Exercise 1 (Nature of an integral). Set $d = 1$. Let us consider the following integral, for $t \geq 0$,

$$I_t = \int_0^t B_s ds.$$ 

2. Show that $d(tB_t) = B_t dt + tdB_t$.
3. Deduce from the preceding question that $I_t = \int_0^t (t-s)dB_s$ for all $t \geq 0$.
4. Deduce from the preceding question that $I_t \sim \mathcal{N}(0, \frac{1}{3} t^3)$ for all $t \geq 0$.
5. For all $t \geq 0$, $n \geq 1$, $0 \leq k \leq n$, let us define $t_k = \frac{k}{n} t$. Show that

$$\sum_{k=0}^{n-1} B_{t_k}(t_{k+1} - t_k) = \frac{t}{n} \sum_{j=0}^{n-2} (n-j-1)(B_{t_{j+1}} - B_{t_j}).$$

6. Deduce from the preceding question another proof that $I_t \sim \mathcal{N}(0, \frac{1}{3} t^3)$ for all $t \geq 0$.
7. Is the process $(I_t)_{t \geq 0}$ a martingale?

Elements of solution for Exercise 1.

1. Since the integrator is of finite variation and the integrand is bounded and measurable (actually continuous), it is a Lebesgue–Stieltjes integral, and in particular an Itô integral with respect to a semi-martingale without martingale part. However it is not a Wiener integral.

2. The Itô formula for $f(x, y) = xy$ and $X_t = (t, B_t)$ gives

$$tB_t = 0 + \int_0^t B_s ds + \int_0^t sdB_s,$$

3. From the preceding question (actually, it is an integration by parts)

$$\int_0^t B_s ds = tB_t - \int_0^t s dB_s = \int_0^t (t-s)dB_s,$$

4. The integral in the right hand side is a Wiener integral. Thus it is Gaussian with mean zero and variance equal to the squared $L^2$ norm of the integrand:

$$\mathbb{E} \left( \int_0^t B_s ds \right) = 0 \quad \text{and} \quad \mathbb{E} \left( \left( \int_0^t B_s ds \right)^2 \right) = \int_0^t (t-s)^2 ds = \frac{t^3}{3}.$$

5. With $t_k = \frac{k}{n} t$ for all $0 \leq k \leq n$, we have

$$S_n = \sum_{k=0}^{n-1} B_{t_k}(t_{k+1} - t_k) = \frac{t}{n} \sum_{k=0}^{n-1} B_{t_k} = \frac{t}{n} \sum_{k=0}^{n-1} \sum_{j=0}^{k-1} (B_{t_{j+1}} - B_{t_j}) = \frac{t}{n} \sum_{j=0}^{n-2} (n-j-1)(B_{t_{j+1}} - B_{t_j}).$$
6. Fix \( t \geq 0 \). Since \( I_t \) is a Lebesgue–Stieltjes integral with continuous integrand, we have \( \lim_{n \to \infty} S_n = I_t \) almost surely and thus in law. For all \( n \) and \( j \), since \( B_{t+j} - B_t \) are independent and Gaussian, we get that \( S_n \sim N(\mathbb{E}(S_n), \mathbb{E}(S_n^2) - \mathbb{E}(S_n)^2) \). But the convergence in law of Gaussians is equivalent to the convergence of the first two moments. Now it remains to note that we have \( \mathbb{E}(S_n) = 0 \) and
\[
\mathbb{E}(S_n^2) = \frac{t^2}{n^2} \sum_{j=0}^{n-2} (n-j-1)^2 \mathbb{E}((B_{t+j} - B_t)^2) = \frac{t^3}{n^3} \sum_{j=1}^{n-1} j^2 = \frac{t^3}{n} \sum_{j=1}^{n-1} \frac{j^2}{n} \to t^3 \int_0^1 x^2 \, dx = \frac{t^3}{3}.
\]

7. Beware that the integrand in \( f(t) = \int_0^t (t-s) \, dB_s \) depends on \( t \). The process \(( -\int_0^t s \, dB_s )_{t \geq 0} \) is a martingale, however the process \(( \int_0^t t \, dB_s )_{t \geq 0} = ( t \, B_t )_{t \geq 0} \) is not a martingale: for all \( 0 \leq s \leq t \),
\[
\mathbb{E}((t+s) B_{t+s} \mid \mathcal{F}_t) = (t+s) B_s \neq s B_s.
\]

**Exercise 2** (Study of a special process). Set \( d = 2 \). For all \( t \geq 0 \), we write \( B_t = (X_t, Y_t) \) and
\[
A_t = \int_0^t X_s \, dY_s - \int_0^t Y_s \, dX_s.
\]

1. Show that \( \langle A \rangle = \int_0^t (X_s^2 + Y_s^2) \, ds \) and that the process \( A \) is a square integrable martingale

2. From now on let \( \lambda > 0 \). Show that for all \( t \geq 0 \),
\[
\mathbb{E}e^{i\lambda A_t} = \mathbb{E} \cos(\lambda A_t).
\]

3. From now on, let \( f : \mathbb{R}_+ \to \mathbb{R} \) be \( \mathcal{C}^2 \), and let us define the continuous semi-martingales
\[
(Z_t)_{t \geq 0} = (\cos(\lambda A_t))_{t \geq 0} \quad \text{and} \quad (W_t)_{t \geq 0} = \left( -\frac{f'(t)}{2} (X_t^2 + Y_t^2) + f(t) \right)_{t \geq 0}.
\]

Show that for all \( t \geq 0 \),
\[
Z_t = 1 - \lambda \int_0^t \sin(\lambda A_s) \, dA_s - \frac{\lambda^2}{2} \int_0^t (X_s^2 + Y_s^2) \, Z_s \, ds.
\]

and
\[
W_t = f(0) - \int_0^t f'(s) X_s \, dY_s - \int_0^t f'(s) Y_s \, dX_s - \frac{1}{2} \int_0^t f''(s) (X_s^2 + Y_s^2) \, ds,
\]

and deduce that
\[
\langle Z, W \rangle = 0.
\]

4. Show that if \( f \) solves \( f'' = f'^2 - \lambda^2 \) then \( Z \, e^{W} \) is a continuous local martingale and
\[
Z \, e^{W} = e^{f(0)} - \lambda \int_0^t \sin(\lambda A_s) e^{W_s} \, dA_s - \int_0^t f'(s) Z_s e^{W_s} X_s \, dX_s - \int_0^t f'(s) Z_s e^{W_s} Y_s \, dY_s.
\]

5. Let \( r > 0 \). By using \( f(t) = -\log \cosh(\lambda (r-t)) \) deduce from the previous question that
\[
\mathbb{E}e^{i\lambda A_t} = \frac{1}{\cosh(\lambda r)}.
\]

**Elements of solution for Exercise 2.** For all \( t \geq 0 \), \( A_t \) is the algebraic area between planar Brownian motion and its chord, and the process \( A \) is the *Lévy area*. This exercise is a slightly more detailed version of [1, Exercise 5.30 pages 144–145]. Its goal is to compute the characteristic function or Fourier transform of \( A_t \).
1. Since \( \langle X \rangle_t = \langle Y \rangle_t = t \) and \( \langle X, Y \rangle_t = 0 \), we get

\[
\langle A \rangle_t = \langle A, A \rangle_t = \left( \int_0^t X_s dY_s \right)_t + \left( \int_0^t Y_s dX_s \right)_t + 2 \left( \int_0^t X_s dY_s \right)_t = \int_0^t X_s^2 d\langle X \rangle_s + \int_0^t Y_s^2 d\langle Y \rangle_s + \int_0^t X_s Y_s d\langle X, Y \rangle_s = \int_0^t (X_s^2 + Y_s^2) ds.
\]

It follows by the Fubini–Tonelli theorem that

\[
\mathbb{E}(A)_t = \int_0^t \mathbb{E}(X_s^2 + Y_s^2) ds = \int_0^t 2s ds = t^2 < \infty
\]

and thus, by a famous martingale criterion, the process \( A \) is a square integrable martingale.

Alternatively, since for all \( t \geq 0, \mathbb{E} \int_0^t X_s^2 d\langle Y \rangle_s = \int_0^t \mathbb{E}(X_s^2) ds < \infty \) the process \( \int_0^t X_s dY_s \) and by symmetry the process \( \int_0^t Y_s dX_s \) are both square integrable martingales, and thus the process \( A \) is also a square integrable martingales as being the difference of two square integrable martingales.

2. For all \( \lambda \in \mathbb{R}, \ t \geq 0, \mathbb{E}(e^{i \lambda A}_t) = \mathbb{E}(\cos(\lambda A_t)) + i \mathbb{E}(\sin(\lambda A_t)). \) Since \( \langle X, Y \rangle \overset{d}{=} \langle Y, X \rangle \), we get, for all \( t \geq 0, \)

\[
-A_t = \int_0^t Y_s dX_s - \int_0^t X_s dY_s - \int_0^t X_s dY_s = \int_0^t Y_s dX_s = A_t,
\]

and thus the characteristic function or Fourier transform of \( A_t \) is real.

3. The canonical decompositions are given by the Itô formula. Namely, for \( Z \),

\[
Z_t = 1 - \lambda \int_0^t \sin(\lambda A_s) dA_s - \frac{\lambda^2}{2} \int_0^t \cos(\lambda A_s) d\langle A \rangle_s
= 1 - \lambda \int_0^t \sin(\lambda A_s) dA_s - \frac{\lambda^2}{2} \int_0^t (X_s^2 + Y_s^2) Z_s ds.
\]

Similarly, for \( W \), by the Itô formula for the function \( g(x, y, t) = - \frac{E(t)}{2} (x^2 + y^2) + f(t) \) and the vector of semi-martingale \( S_t = (X_t, Y_t, t) \) with the martingale part \( (X_t, Y_t) \),

\[
W_t = g(0, 0, 0) + \int_0^t \partial_1 g(S_s) dX_s + \int_0^t \partial_2 g(S_s) dY_s + \int_0^t \partial_3 g(S_s) ds + \frac{1}{2} \int_0^t (\partial_1^2 g + \partial_2^2 g)(S_s) ds
= f(0) - \int_0^t f'(s) X_s dX_s - \int_0^t f'(s) Y_s dY_s + \int_0^t \left( - \frac{E''(s)}{2} (X_s^2 + Y_s^2) + f''(s) \right) ds - \int_0^t f''(s) ds
= f(0) - \int_0^t f'(s) X_s dX_s - \int_0^t f'(s) Y_s dY_s - \frac{1}{2} \int_0^t f''(s)(X_s^2 + Y_s^2) ds.
\]

The computation of \( \langle Z, W \rangle \) involves only the local martingale parts, namely

\[
\langle Z, W \rangle_t = \lambda \left( \int_0^t \sin(\lambda A_s) dA_s, \int_0^t f'(s) X_s dX_s + \int_0^t f'(s) Y_s dY_s \right)_t
= \lambda \int_0^t f'(s) \sin(\lambda A_s) X_s d\langle A, X \rangle_s + \lambda \int_0^t f'(s) \sin(\lambda A_s) Y_s d\langle A, Y \rangle_s.
\]

Now since \( \langle A, X \rangle_t = - \int_0^t Y_s dA_s \) and \( \langle A, Y \rangle_t = \int_0^t X_s dA_s \), we get

\[
\langle Z, W \rangle_t = \lambda \int_0^t f'(s)(-X_s Y_s + X_s Y_s) \sin(\lambda A_s) ds = 0.
\]

4. The Itô formula gives (we benefit from the fact that \( \langle Z, W \rangle = 0 \) from the previous question)

\[
Z_t e^W = e^{f(0)} + \int_0^t e^W dZ_s + \int_0^t Z_s e^W dW_s + \frac{1}{2} \int_0^t Z_s e^W d\langle W \rangle_s.
\]
By collecting the finite variation parts from $dZ$ and $dW$ from a previous question we get
\[-\frac{\lambda^2}{2} \int_0^t (X^2 + Y^2) Z dW ds - \frac{1}{2} \int_0^t f''(s)(X^2 + Y^2) Z dW ds + \frac{1}{2} \int_0^t Z dW d(W)_s.\]

Now from a previous question
\[\langle W \rangle_t = \left( \int_0^t f'(s) X dX_s + \int_0^t f'(s) Y dY_s \right)_t = \int_0^t f'^2(s)(X^2 + Y^2) ds.\]

It follows that the finite variation part of $Ze^W$ vanishes when $f'' = f'^2 - \lambda^2$.

5. With $f(t) = -\log\cosh(\lambda(r-t))$, we have
\[f'(t) = \frac{\lambda \sinh(\lambda(r-t))}{\cosh(\lambda(r-t))} = \lambda \tanh(\lambda(r-t))\]
and
\[f''(t) = -\frac{\lambda^2}{\cosh(\lambda(r-t))^2} = -\lambda^2(1 - \tanh(\lambda(r-t))^2) = -\lambda^2 + f'^2(t).\]

It follows from the previous question that $Ze^W$ is a continuous local martingale. Note that $f'(r) = f''(r) = 0$ and $W_t = 0$, and by using previous questions,
\[\mathbb{E}e^{\lambda A_t} = \mathbb{E}\cos(\lambda A_t) = \mathbb{E}Z_t = \mathbb{E}(Ze^{W_t}).\]

On the other hand, since $f(0) = -\log\cosh(\lambda r)$, $Z_0 = 1$, $W_0 = f(0)$, we get
\[\mathbb{E}(Z_0 e^{W_t}) = e^{f(0)} = \frac{1}{\cosh(\lambda r)}.\]

It remains to show that the local martingale $Ze^W$ is a martingale on the time interval $[0, r]$. From the previous question, since $f$, $\cos$, and $\sin$ are bounded, it suffices to show that
\[\mathbb{E} \int_0^t e^{2W_s} d(A)_s < \infty \quad \text{and} \quad \mathbb{E} \int_0^t e^{2W_s}(X^2 + Y^2) ds < \infty.\]

But the first condition follows from the second thanks to the formula for $\langle A \rangle$ provided by a previous question. On the other hand, if $t \in [0, r]$ then $f'(t) \geq 0$ and thus $W_s \leq f(t)$ for all $s \in [0, t]$, which implies that the second condition is satisfied by using $\mathbb{E}(X^2 + Y^2) = 2s$.

**Exercise 3** (Criterion for a stochastic differential equation). Set $d = 1$. Let $\sigma, b$ be two functions $\mathbb{R} \to \mathbb{R}$ such that for some finite constant $C < \infty$ and for all $x, y \in \mathbb{R},$
\[|\sigma(x) - \sigma(y)| \leq C|x - y| \quad \text{and} \quad |b(x) - b(y)| \leq C|x - y|\]

The goal of this exercise is to prove pathwise uniqueness for the stochastic differential equation
\[dX_t = \sigma(X_t) dB_t + b(X_t) dt.\]

A solution $X$ is a continuous semi-martingale with canonical decomposition $X = X_0 + M + V$ with $X_0 \in L^2$, local martingale part $M = \int_0^\tau \sigma(X_s) dB_s$, and finite variation part $V = \int_0^\tau b(X_s) ds$. Note that the continuity of $\sigma, X, b$ gives that almost surely, for all $t \geq 0, s \mapsto \sigma(X_s) + b(X_s)$ is locally bounded.

1. Let $Z$ be a continuous semi-martingale such that $\langle Z \rangle = \int_0^\tau \varphi ds$ for a progressive process $\varphi$ such that $0 \leq \varphi \leq C|Z|$ for some constant $C < \infty$. Prove that for all $t \geq 0$ and all $a > 0$,
\[\mathbb{E} \int_0^t 1_{0 < |Z_s| \leq a} d\langle Z \rangle_s \leq Ct.\]
2. Deduce from the preceding question that for all $t \geq 0$,

$$\lim_{n \to \infty} n \mathbb{E} \int_0^t 1_{|Z_s| \leq \frac{1}{n}} d\langle Z \rangle_s = 0.$$ 

3. For all $n \geq 1$, $x \in \mathbb{R}$, let us define $g_n(x) = 2n(1 + nx)1_{x \in [-\frac{1}{n}, 0]} + 2n1_{x = 0} + 2n(1 - nx)1_{x \in (0, \frac{1}{n})}$.

Let $f_n : \mathbb{R} \to \mathbb{R}$ be the twice differentiable function such that $f''_n = g_n$ and $f_n(0) = f''_n(0) = 0$.

Show that for all $x \in \mathbb{R}$, the following properties hold true:

(a) $f'_n(x) \in [-1, 1]$ and $\lim_{n \to \infty} f'_n(x) = \text{sign}(x) = 1_{x > 0} - 1_{x < 0}$

(b) $|f_n(x)| \leq |x|$ and $\lim_{n \to \infty} f_n(x) = |x|$.

4. By using Itô formula, prove that for all continuous semi-martingale $Z = (Z_t)_{t \geq 0}$, all $t \geq 0$,

$$\int_0^t 1_{Z_s = 0} d\langle Z \rangle_s = 0.$$ 

5. From now on, let $X$ and $X'$ be two solutions of (SDE) on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ and with respect to the Brownian motion $B$. Show that for all $t \geq 0$,

$$\langle X - X' \rangle_t = \int_0^t (\sigma(X_s) - \sigma(X'_s))^2 ds.$$ 

6. By using the assumption on $\sigma$, deduce from the preceding questions that for all $t \geq 0$,

$$\lim_{n \to \infty} \mathbb{E} \int_0^t g_n(X_s - X'_s) d\langle X - X' \rangle_s = 0.$$ 

7. Set $Z = X - X'$. From now on, let $T$ be a stopping time such that the semi-martingale $(Z_{t \wedge T})_{t \geq 0}$ is bounded. By using notably the assumption on $\sigma$, prove that for all $t \geq 0$, $n \geq 1$,

$$\mathbb{E}(f_n(Z_{t \wedge T})) = \mathbb{E}(f_n(Z_0)) + \mathbb{E} \int_0^{t \wedge T} f''_n(Z_s)(b(X_s) - b(X'_s))ds + \frac{1}{2} \mathbb{E} \int_0^{t \wedge T} f''_n(Z_s)d\langle Z \rangle_s.$$ 

8. Deduce from the preceding questions and the assumption on $b$ that for all $t \geq 0$,

$$\mathbb{E}(|X_{t \wedge T} - X'_{t \wedge T}|) = \mathbb{E}(|X_0 - X'_0|) + \mathbb{E} \int_0^{t \wedge T} (b(X_s) - b(X'_s))\text{sign}(X_s - X'_s)ds.$$ 

9. By using the Grönwall lemma, deduce that if $X_0 = X'_0$ then $X_t = X'_t$ for all $t \geq 0$.

**Elements of solution for Exercise 3.** The result is known as the Yamada–Watanabe criterion. This is a slightly more detailed version of [1, Exercise 8.14 pages 231–232].

1. We have, using the properties of $Z$ and $\varphi$,

$$\int_0^t 1_{0 < |Z_s| \leq \frac{a}{n}} d\langle Z \rangle_s = \int_0^t \frac{1_{0 < |Z_s| \leq \frac{a}{n}}}{|Z_s|} \varphi_s ds \leq \int_0^t C ds = Ct.$$ 

2. For all $n \geq 1$, we have $n1_{0 < |Z_s| \leq \frac{1}{n}} \leq \frac{1_{0 < |Z_s| \leq 1}}{|Z_s|} \leq \frac{1_{0 < |Z_s| \leq 1}}{|Z_s|}$, which is integrable on $[0, t]$ by the preceding question used with $a = 1$, and thus the desired result follows then by dominated convergence.

3. The function $g_n$ is $0$ on $(-\infty, -\frac{1}{n})$, then increases from $0$ to $2$ on $[-\frac{1}{n}, 0]$, then decreases from $2$ to $0$ on $[0, \frac{1}{n}]$, then stays at $0$ on $(\frac{1}{n}, +\infty)$. Since $\int_{-\infty}^{0} g_n(y) dy = 1$, we have, for all $x \in \mathbb{R}$,

$$f'_n(x) = \int_0^x g_n(u)du, \quad \text{in such a way that } f'_n(0) = 0 \text{ and } f''_n = g_n.$$ 

5/7
The function $f_n'$ is $-1$ on $(-\infty, -\frac{1}{n}]$, $0$ at $0$, and $1$ on $[\frac{1}{n}, +\infty)$. Also for all $x \in \mathbb{R}$,

$$\lim_{n \to \infty} f_n'(x) = 1_{x > 0} - 1_{x < 0} =: \text{sign}(x).$$

Next, for all $x \in \mathbb{R}$, we have

$$f_n(x) = \int_0^x f_n'(u)du \quad \text{in such a way that } f_n(0) = 0 \text{ and } f_n'' = g_n.$$ 

Since $g_n \geq 0$, we have that $f_n'$ is non-decreasing, and thus $f_n'$ takes actually its values in $[-1, 1]$, and is in particular bounded. It follows by dominated convergence that for all $x \in \mathbb{R}$,

$$\lim_{n \to \infty} f_n(x) = \int_0^x \lim_{n \to \infty} f_n'(u)du = \int_0^x \text{sign}(u)du = |x|.$$

Finally, for all $x \in \mathbb{R}$, $|f_n(x)| \leq f_0 |x|du = |x|$.

4. The Itô formula for function $f_n$ of question 3 and semi-martingale $Z$ gives, for all $t \geq 0$,

$$f_n(Z_t) = f_n(Z_0) + \int_0^t f_n'(Z_s)dZ_s + \frac{1}{2} \int_0^t f_n''(Z_s)d(Z)_s.$$ 

Since $|f_n'| \leq 1$ and $\lim_{n \to \infty} \frac{f_n'(x)}{2n} = 1_{x=0}$ for all $x \in \mathbb{R}$, by dominated convergence,

$$\lim_{n \to \infty} \frac{1}{2n} \int_0^t f_n''(Z_s)d(Z)_s = \int_0^t 1_{Z_s=0}d(Z)_s \quad \text{a.s.}$$

On the other hand, since by question 3, $\lim_{n \to \infty} \frac{f_n(x)}{2n} = 0$ for all $x \in \mathbb{R}$, it follows that a.s.

$$\lim_{n \to \infty} \frac{f_n(Z_t)}{2n} = 0 \quad \text{and} \quad \lim_{n \to \infty} \frac{f_n(Z_0)}{2n} = 0.$$ 

Finally since by question 3, $\frac{1}{2n} |f_n'\leq 1$ and $\lim_{n \to \infty} \frac{1}{2n} |f_n'(x)| = 0$ for all $x \in \mathbb{R}$, dominated convergence for stochastic integrals gives

$$\frac{1}{2n} \int_0^t f_n'(Z_s)dZ_s \xrightarrow{n \to \infty} 0.$$ 

5. Since $X$ and $X'$ are both solutions on the same space and for the same Brownian motion, we have, for all $X_0$ and $X'_0$ and for all $t \geq 0$,

$$X_t - X'_t = X_0 - X'_0 + \int_0^t (\sigma(X_s) - \sigma(X'_s))dB_s + \int_0^t (b(X_s) - b(X'_s))ds.$$ 

The right hand side gives the canonical decomposition of the semi-martingale $X - X'$. In this decomposition, the first integral is the local martingale part, and

$$\langle X - X' \rangle = \int_0^t (\sigma(X_s) - \sigma(X'_s))^2ds.$$ 

6. By assumption on $\sigma$, the process $\nu = (\sigma(X) - \sigma(X'))^2$ satisfies $0 \leq \nu \leq C|X - X'|$. We can then use question 5 and question 2 with $Z = X - X'$ to get, for all $t \geq 0$,

$$\lim_{n \to \infty} n\mathbb{E} \int_0^t 1_{\nu < |Z_s| \leq \frac{1}{n}}d(Z)_s = 0.$$ 

If $g_n$ is as in question 3, then for all $x \in \mathbb{R}$, $0 \leq g_n(x) \leq 2n1_{0 < |x| \leq \frac{1}{n}} + 2n1_{x=0}$. Thus, for all $t \geq 0$,

$$0 \leq \mathbb{E} \int_0^t g_n(Z_s)d(Z)_s \leq 2n\mathbb{E} \int_0^t 1_{0 < |Z_s| \leq \frac{1}{n}}d(Z)_s + 2n\mathbb{E} \int_0^t 1_{Z_s=0}d(Z)_s \xrightarrow{n \to \infty} 0,$$ 

where we have used question 2 and question 4.
7. The Itô formula for the $\mathcal{C}^2$ function $f_n$ and the continuous semi-martingale $Z^T$ gives

\[ f_n(Z^T_t) = f_n(Z^T_0) + \int_0^{t\wedge T} f_n'(Z_s) dZ_s + \frac{1}{2} \int_0^{t\wedge T} f_n''(Z_s) d\langle Z \rangle_s, \]

and since $dZ_s = (\sigma(X_s) - \sigma(X'_s)) dB_s + (b(X_s) - b(X'_s)) ds$, we get

\[ \int_0^{t\wedge T} f'(Z_s) dZ_s = \int_0^t f'(Z^T_s)(\sigma(X_s) - \sigma(X'_s)) 1_{s\leq T} dB_s + \int_0^{t\wedge T} f'(Z_s)(b(X_s) - b(X'_s)) ds. \]

Now, by the assumptions on $\sigma$ and $T$, we get

\[ |\sigma(X_s) - \sigma(X'_s)| 1_{s\leq T} \leq C \sqrt{|Z^T_s|} \leq C'. \]

This boundedness, together with the one of $f''_n$, imply that the first integral in the right hand side above (the $dB_s$ one) is a martingale. Since this martingale is issued from the origin, its expectation vanishes for all times. On the other hand, since $f_n$ is continuous and $Z^T$ is bounded, the random variables $f_n(Z^T_t)$ and $f_n(Z^T_0)$ are integrable. All in all, we obtain

\[ \mathbb{E}(f_n(Z^T_t)) = \mathbb{E}(f_n(Z^T_0)) + \mathbb{E} \int_0^{t\wedge T} f_n'(Z_s)(b(X_s) - b(X'_s)) ds + \frac{1}{2} \mathbb{E} \int_0^{t\wedge T} f_n''(Z_s) d\langle Z \rangle_s. \]

8. Since $f''_n = g_n \geq 0$, we get, by using question 6, that

\[ 0 \leq \mathbb{E} \int_0^{t\wedge T} f''_n(Z_s) d\langle Z \rangle_s \leq \mathbb{E} \int_0^t g_n(Z_s) d\langle Z \rangle_s \xrightarrow{n \to \infty} 0. \]

On the other hand, by the assumption on $b$ and the boundedness of $Z^T$, we have, on $\{s \leq T\}$,

\[ |b(X_s) - b(X'_s)|^2 \leq C^2 |X_s - X'_s|^2 = C^2 |Z^T_s| \leq C'. \]

But since $f'_n$ is bounded (takes its values in $[-1, 1]$), we get, by dominated convergence

\[ \lim_{n \to \infty} \int_0^{t\wedge T} f'_n(Z_s)(b(X_s) - b(X'_s)) ds = \int_0^{t\wedge T} \text{sign}(Z_s)(b(X_s) - b(X'_s)) ds. \]

Finally, since $Z^T_t$ is bounded, and since from question 3, for all $x \in \mathbb{R}$, $|f_n(x)| \leq |x|$ and $\lim_{n \to \infty} f_n(x) = |x|$, we get, by dominated convergence, $\lim_{n \to \infty} \mathbb{E}(f_n(Z^T_t)) = \mathbb{E}(|Z^T_t|)$. Finally

\[ \mathbb{E}(|X_{t\wedge T} - X'_{t\wedge T}|) = \mathbb{E}(|X_0 - X'_0|) + \mathbb{E} \int_0^{t\wedge T} (b(X_s) - b(X'_s)) \text{sign}(X_s - X'_s) ds. \]

9. From the preceding question, we get, by using the assumption on $b$,

\[ \alpha(t) = \mathbb{E}(|X_{t\wedge T} - X'_{t\wedge T}|) \leq \mathbb{E}(|X_0 - X'_0|) + C \mathbb{E} \int_0^t |X_{s\wedge T} - X'_{s\wedge T}| ds = \alpha(0) + C \int_0^t \alpha(s) ds. \]

By the Grönwall lemma, we obtain $\alpha(t) \leq \alpha(0) e^{Ct}$ for all $t \geq 0$. It follows that if $\alpha(0) = 0$ then $\alpha(t) = 0$ for all $t \geq 0$. This means that if $X_0 = X'_0$ then $X_{t\wedge T} = X'_{t\wedge T}$ for all $t \geq 0$. By writing this for $t \in \mathbb{Q}^+$, and by taking $T = T_m$ such that $\lim_{m \to \infty} T_m = +\infty$ almost surely, we get that $X_t = X'_t$ for all $t \in \mathbb{Q}^+$, and thus for all $t \geq 0$ since $X$ and $X'$ are continuous.

References