d\langle Z \rangle_s \xrightarrow{n \to \infty} 0,$$

where we have used question 2 and question 4.
The Itô formula for the \( e^2 \) function \( f_n \) and the continuous semi-martingale \( Z^T \) gives
\[
f_n(Z^T_\alpha) = f_n(Z^T_0) + \int_0^{t\wedge T} f''_n(Z_s) \, dB_s + \frac{1}{2} \int_0^{t\wedge T} f'''_n(Z_s) \, dB_s^2,
\]
and since \( dZ_s = (\sigma(X_s) - \sigma(X'_s)) \, dB_s + (b(X_s) - b(X'_s)) \, ds \), we get
\[
\int_0^{t\wedge T} f'(Z_s) \, dZ_s = \int_0^{t\wedge T} f'(Z^T_s)(\sigma(X_s) - \sigma(X'_s)) \mathbf{1}_{s \leq T} \, dB_s + \int_0^{t\wedge T} f'(Z_s)(b(X_s) - b(X'_s)) \, ds.
\]
Now, by the assumptions on \( \sigma \) and \( T \), we get
\[
|\sigma(X_s) - \sigma(X'_s)| \mathbf{1}_{s \leq T} \leq C|Z^T_s| \leq C'.
\]
This boundedness, together with the one of \( f'_n \), imply that the first integral in the right hand side above (the \( dB_s \) one) is a martingale. Since this martingale is issued from the origin, its expectation vanishes for all times. On the other hand, since \( f_n \) is continuous and \( Z^T \) is bounded, the random variables \( f_n(Z^T_\alpha) \) and \( f_n(Z^T_0) \) are integrable. All in all, we obtain
\[
E(f_n(Z^T_\alpha)) = E(f_n(Z^T_0)) + E \int_0^{t\wedge T} f''_n(Z_s)(b(X_s) - b(X'_s)) \, ds + \frac{1}{2} E \int_0^{t\wedge T} f'''_n(Z_s) \, dB_s^2.
\]
8. Since \( f''_n = g_n \geq 0 \), we get, by using question 6, that
\[
0 \leq E \int_0^{t\wedge T} f''_n(Z_s) \, dB_s^2 \leq E \int_0^{t\wedge T} g_n(Z_s) \, dB_s \underset{n \to \infty}{\longrightarrow} 0.
\]
On the other hand, by the assumption on \( b \) and the boundedness of \( Z^T \), we have, on \( \{s \leq T\} \),
\[
|b(X_s) - b(X'_s)|^2 \leq C^2|X_s - X'_s|^2 = C^2|Z^T_s| \leq C'.
\]
But since \( f'_n \) is bounded (takes its values in \([-1, 1]\)), we get, by dominated convergence
\[
\lim_{n \to \infty} \int_0^{t\wedge T} f'_n(Z_s)(b(X_s) - b(X'_s)) \, ds = \int_0^{t\wedge T} \text{sign}(Z_s)(b(X_s) - b(X'_s)) \, ds.
\]
Finally, since \( Z^T_\alpha \) is bounded, and since from question 3, for all \( x \in \mathbb{R}, |f_n(x)| \leq |x| \) and \( \lim_{n \to \infty} f_n(x) = |x| \), we get, by dominated convergence, \( \lim_{n \to \infty} E(f_n(Z^T_\alpha)) = E(|Z^T_\alpha|) \). Finally
\[
E(|X_{t\wedge T} - X'_{t\wedge T}|) = E(|X_0 - X'_0|) + E \int_0^{t\wedge T} (b(X_s) - b(X'_s)) \text{sign}(X_s - X'_s) \, ds.
\]
9. From the preceding question, we get, by using the assumption on \( b \),
\[
\alpha(t) = E(|X_{t\wedge T} - X'_{t\wedge T}|) \leq E(|X_0 - X'_0|) + CE \int_0^{t\wedge T} |X_{s\wedge T} - X'_{s\wedge T}| \, ds = \alpha(0) + C \int_0^{t\wedge T} \alpha(s) \, ds.
\]
By the Grönwall lemma, we obtain \( \alpha(t) \leq \alpha(0)e^{Ct} \) for all \( t \geq 0 \). It follows that if \( \alpha(0) = 0 \) then \( \alpha(t) = 0 \) for all \( t \geq 0 \). This means that if \( X_0 = X'_0 \) then \( X_{t\wedge T} = X'_{t\wedge T} \) for all \( t \geq 0 \). By writing this for \( t \in \mathbb{Q}_+ \), and by taking \( T = T_m \) such that \( \lim_{m \to \infty} T_m = +\infty \) almost surely, we get that \( X_t = X'_t \) for all \( t \in \mathbb{Q}_+ \), and thus for all \( t \geq 0 \) since \( X \) and \( X' \) are continuous.

References