Exercise 1 (Nature of an integral). Set $d = 1$. Let us consider the following integral, for $t ≥ 0$,

$$I_t = \int_0^t B_s \, ds.$$  


2. Show that $d(tB_t) = B_t \, dt + t \, dB_t$;

3. Deduce from the preceding question that $I_t = \int_0^t (t - s) \, dB_s$ for all $t ≥ 0$;

4. Deduce from the preceding question that $I_t \sim \mathcal{N}(0, \frac{1}{3} t^3)$ for all $t ≥ 0$;

5. For all $t ≥ 0$, $n ≥ 1$, $0 ≤ k ≤ n$, let us define $t_k = \frac{k}{n} t$. Show that

$$\sum_{k=0}^{n-1} B_{t_k} (t_{k+1} - t_k) = \frac{t}{n} \sum_{j=0}^{n-2} (n - j - 1) (B_{t_{j+1}} - B_{t_j}).$$

6. Deduce from the preceding question another proof that $I_t \sim \mathcal{N}(0, \frac{1}{3} t^3)$ for all $t ≥ 0$;

7. Is the process $(I_t)_{t ≥ 0}$ a martingale?

Elements of solution for Exercise 1.

1. Since the integrator is of finite variation and the integrand is bounded and measurable (actually continuous), it is a Lebesgue–Stieltjes integral, and in particular an Itô integral with respect to a semi-martingale without martingale part. However it is not a Wiener integral.

2. The Itô formula for $f(x, y) = xy$ and $X_t = (t, B_t)$ gives

$$tB_t = 0 + \int_0^t B_s \, ds + \int_0^t s \, dB_s,$$

3. From the preceding question (actually, it is an integration by parts)

$$\int_0^t B_s \, ds = tB_t - \int_0^t s \, dB_s = \int_0^t (t - s) \, dB_s,$$

4. The integral in the right hand side is a Wiener integral. Thus it is Gaussian with mean zero and variance equal to the squared $L^2$ norm of the integrand:

$$\mathbb{E} \left[ \int_0^t B_s \, ds \right] = 0 \quad \text{and} \quad \mathbb{E} \left[ \left( \int_0^t B_s \, ds \right)^2 \right] = \int_0^t (t - s)^2 \, ds = \frac{t^3}{3}.$$

5. With $t_k = \frac{k}{n} t$ for all $0 ≤ k ≤ n$, we have

$$S_n = \sum_{k=0}^{n-1} B_{t_k} (t_{k+1} - t_k) = \frac{t}{n} \sum_{k=0}^{n-1} B_{t_k} = \frac{t}{n} \sum_{k=0}^{n-1} \sum_{j=0}^{k-1} (B_{t_{j+1}} - B_{t_j}) = \frac{t}{n} \sum_{j=0}^{n-2} (n - j - 1) (B_{t_{j+1}} - B_{t_j}).$$
6. Fix $t \geq 0$. Since $I_t$ is a Lebesgue–Stieltjes integral with continuous integrand, we have $\lim_{n \to \infty} S_n = I_t$ almost surely and thus in law. For all $n$ and $j$, since $B_{t_{j+1}} - B_{t_j}$ are independent and Gaussian, we get that $S_n \sim \mathcal{N}(\mathbb{E}(S_n), \mathbb{E}(S_n^2) - \mathbb{E}(S_n)^2)$. But the convergence in law of Gaussians is equivalent to the convergence of the first two moments. Now it remains to note that we have $\mathbb{E}(S_n) = 0$ and

$$
\mathbb{E}(S_n^2) = \frac{t^2}{n^2} \sum_{j=0}^{n-2} (n-j-1)^2 \mathbb{E}((B_{t_{j+1}} - B_{t_j})^2) = \frac{t^4}{n^4} \sum_{j=1}^{n-1} \left( \frac{j}{n} \right)^2 \frac{1}{n} \xrightarrow{n \to \infty} \frac{t^4}{3} \int_0^t x^2 \, dx = \frac{t^4}{3}.
$$

7. Beware that the integrand in $\int_0^t (t-s) \, dB_s$ depends on $t$. The process $(-\int_0^t s \, dB_s)_{t \geq 0}$ is a martingale, however the process $(\int_0^t t \, dB_s)_{t \geq 0} = (tB_t)_{t \geq 0}$ is not a martingale: for all $0 \leq s \leq t$,

$$
\mathbb{E}((t+s)B_{t+s} \mid \mathcal{F}_t) = (t+s)B_s \neq sB_s.
$$

**Exercise 2** (Study of a special process). Set $d = 2$. For all $t \geq 0$, we write $B_t = (X_t, Y_t)$ and

$$
A_t = \int_0^t X_s \, dY_s - \int_0^t Y_s \, dX_s.
$$

1. Show that $(A) = \int_0^t (X_s^2 + Y_s^2) \, ds$ and that the process $A$ is a square integrable martingale;

2. From now on let $\lambda > 0$. Show that for all $t \geq 0$,

$$
\mathbb{E}e^{\lambda A_t} = \mathbb{E}\cos(\lambda A_t).
$$

3. From now on, let $f : \mathbb{R}_+ \to \mathbb{R}$ be ‘$\mathcal{C}^2$’, and let us define the continuous semi-martingales

$$(Z_t)_{t \geq 0} = (\cos(\lambda A_t))_{t \geq 0} \quad \text{and} \quad (W_t)_{t \geq 0} = \left( -\frac{f'(t)}{2} (X_t^2 + Y_t^2) + f(t) \right)_{t \geq 0}.
$$

Show that for all $t \geq 0$,

$$
Z_t = 1 - \lambda \int_0^t \sin(\lambda A_s) \, dA_s - \frac{\lambda^2}{2} \int_0^t (X_s^2 + Y_s^2) \, Z_s \, ds.
$$

and

$$
W_t = f(0) - \int_0^t f'(s) X_s \, dX_s - \int_0^t f'(s) Y_s \, dY_s - \frac{1}{2} \int_0^t f''(s) (X_s^2 + Y_s^2) \, ds,
$$

and deduce that

$$(Z, W) = 0.
$$

4. Show that if $f$ solves $f'' = f' - \lambda^2$ then $Ze^W$ is a continuous local martingale and

$$
Ze^W = e^{f(0)} - \lambda \int_0^t \sin(\lambda A_s) e^{W_s} \, dA_s - \int_0^t f'(s) Z_s e^{W_s} X_s \, dX_s - \int_0^t f'(s) Z_s e^{W_s} Y_s \, dY_s.
$$

5. Let $r > 0$. By using $f(t) = -\log \cosh(\lambda(r-t))$ deduce from the previous question that

$$
\mathbb{E}e^{\lambda A_t} = \frac{1}{\cosh(\lambda r)}.
$$

**Elements of solution for Exercise 2.** For all $t \geq 0$, $A_t$ is the algebraic area between planar Brownian motion and its chord, and the process $A$ is the Lévy area. This exercise is a slightly more detailed version of [1, Exercise 5.30, pages 144–145]. Its goal is to compute the characteristic function or Fourier transform of $A_t$. 

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1. Since \( \langle X \rangle_t = \langle Y \rangle_t = t \) and \( \langle X, Y \rangle_t = 0 \), we get
\[
\langle A \rangle_t = \langle A, A \rangle_t = \left( \int_0^t X_s \, dY_s \right)_t + \left( \int_0^t Y_s \, dX_s \right)_t + 2 \left( \int_0^t X_s \, dY_s \right)_t = \int_0^t X_s \, dX_s + \int_0^t Y_s \, dY_s + \int_0^t X_s \, dY_s + \int_0^t Y_s \, dX_s = \int_0^t (X_s^2 + Y_s^2) \, ds.
\]
It follows by the Fubini–Tonelli theorem that
\[
\mathbb{E}(A)_t = \int_0^t \mathbb{E}(X_s^2 + Y_s^2) \, ds = \int_0^t 2ds = t^2 < \infty
\]
and thus, by a famous martingale criterion, the process \( A \) is a square integrable martingale.

Alternatively, since for all \( t \geq 0 \), \( \mathbb{E} \int_0^t X_s^2 \, dY_s = \int_0^t \mathbb{E}(X_s^2) \, ds < \infty \) the process \( \int_0^t X_s \, dY_s \) and by symmetry the process \( \int_0^t Y_s \, dX_s \) are both square integrable martingales, and thus the process \( A \) is also a square integrable martingales as being the difference of two square integrable martingales.

2. For all \( \lambda \in \mathbb{R} \), \( t \geq 0 \), \( \mathbb{E}(e^{\lambda A_t}) = \mathbb{E}(\cos(\lambda A_t)) + i \mathbb{E}(\sin(\lambda A_t)) \). Since \( \langle X, Y \rangle = \langle Y, X \rangle \), we get, for all \( t \geq 0 \),
\[
-A_t = \int_0^t X_s \, dX_s - \int_0^t X_s \, dY_s - \int_0^t X_s \, dY_s - \int_0^t Y_s \, dX_s = A_t,
\]
and thus the characteristic function or Fourier transform of \( A_t \) is real.

3. The canonical decompositions are given by the Itô formula. Namely, for \( Z \),
\[
Z_t = 1 - \lambda \int_0^t \sin(\lambda A_s) \, dA_s - \frac{\lambda^2}{2} \int_0^t \cos(\lambda A_s) \, d\langle A \rangle_s
= 1 - \lambda \int_0^t \sin(\lambda A_s) \, dA_s - \frac{\lambda^2}{2} \int_0^t (X_s^2 + Y_s^2) \, Z_s \, ds.
\]
Similarly, for \( W \), by the Itô formula for the function \( g(x, y, t) = -\frac{[f(t)]}{2}(x^2 + y^2) + f(t) \) and the vector of semi-martingale \( S_t = (X_t, Y_t, t) \) with the martingale part \((X_t, Y_t)\),
\[
W_t = g(0, 0, 0) + \int_0^t \partial_1 g(S_s) \, dX_s + \int_0^t \partial_2 g(S_s) \, dY_s + \int_0^t \partial_3 g(S_s) \, ds + \frac{1}{2} \int_0^t \partial_3^2 g(S_s) \, ds.
= f(0) - \int_0^t f'(s) X_s \, dX_s - \int_0^t f'(s) Y_s \, dY_s + \int_0^t \left( -\frac{[f'(t)]}{2}(X_s^2 + Y_s^2) + f'(t) \right) \, ds - \int_0^t f''(t) ds
= f(0) - \int_0^t f'(s) X_s \, dX_s - \int_0^t f'(s) Y_s \, dY_s - \frac{1}{2} \int_0^t f''(s)(X_s^2 + Y_s^2) \, ds.
\]
The computation of \( \langle Z, W \rangle \) involves only the local martingale parts, namely
\[
\langle Z, W \rangle_t = \lambda \left( \int_0^t \sin(\lambda A_s) \, dA_s, \int_0^t f'(s) X_s \, dX_s + \int_0^t f'(s) Y_s \, dY_s \right)_t
= \lambda \int_0^t f'(s) \sin(\lambda A_s) \, dA_s + \lambda \int_0^t f'(s) \sin(\lambda A_s) \, d(A, Y)_s.
\]
Now since \( \langle A, X \rangle_t = -\int_0^t Y_s \, ds \) and \( \langle A, Y \rangle_t = \int_0^t X_s \, ds \), we get
\[
\langle Z, W \rangle_t = \lambda \int_0^t f'(s)(-X_s Y_s + X_s Y_s) \, \sin(\lambda A_s) \, ds = 0.
\]
4. The Itô formula gives (we benefit from the fact that \( \langle Z, W \rangle = 0 \) from the previous question)
\[
Z_t e^{W_t} = e^{f(0)} + \int_0^t e^{W_s} \, dB_s + \frac{1}{2} \int_0^t Z_s e^{W_s} \, ds.
\]
By collecting the finite variation parts from \(dZ\) and \(dW\) from a previous question we get

\[
-\frac{\lambda^2}{2} \int_0^t (X_r^2 + Y_r^2) Z_\sigma e^{W_\sigma} \, ds - \frac{1}{2} \int_0^t f''(s)(X_r^2 + Y_r^2) Z_\sigma e^{W_\sigma} \, ds + \frac{1}{2} \int_0^t Z_\sigma e^{W_\sigma} d(W)_s.
\]

Now from a previous question

\[
\langle W \rangle_t = \left( \int_0^t f'(s) X_s \, dX_s + \int_0^t f'(s) Y_s \, dY_s \right)_t = \int_0^t f''(s)(X_r^2 + Y_r^2) \, ds.
\]

It follows that the finite variation part of \(Z e^{W}\) vanishes when \(f'' = f'^2 - \lambda^2\).

5. With \(f(t) = -\log\cosh(\lambda(r-t))\), we have

\[
f'(t) = \frac{\lambda \sinh(\lambda(r-t))}{\cosh(\lambda(r-t))} = \lambda \tanh(\lambda(r-t))
\]

and

\[
f''(t) = -\frac{\lambda^2}{\cosh(\lambda(r-t))^2} = -\lambda^2 (1 - \tanh(\lambda(r-t))^2) = -\lambda^2 + f'^2(t).
\]

It follows from the previous question that \(Z e^{W}\) is a continuous local martingale. Note that \(f(r) = f'(r) = 0\) and \(W_0 = 0\), and by using previous questions,

\[
\mathbb{E}e^{\lambda A_r} = \mathbb{E}\cos(\lambda A_r) = \mathbb{E}Z_r = \mathbb{E}(Z_r e^{W_t}).
\]

On the other hand, since \(f(0) = -\log\cosh(\lambda r)\), \(Z_0 = 1\), \(W_0 = f(0)\), we get

\[
\mathbb{E}(Z_0 e^{W_0}) = e^{f(0)} = \frac{1}{\cosh(\lambda r)}.
\]

It remains to show that the local martingale \(Z e^{W}\) is a martingale on the time interval \([0, r]\). From the previous question, since \(f\), \(\cos\), and \(\sin\) are bounded, it suffices to show that

\[
\mathbb{E} \int_0^t e^{2W_s} \, d\langle A \rangle_s < \infty \quad \text{and} \quad \mathbb{E} \int_0^t e^{2W_s}(X_r^2 + Y_r^2) \, ds < \infty.
\]

But the first condition follows from the second thanks to the formula for \(\langle A \rangle\) provided by a previous question. On the other hand, if \(t \in [0, r]\) then \(f'(t) \geq 0\) and thus \(W_s \leq f(t)\) for all \(s \in [0, t]\), which implies that the second condition is satisfied by using \(\mathbb{E}(X_r^2 + Y_r^2) = 2s\).

**Exercise 3** (Criterion for a stochastic differential equation). Set \(d = 1\). Let \(\sigma, b\) be two functions \(\mathbb{R} \rightarrow \mathbb{R}\) such that for some finite constant \(C < \infty\) and for all \(x, y \in \mathbb{R}\),

\[
|\sigma(x) - \sigma(y)| \leq C \sqrt{x - y} \quad \text{and} \quad |b(x) - b(y)| \leq C|x - y|.
\]

The goal of this exercise is to prove pathwise uniqueness for the stochastic differential equation

\[
dX_t = \sigma(X_t) \, dB_t + b(X_t) \, dt.
\]

A solution \(X\) is a continuous semi-martingale with canonical decomposition \(X = X_0 + M + V\) with \(X_0 \in L^2\), local martingale part \(M = \int_0^\cdot \sigma(X_s) \, dB_s\), and finite variation part \(V = \int_0^\cdot b(X_s) \, ds\). Note that the continuity of \(\sigma, X, b\) gives that almost surely, for all \(t \geq 0\), \(s \mapsto \sigma(X_s) + b(X_s)\) is locally bounded.

1. Let \(Z\) be a continuous semi-martingale such that \(\langle Z \rangle = \int_0^\cdot \varphi \, ds\) for a progressive process \(\varphi\) such that \(0 \leq \varphi \leq C|Z|\) for some constant \(C < \infty\). Prove that for all \(t \geq 0\) and all \(a > 0\),

\[
\mathbb{E} \int_0^t 1_{0 < |Z_s| \leq a} \, d\langle Z \rangle_s \leq Ct.
\]
2. Deduce from the preceding question that for all \( t \geq 0 \),
\[
\lim_{n \to \infty} nE \int_0^t 1_{|Z_s| \leq \frac{1}{n}} d\langle Z \rangle_s = 0.
\]

3. For all \( n \geq 1 \), let us define \( g_n(x) = 2n(1 + nx)1_{x \leq -\frac{1}{2}} + 2n1_{x = 0} + 2n(1 - nx)1_{x \geq 0} \).
Let \( f_n : \mathbb{R} \to \mathbb{R} \) be the twice differentiable function such that \( f_n'' = g_n \) and \( f_n(0) = f_n'(0) = 0 \). Show that for all \( x \in \mathbb{R} \), the following properties hold true:
\begin{itemize}
  \item[(a)] \( f_n'(x) \in [-1, 1] \) and \( \lim_{n \to \infty} f_n'(x) = \text{sign}(x) = 1_{x > 0} - 1_{x < 0} \);
  \item[(b)] \( |f_n(x)| \leq |x| \) and \( \lim_{n \to \infty} f_n(x) = |x| \).
\end{itemize}

4. By using Itô formula, prove that for all continuous semi-martingale \( Z = (Z_t)_{t \geq 0} \), all \( t \geq 0 \),
\[
\int_0^t 1_{Z_s = 0} d\langle Z \rangle_s = 0.
\]

5. From now on, let \( X \) and \( X' \) be two solutions of (SDE) on \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})\) and with respect to the Brownian motion \( B \). Show that for all \( t \geq 0 \),
\[
\langle X - X' \rangle_t = \int_0^t (\sigma(X_s) - \sigma(X'_s))^2 ds.
\]

6. By using the assumption on \( \sigma \), deduce from the preceding questions that for all \( t \geq 0 \),
\[
\lim_{n \to \infty} E \int_0^t g_n(X_s - X'_s) d\langle X - X' \rangle_s = 0.
\]

7. Set \( Z = X - X' \). From now on, let \( T \) be a stopping time such that the semi-martingale \((Z_t)_{t \geq 0}\) is bounded. By using notably the assumption on \( \sigma \), prove that for all \( t \geq 0 \), \( n \geq 1 \),
\[
E(f_n(Z_{t \wedge T})) = E(f_n(Z_0)) + E \int_0^{t \wedge T} f_n'(Z_s)(b(X_s) - b(X'_s)) ds + \frac{1}{2} E \int_0^{t \wedge T} f_n''(Z_s) d\langle Z \rangle_s.
\]

8. Deduce from the preceding questions and the assumption on \( b \) that for all \( t \geq 0 \),
\[
E(|X_{t \wedge T} - X'_{t \wedge T}|) = E(|X_0 - X'_0|) + E \int_0^{t \wedge T} (b(X_s) - b(X'_s)) \text{sign}(X_s - X'_s) ds.
\]

9. By using the Grönwall lemma, deduce that if \( X_0 = X'_0 \) then \( X_t = X'_t \) for all \( t \geq 0 \).

**Elements of solution for Exercise 3.** The result is known as the Yamada–Watanabe criterion. This is a slightly more detailed version of [1, Exercise 8.14 pages 231–232].

1. We have, using the properties of \( Z \) and \( \varphi \),
\[
\int_0^t \frac{1_{|Z_s| \leq \alpha}}{|Z_s|} d\langle Z \rangle_s = \int_0^t \frac{1_{\alpha < |Z_s| \leq \frac{1}{n}}}{|Z_s|} \varphi_s ds \leq \int_0^t C ds = Ct.
\]

2. For all \( n \geq 1 \), we have \( n1_{\alpha < |Z_s| \leq \frac{1}{n}} \leq 1_{\alpha < |Z_s| \leq \frac{1}{n}} \leq 1_{\frac{1}{n} < |Z_s| \leq \frac{1}{\alpha}} \), which is integrable on \([0, t]\) by the preceding question used with \( \alpha = 1 \), and thus the desired result follows then by dominated convergence.

3. The function \( g_n(t) = 0 \) on \((-\infty, -\frac{1}{n}]\), then increases from 0 to 2 on \([-\frac{1}{n}, 0] \), then decreases from 2 to 0 on \([0, \frac{1}{n}]\), then stays at 0 on \([\frac{1}{n}, +\infty)\). Since \( f_n^{0,\infty} = 1 \), we have, for all \( x \in \mathbb{R} \),
\[
f_n'(x) = \int_0^x g_n(u) du, \quad \text{in such a way that } f_n'(0) = 0 \text{ and } f_n'' = g_n.
\]
The function \( f_n \) is \(-1\) on \((-\infty, -\frac{1}{n}]\), \(0\) at \(0\), and \(1\) on \([\frac{1}{n}, +\infty)\). Also for all \(x \in \mathbb{R}\),
\[
\lim_{n \to \infty} f'_n(x) = 1_{x > 0} - 1_{x < 0} =: \text{sign}(x).
\]

Next, for all \(x \in \mathbb{R}\), we have
\[
f_n(x) = \int_0^x f'_n(u) \, du \quad \text{in such a way that} \quad f_n(0) = 0 \quad \text{and} \quad f''_n = g_n.
\]

Since \(g_n \geq 0\), we have that \(f'_n\) is non-decreasing, and thus \(f'_n\) takes actually its values in \([-1, 1]\), and is in particular bounded. It follows by dominated convergence that for all \(x \in \mathbb{R}\),
\[
\lim_{n \to \infty} f_n(x) = \int_0^x \lim_{n \to \infty} f'_n(u) \, du = \int_0^x \text{sign}(u) \, du = |x|.
\]

Finally, for all \(x \in \mathbb{R}\), \(|f_n(x)| \leq f_0^{|x|} \, du = |x|).

4. The Itô formula for function \(f_n\) of question 3 and semi-martingale \(Z\) gives, for all \(t \geq 0\),
\[
f_n(Z_t) = f_n(Z_0) + \int_0^t f'_n(Z_s) \, dZ_s + \frac{1}{2} \int_0^t f''_n(Z_s) \, d\langle Z \rangle_s.
\]

Since \(\frac{f''_n(x)}{2n} \leq 1\) and \(\lim_{n \to \infty} \frac{f''_n(x)}{2n} = 1_{x=0}\) for all \(x \in \mathbb{R}\), by dominated convergence,
\[
\lim_{n \to \infty} \int_0^t f''_n(Z_s) \, d\langle Z \rangle_s = \int_0^t 1_{Z_s=0} \, d\langle Z \rangle_s \quad \text{a.s.}
\]

On the other hand, since by question 3, \(\lim_{n \to \infty} \frac{f_n(Z_t)}{2n} = 0\) for all \(x \in \mathbb{R}\), it follows that a.s.
\[
\lim_{n \to \infty} \frac{f_n(Z_t)}{2n} = 0 \quad \text{and} \quad \lim_{n \to \infty} \frac{f_n(Z_0)}{2n} = 0.
\]

Finally since by question 3, \(\frac{f_n(x)}{2n} \leq \frac{1}{2n} \leq 1\) and \(\lim_{n \to \infty} \frac{f_n(x)}{2n} = 0\) for all \(x \in \mathbb{R}\), dominated convergence for stochastic integrals gives
\[
\frac{1}{2n} \int_0^t f''_n(Z_s) \, dZ_s \xrightarrow{n \to \infty} 0.
\]

5. Since \(X\) and \(X'\) are both solutions on the same space and for the same Brownian motion, we have, for all \(X_0\) and \(X'_0\) and for all \(t \geq 0\),
\[
X_t - X'_t = X_0 - X'_0 + \int_0^t (\sigma(X_s) - \sigma(X'_s)) \, dB_s + \int_0^t (b(X_s) - b(X'_s)) \, ds.
\]

The right hand side gives the canonical decomposition of the semi-martingale \(X - X'\). In this decomposition, the first integral is the local martingale part, and
\[
\langle X - X' \rangle = \int_0^t (\sigma(X_s) - \sigma(X'_s))^2 \, ds.
\]

6. By assumption on \(\sigma\), the process \(\varphi = (\sigma(X) - \sigma(X'))^2\) satisfies \(0 \leq \varphi \leq C|X - X'|\). We can then use question 5 and question 2 with \(Z = X - X'\) to get, for all \(t \geq 0\),
\[
\lim_{n \to \infty} n \mathbb{E} \int_0^t 1_{0 < |Z_s| \leq \frac{1}{n}} \, d\langle Z \rangle_s = 0.
\]

If \(g_n\) is as in question 3, then for all \(x \in \mathbb{R}\), \(0 \leq g_n(x) \leq 2n 1_{0 < |x| \leq \frac{1}{n}} + 2n 1_{x=0}\). Thus, for all \(t \geq 0\),
\[
0 \leq \mathbb{E} \int_0^t g_n(Z_s) \, d\langle Z \rangle_s \leq 2n \mathbb{E} \int_0^t 1_{0 < |Z_s| \leq \frac{1}{n}} \, d\langle Z \rangle_s + 2n \mathbb{E} \int_0^t 1_{Z_s=0} \, d\langle Z \rangle_s \xrightarrow{n \to \infty} 0,
\]
where we have used question 2 and question 4.
7. The Itô formula for the $e^T$ function $f_n$ and the continuous semi-martingale $Z^T$ gives
\[ f_n(Z^T_t) = f_n(Z_0^T) + \int_0^T f''_n(Z_s)dB_s + \frac{1}{2} \int_0^T f'''_n(Z_s)d\langle B \rangle_s, \]
and since $dZ_t = (\sigma(X_t) - \sigma(X'_t))dB_t + (b(X_t) - b(X'_t))ds$, we get
\[ \int_0^T f'(Z_s)dZ_s = \int_0^T f'(Z^T_s)(\sigma(X_s) - \sigma(X'_s))1_{s \leq T}dB_s + \int_0^T f'(Z_s)(b(X_s) - b(X'_s))ds. \]

Now, by the assumptions on $\sigma$ and $T$, we get
\[ |\sigma(X_s) - \sigma(X'_s)|1_{s \leq T} \leq C|Z^T_s| \leq C'. \]

This boundedness, together with the one of $f''_n$, imply that the first integral in the right hand side above (the $dB_s$ one) is a martingale. Since this martingale is issued from the origin, its expectation vanishes for all times. On the other hand, since $f_n$ is continuous and $Z^T$ is bounded, the random variables $f_n(Z^T_s)$ and $f_n(Z'_s)$ are integrable. All in all, we obtain
\[ E(f_n(Z^T_T)) = E(f_n(Z_0^T)) + \mathbb{E}\int_0^T f''_n(Z_s)(b(X_s) - b(X'_s))ds + \frac{1}{2} \mathbb{E}\int_0^T f'''_n(Z_s)d\langle B \rangle_s. \]

8. Since $f'''_n = g_n \geq 0$, we get, by using question 6, that
\[ 0 \leq \mathbb{E}\int_0^T f'''_n(Z_s)d\langle B \rangle_s \leq \mathbb{E}\int_0^T g_n(Z_s)d\langle B \rangle_s \xrightarrow{n \to \infty} 0. \]
On the other hand, by the assumption on $b$ and the boundedness of $Z^T$, we have, on $[s \leq T]$,
\[ |b(X_s) - b(X'_s)|^2 \leq C^2|X_s - X'_s|^2 = C^2|Z^T_s| \leq C'. \]

But since $f'_n$ is bounded (takes its values in $[-1, 1]$), we get, by dominated convergence
\[ \lim_{n \to \infty} \int_0^T f'_n(Z_s)(b(X_s) - b(X'_s))ds = \int_0^T \text{sign}(Z_s)(b(X_s) - b(X'_s))ds. \]

Finally, since $Z^T_t$ is bounded, and since from question 3, for all $x \in \mathbb{R}$, $|f_n(x)| \leq |x|$ and $\lim_{n \to \infty} f_n(x) = |x|$, we get, by dominated convergence, $\lim_{n \to \infty} E(f_n(Z^T_T)) = E(|Z^T_T|)$. Finally
\[ E(|X_{t\wedge T} - X'_{t\wedge T}|) = E(|X_0 - X'_0|) + \mathbb{E}\int_0^T (b(X_s) - b(X'_s))\text{sign}(X_s - X'_s)ds. \]

9. From the preceding question, we get, by using the assumption on $b$,
\[ \alpha(t) = E(|X_{t\wedge T} - X'_{t\wedge T}|) \leq E(|X_0 - X'_0|) + CE\int_0^T |X_{s\wedge T} - X'_{s\wedge T}|ds = \alpha(0) + C\int_0^T \alpha(s)ds. \]

By the Grönwall lemma, we obtain $\alpha(t) \leq \alpha(0)e^{Ct}$ for all $t \geq 0$. It follows that if $\alpha(0) = 0$ then $\alpha(t) = 0$ for all $t \geq 0$. This means that if $X_0 = X'_0$, then $X_{t\wedge T} = X'_{t\wedge T}$ for all $t \geq 0$. By writing this for $t \in Q_+$, and by taking $T = T_m$ such that $\lim_{m \to \infty} T_m = +\infty$ almost surely, we get that $X_t = X'_t$ for all $t \in Q_+$, and thus for all $t \geq 0$ since $X$ and $X'$ are continuous.

References