(Ω, ℋ, (ℱₜ)ₜ≥0, ℙ) is a filtered probability space, with complete and right continuous filtration.  
\( B = (Bₜ)_{t ≥ 0} \) is a \( d \)-dimensional Brownian motion issued from the origin, \( d ≥ 1 \).  
If \( Z \) is a semi-martingale, we denote by \( (Z) \) the increasing process of its local martingale part.  
If \( Z = Z₀ + M + V \), do not confuse \( (Z) = (M) \) with the finite variation part \( V \) of \( Z \).

**Exercise 1** (Nature of an integral).  
Set \( d = 1 \). Let us consider the following integral, for \( t ≥ 0 \),  
\[ Iₜ = \int_0^t Bₛ ds. \]

2. Show that \( d(tBₜ) = Bₜ dt + tdBₜ \).
3. Deduce from the preceding question that \( Iₜ = \int₀^t (t - s)dBₛ \) for all \( t ≥ 0 \).
4. Deduce from the preceding question that \( Iₜ \sim ℕ(0, \frac{1}{3} t³) \) for all \( t ≥ 0 \).
5. For all \( t ≥ 0 \), \( n ≥ 1 \), \( 0 ≤ k ≤ n \), let us define \( t_k = \frac{k}{n} t \). Show that  
\[ \sum_{k=0}^{n-1} B_{t_k}(t_{k+1} - t_k) = \frac{1}{n} \sum_{j=0}^{n-2} (n - j - 1)(B_{t_{j+1}} - Bₜ). \]
6. Deduce from the preceding question another proof that \( Iₜ \sim ℕ(0, \frac{1}{3} t³) \) for all \( t ≥ 0 \).
7. Is the process \((Iₜ)_{t ≥ 0}\) a martingale?

**Elements of solution for Exercise 1.**

1. Since the integrator is of finite variation and the integrand is bounded and measurable (actually continuous), it is a Lebesgue–Stieltjes integral, and in particular an Itô integral with respect to a semi-martingale without martingale part. However it is not a Wiener integral.
2. The Itô formula for \( f(x, y) = xy \) and \( Xₜ = (t, Bₜ) \) gives  
\[ tBₜ = 0 + \int₀^t Bₛ ds + \int₀^t s dBₛ, \]
3. From the preceding question (actually, it is an integration by parts)  
\[ \int₀^t Bₛ ds = tBₜ - \int₀^t s dBₛ = \int₀^t (t - s) dBₛ, \]
4. The integral in the right hand side is a Wiener integral. Thus it is Gaussian with mean zero and variance equal to the squared \( L² \) norm of the integrand:  
\[ \mathbb{E} \int₀^t Bₛ ds = 0 \quad \text{and} \quad \mathbb{E} \left[ \left( \int₀^t Bₛ ds \right)^2 \right] = \int₀^t (t - s)^² ds = \frac{t³}{3}. \]
5. With \( t_k = \frac{k}{n} t \) for all \( 0 ≤ k ≤ n \), we have  
\[ Sₙ = \sum_{k=0}^{n-1} B_{t_k}(t_{k+1} - t_k) = \frac{t}{n} \sum_{k=0}^{n-1} Bₜ = \frac{t}{n} \sum_{k=0}^{n-1} \sum_{j=0}^{k-1} (B_{t_{j+1}} - Bₜ) = \frac{t}{n} \sum_{j=0}^{n-2} (n - j - 1)(B_{t_{j+1}} - Bₜ). \]
6. Fix \( t \geq 0 \). Since \( I_t \) is a Lebesgue–Stieltjes integral with continuous integrand, we have \( \lim_{n \to \infty} S_n = I_t \) almost surely and thus in law. For all \( n \) and \( j \), since \( B_{t_{j+1}} - B_{t_j} \) are independent and Gaussian, we get that \( S_n \sim \mathcal{N}(E(S_n), E(S_n^2) - E(S_n)^2) \). But the convergence in law of Gaussians is equivalent to the convergence of the first two moments. Now it remains to note that we have \( E(S_n) = 0 \) and
\[
E(S_n^2) = \frac{t^2}{n^2} \sum_{j=0}^{n-2} (n - j - 1)^2 E((B_{t_{j+1}} - B_{t_j})^2) = \frac{t^4}{n^3} \sum_{j=1}^{n-1} j^2 = \frac{t^4}{n} \sum_{j=1}^{n-1} \left( \frac{j^2}{n} \right) = \frac{t^4}{n} \sum_{j=1}^{n-1} \left( \frac{j^2}{n} \to 1 \right) = \frac{1}{6} \int_0^1 x^2 \, dx = \frac{t^4}{3}.
\]

7. Beware that the integrand in \( f_0^1(t - s) \, dB_s \) depends on \( t \). The process \( (-f_0^1(s) \, dB_s)_{t \geq 0} \) is a martingale, however the process \( (\int_0^t t \, dB_s)_{t \geq 0} = \tau_B t_{t \geq 0} \) is not a martingale: for all \( 0 \leq s \leq t \),
\[
E((t + s) \, B_{t+s} + \mathcal{F}_t) = (t + s) B_s \neq s B_s.
\]

**Exercise 2** (Study of a special process). Set \( d = 2 \). For all \( t \geq 0 \), we write \( B_t = (X_t, Y_t) \) and
\[
A_t = \int_0^t X_s \, dY_s - \int_0^t Y_s \, dX_s.
\]

1. Show that \( (A) = \int_0^t (X_s^2 + Y_s^2) \, ds \) and that the process \( A \) is a square integrable martingale.

2. From now on let \( \lambda > 0 \). Show that for all \( t \geq 0 \),
\[
E e^{\lambda A_t} = E \cos(\lambda A_t).
\]

3. From now on, let \( f : \mathbb{R}_+ \to \mathbb{R} \) be \( \mathcal{C}^2 \), and let us define the continuous semi-martingales
\[
(Z_t)_{t \geq 0} = (\cos(\lambda A_t))_{t \geq 0} \quad \text{and} \quad (W_t)_{t \geq 0} = \left(-\frac{f'(t)}{2}(X_t^2 + Y_t^2) + f(t)\right)_{t \geq 0}.
\]
Show that for all \( t \geq 0 \),
\[
Z_t = 1 - \lambda \int_0^t \sin(\lambda A_s) \, dA_s - \frac{\lambda^2}{2} \int_0^t (X_s^2 + Y_s^2) \, Z_s \, ds.
\]
and
\[
W_t = f(0) - \int_0^t f'(s)X_s \, dY_s - \int_0^t f'(s)Y_s \, dX_s - \frac{\lambda}{2} \int_0^t f''(s)(X_s^2 + Y_s^2) \, ds,
\]
and deduce that
\[
(Z, W) = 0.
\]

4. Show that if \( f \) solves \( f'' = f'^2 - \lambda^2 \) then \( Ze^{W_t} \) is a continuous local martingale and
\[
Ze^{W_t} = e^{f(0)} - \lambda \int_0^t \sin(\lambda A_s) e^{W_s} \, dA_s - \int_0^t f'(s) Ze^{W_s} X_s \, dX_s - \int_0^t f'(s) Z_s e^{W_s} Y_s \, dY_s.
\]

5. Let \( r > 0 \). By using \( f(t) = -\log \cosh(\lambda(r - t)) \) deduce from the previous question that
\[
E e^{\lambda A_t} = \frac{1}{\cosh(\lambda r)}.
\]

**Elements of solution for Exercise 2.** For all \( t \geq 0 \), \( A_t \) is the algebraic area between planar Brownian motion and its chord, and the process \( A \) is the Lévy area. This exercise is a slightly more detailed version of [1, Exercise 5.30 pages 144–145]. Its goal is to compute the characteristic function or Fourier transform of \( A_t \).
1. Since \( \langle X \rangle_t = \langle Y \rangle_t = t \) and \( \langle X, Y \rangle_t = 0 \), we get
\[
\langle A \rangle_t = \langle A, A \rangle_t = \left( \int_0^t X_s dY_s \right)^2 + \left( \int_0^t Y_s dX_s \right)^2 + 2 \left( \int_0^t X_s dY_s \right) \left( \int_0^t Y_s dX_s \right)
\]
\[
= \int_0^t X_s^2 dY_s + \int_0^t Y_s^2 dX_s + \int_0^t X_s Y_s d\langle X, Y \rangle_s
\]
\[
= \int_0^t (X_s^2 + Y_s^2) ds.
\]
It follows by the Fubini–Tonelli theorem that
\[
\mathbb{E}(A)_t = \int_0^t \mathbb{E}(X^2_s + Y^2_s) ds = \int_0^t 2s ds = t^2 < \infty
\]
and thus, by a famous martingale criterion, the process \( A \) is a square integrable martingale.

Alternatively, since for all \( t \geq 0 \), \( \mathbb{E} \int_0^t X_s^2 dY_s = \int_0^t \mathbb{E}(X_s^2) ds < \infty \) the process \( \int_0^t X_s dY_s \) and by symmetry the process \( \int_0^t Y_s dX_s \) are both square integrable martingales, and thus the process \( A \) is also a square integrable martingale as being the difference of two square integrable martingales.

2. For all \( \lambda \in \mathbb{R}, \ t \geq 0, \mathbb{E}(e^{tA}) = \mathbb{E}(\cos(\lambda A_t)) + i \mathbb{E}(\sin(\lambda A_t)). \) Since \( \langle X, Y \rangle = (Y, X) \), we get, for all \( t \geq 0 \),
\[
-A_t = \int_0^t Y_s dX_s - \int_0^t X_s dY_s \overset{d}{=} \int_0^t X_s dY_s - \int_0^t Y_s dX_s = A_t,
\]
and thus the characteristic function or Fourier transform of \( A_t \) is real.

3. The canonical decompositions are given by the Itô formula. Namely, for \( Z \),
\[
Z_t = 1 - \lambda \int_0^t \sin(\lambda A_s) dA_s - \frac{\lambda^2}{2} \int_0^t \cos(\lambda A_s) d\langle A \rangle_s
\]
\[
= 1 - \lambda \int_0^t \sin(\lambda A_s) dA_s - \frac{\lambda^2}{2} \int_0^t (X_s^2 + Y_s^2) Z_s ds.
\]
Similarly, for \( W \), by the Itô formula for the function \( g(x, y, t) = -\frac{1}{2} (x^2 + y^2) + f(t) \) and the vector of semi-martingale \( S_t = (X_t, Y_t, t) \) with the martingale part \( (X_t, Y_t) \),
\[
W_t = g(0, 0, 0) + \int_0^t \partial_x g(S_s) dX_s + \int_0^t \partial_y g(S_s) dY_s + \int_0^t \partial_t g(S_s) ds + \frac{1}{2} \int_0^t (\partial^2_x g + \partial^2_y g)(S_s) ds
\]
\[
= f(0) - \int_0^t f'(s) X_s dX_s - \int_0^t f'(s) Y_s dY_s + \int_0^t \left( -\frac{f''(s)}{2} (X_s^2 + Y_s^2) + f'(s) \right) ds - \int_0^t f'(s) ds
\]
\[
= f(0) - \int_0^t f'(s) X_s dX_s - \int_0^t f'(s) Y_s dY_s - \frac{1}{2} \int_0^t f''(s) (X_s^2 + Y_s^2) ds.
\]
The computation of \( \langle Z, W \rangle \) involves only the local martingale parts, namely
\[
\langle Z, W \rangle_t = \lambda \left( \int_0^t \sin(\lambda A_s) dA_s, \int_0^t f'(s) X_s dX_s + \int_0^t f'(s) Y_s dY_s \right)_t
\]
\[
= \lambda \int_0^t f'(s) \sin(\lambda A_s) X_s d\langle A, X \rangle_s + \lambda \int_0^t f'(s) \sin(\lambda A_s) Y_s d\langle A, Y \rangle_s.
\]
Now since \( \langle X, A \rangle_t = -\int_0^t Y_s ds \) and \( \langle A, Y \rangle_t = \int_0^t X_s ds \), we get
\[
\langle Z, W \rangle_t = \lambda \int_0^t f'(s) (-X_s Y_s + X_s Y_s) \sin(\lambda A_s) ds = 0.
\]
4. The Itô formula gives (we benefit from the fact that \( \langle Z, W \rangle = 0 \) from the previous question)
\[
Z_t e^{W_t} = e^{W(0)} + \int_0^t e^{W_s} dZ_s + \int_0^t Z_s e^{W_s} dW_s + \frac{1}{2} \int_0^t Z_s e^{W_s} d\langle W \rangle_s.
\]
Exercise 3 (Criterion for a stochastic differential equation). Set $d = 1$. Let $\sigma, b$ be two functions $\mathbb{R} \rightarrow \mathbb{R}$ such that for some finite constant $C < \infty$ and for all $x, y \in \mathbb{R}$,

$$|\sigma(x) - \sigma(y)| \leq C|x - y| \quad \text{and} \quad |b(x) - b(y)| \leq C|x - y|$$

The goal of this exercise is to prove pathwise uniqueness for the stochastic differential equation

$$dX_t = \sigma(X_t)dB_t + b(X_t)dt. \quad \text{(SDE)}$$

A solution $X$ is a continuous semi-martingale with canonical decomposition $X = X_0 + M + V$ with $X_0 \in L^2$, local martingale part $M = \int_0^\cdot \sigma(X_s)dB_s$, and finite variation part $V = \int_0^\cdot b(X_s)ds$. Note that the continuity of $\sigma, X, b$ gives that almost surely, for all $t \geq 0$, $s \mapsto \sigma(X_s) + b(X_s)$ is locally bounded.

1. Let $Z$ be a continuous semi-martingale such that $\langle Z \rangle = \int_0^\cdot \varphi ds$ for a progressive process $\varphi$ such that $0 \leq \varphi \leq C|Z|$ for some constant $C < \infty$. Prove that for all $t \geq 0$ and all $a > 0$,

$$\mathbb{E} \int_0^t 1_{0<|Z_s|\leq a} \frac{d\langle Z \rangle_s}{|Z_s|} \leq Ct.$$
2. Deduce from the preceding question that for all \( t \geq 0 \),
\[
\lim_{n \to \infty} nE \int_0^t \mathbf{1}_{|Z_s| \leq \frac{1}{n}} d\langle Z \rangle_s = 0.
\]

3. For all \( n \geq 1 \), \( x \in \mathbb{R} \), let us define \( g_n(x) = 2n(1 + nx)\mathbf{1}_{x \in [-\frac{1}{n}, 0]} + 2n\mathbf{1}_{x = 0} + 2n(1 - nx)\mathbf{1}_{x \in (0, \frac{1}{n})} \).
Let \( f_n : \mathbb{R} \to \mathbb{R} \) be the twice differentiable function such that \( f_n'' = g_n \) and \( f_n(0) = f_n'(0) = 0 \).
Show that for all \( x \in \mathbb{R} \), the following properties hold true:

(a) \( f_n'(x) \in [-1, 1] \) and \( \lim_{n \to \infty} f_n'(x) = \text{sign}(x) = \mathbf{1}_{x > 0} - \mathbf{1}_{x < 0} \)
(b) \( |f_n(x)| \leq |x| \) and \( \lim_{n \to \infty} f_n(x) = |x| \).

4. By using Itô formula, prove that for all continuous semi-martingale \( Z = (Z_t)_{t \geq 0} \), all \( t \geq 0 \),
\[
\int_0^t \mathbf{1}_{Z_s = 0} d\langle Z \rangle_s = 0.
\]

5. From now on, let \( X \) and \( X' \) be two solutions of (SDE) on \( (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}) \) and with respect to the Brownian motion \( B \). Show that for all \( t \geq 0 \),
\[
\langle X - X' \rangle_t = \int_0^t (\sigma(X_s) - \sigma(X'_s))^2 ds.
\]

6. By using the assumption on \( \sigma \), deduce from the preceding questions that for all \( t \geq 0 \),
\[
\lim_{n \to \infty} E \int_0^t g_n(X_s - X'_s) d\langle X - X' \rangle_s = 0.
\]

7. Set \( Z = X - X' \). From now on, let \( T \) be a stopping time such that the semi-martingale \( (Z_{t \wedge T})_{t \geq 0} \) is bounded. By using notably the assumption on \( \sigma \), prove that for all \( t \geq 0 \), \( n \geq 1 \),
\[
E(f_n(Z_{t \wedge T})) = E(f_n(Z_0)) + E \int_0^{t \wedge T} f'_n(Z_s)(b(X_s) - b(X'_s))ds + \frac{1}{2} E \int_0^{t \wedge T} f''_n(Z_s)d\langle Z \rangle_s.
\]

8. Deduce from the preceding questions and the assumption on \( b \) that for all \( t \geq 0 \),
\[
E(|X_{t \wedge T} - X'_{t \wedge T}|) = E(|X_0 - X'_0|) + E \int_0^{t \wedge T} (b(X_s) - b(X'_s))\text{sign}(X_s - X'_s)ds.
\]

9. By using the Grönwall lemma, deduce that if \( X_0 = X'_0 \) then \( X_t = X'_t \) for all \( t \geq 0 \).

**Elements of solution for Exercise 3.** The result is known as the Yamada–Watanabe criterion. This is a slightly more detailed version of [1, Exercise 8.14 pages 231–232].

1. We have, using the properties of \( Z \) and \( \varphi \),
\[
\int_0^t \frac{\mathbf{1}_{0 \leq Z_s \leq a}}{|Z_s|} d\langle Z \rangle_s = \int_0^t \frac{\mathbf{1}_{0 < |Z_s| \leq a}}{|Z_s|} \varphi_s ds \leq \int_0^t Cds = Ct.
\]

2. For all \( n \geq 1 \), we have \( n\mathbf{1}_{0 \leq |Z_s| \leq \frac{1}{n}} \leq \mathbf{1}_{0 < |Z_s| \leq 1} \leq \mathbf{1}_{0 < |Z_s| \leq \frac{1}{n}} \), which is integrable on \([0, t]\) by the preceding question used with \( a = 1 \), and thus the desired result follows then by dominated convergence.

3. The function \( g_n \) is \( 0 \) on \((-\infty, -\frac{1}{n})\), then increases from 0 to 2 on \([-\frac{1}{n}, 0]\), then decreases from 2 to 0 on \([0, \frac{1}{n}]\), then stays at 0 on \([\frac{1}{n}, +\infty)\). Since \( \int_{-\infty}^{0} g_n(y)dy = 1 \), we have, for all \( x \in \mathbb{R} \),
\[
f_n'(x) = \int_{0}^{x} g_n(u)du, \quad \text{in such a way that } f_n'(0) = 0 \text{ and } f_n'' = g_n.
\]
The function \( f'_n \) is \(-1\) on \((-\infty, -\frac{1}{n}]\), \(0\) at \(0\), and \(1\) on \([\frac{1}{n}, +\infty)\). Also for all \(x \in \mathbb{R}\),

\[
\lim_{n \to \infty} f'_n(x) = 1_{x > 0} - 1_{x < 0} =: \text{sign}(x).
\]

Next, for all \(x \in \mathbb{R}\), we have

\[
f_n(x) = \int_0^x f'_n(u) \,du \quad \text{in such a way that } f_n(0) = 0 \text{ and } f''_n = g_n.
\]

Since \(g_n \geq 0\), we have that \(f'_n\) is non-decreasing, and thus \(f'_n\) takes actually its values in \([-1, 1]\), and is in particular bounded. It follows by dominated convergence that for all \(x \in \mathbb{R}\),

\[
\lim_{n \to \infty} f_n(x) = \int_0^x \lim_{n \to \infty} f'_n(u) \,du = \int_0^x \text{sign}(u) \,du = |x|.
\]

Finally, for all \(x \in \mathbb{R}\), \(|f_n(x)| \leq f_0|x| \,du = |x|\).

4. The Itô formula for function \(f_n\) of question 3 and semi-martingale \(Z\) gives, for all \(t \geq 0\),

\[
f_n(Z_t) = f_n(Z_0) + \int_0^t f'_n(Z_s) \,dZ_s + \frac{1}{2} \int_0^t f''_n(Z_s) \,d\langle Z \rangle_s.
\]

Since \(|f''_n(x)| \leq 1\) and \(\lim_{n \to \infty} f''_n(x) = 1_{x=0}\) for all \(x \in \mathbb{R}\), by dominated convergence,

\[
\lim_{n \to \infty} \frac{1}{2n} \int_0^t f''_n(Z_s) \,d\langle Z \rangle_s = \int_0^t 1_{Z_s=0} \,d\langle Z \rangle_s \quad \text{a.s.}
\]

On the other hand, since by question 3, \(\lim_{n \to \infty} f_n(Z_t) = f(Z_t)\), by dominated convergence, \(\lim_{n \to \infty} f_n(Z_t) = 0\) for all \(x \in \mathbb{R}\), it follows that a.s.

\[
\lim_{n \to \infty} \frac{f_n(Z_t)}{2n} = 0 \quad \text{and} \quad \lim_{n \to \infty} \frac{f_n(Z_0)}{2n} = 0.
\]

Finally since by question 3, \(\frac{1}{2n} |f_n(x)| \leq \frac{1}{2n} \leq 1\) and \(\lim_{n \to \infty} \frac{1}{2n} |f_n(x)| = 0\) for all \(x \in \mathbb{R}\), dominated convergence for stochastic integrals gives

\[
\frac{1}{2n} \int_0^t f''_n(Z_s) \,dZ_s \xrightarrow{\text{P}} 0.
\]

5. Since \(X\) and \(X'\) are both solutions on the same space and for the same Brownian motion, we have, for all \(X_0\) and \(X'_0\) and for all \(t \geq 0\),

\[
X_t - X'_t = X_0 - X'_0 + \int_0^t (\sigma(X_s) - \sigma(X'_s)) \,dB_s + \int_0^t (b(X_s) - b(X'_s)) \,ds.
\]

The right hand side gives the canonical decomposition of the semi-martingale \(X - X'\). In this decomposition, the first integral is the local martingale part, and

\[
\langle X - X' \rangle = \int_0^t (\sigma(X_s) - \sigma(X'_s))^2 \,ds.
\]

6. By assumption on \(\sigma\), the process \(\varphi = (\sigma(X) - \sigma(X'))^2\) satisfies \(0 \leq \varphi \leq C|X - X'|\). We can then use question 5 and question 2 with \(Z = X - X'\) to get, for all \(t \geq 0\),

\[
\lim_{n \to \infty} nE\int_0^t 1_{0 < |Z_s| \leq \frac{1}{n}} \,d\langle Z \rangle_s = 0.
\]

If \(g_n\) is as in question 3, then for all \(x \in \mathbb{R}\), \(0 \leq g_n(x) \leq 2n 1_{0 < |x| \leq \frac{1}{n}} + 2n 1_{x=0}\). Thus, for all \(t \geq 0\),

\[
0 \leq E\int_0^t g_n(Z_s) \,d\langle Z \rangle_s \leq 2nE\int_0^t 1_{0 < |Z_s| \leq \frac{1}{n}} \,d\langle Z \rangle_s + 2nE\int_0^t 1_{Z_s=0} \,d\langle Z \rangle_s \xrightarrow{n \to \infty} 0,
\]

where we have used question 2 and question 4.
7. The Itô formula for the \( e^T \) function \( f_n \) and the continuous semi-martingale \( Z^T \) gives
\[
f_n(Z^T_t) = f_n(Z^T_0) + \int_0^t f''(Z^T_s) \, dB_s + \frac{1}{2} \int_0^t f'''(Z^T_s) \, d\langle B \rangle_s,
\]
and since \( dB_s = (\sigma(X_s) - \sigma(X'_s))dB_s + (b(X_s) - b(X'_s)) \, ds \), we get
\[
\int_0^t f'((Z^T_s)) \, dB_s = \int_0^t f'((Z^T_s)) (\sigma(X_s) - \sigma(X'_s)) \, dB_s + \int_0^t f'((Z^T_s)) (b(X_s) - b(X'_s)) \, ds.
\]

Now, by the assumptions on \( \sigma \) and \( T \), we get
\[
|\sigma(X_s) - \sigma(X'_s)| |1_{s \leq T}| \leq C \sqrt{|Z^T_s|} \leq C'.
\]

This boundedness, together with the one of \( f''_n \), imply that the first integral in the right hand side above (the \( dB_s \) one) is a martingale. Since this martingale is issued from the origin, its expectation vanishes for all times. On the other hand, since \( f_n \) is continuous and \( Z^T \) is bounded, the random variables \( f_n(Z^T_t) \) and \( f_n(Z^T_0) \) are integrable. All in all, we obtain
\[
E(f_n(Z^T_t)) = E(f_n(Z^T_0)) + \int_0^t f''(Z^T_s) (b(X_s) - b(X'_s)) \, ds + \frac{1}{2} \int_0^t f'''(Z^T_s) \, d\langle B \rangle_s.
\]

8. Since \( f''_n = g_n \geq 0 \), we get, by using question 6, that
\[
0 \leq E \int_0^T f''_n(Z^T_s) \, d\langle Z^T \rangle_s \leq \int_0^T g_n(Z^T_s) \, d\langle Z^T \rangle_s \xrightarrow{n \to \infty} 0.
\]

On the other hand, by the assumption on \( b \) and the boundedness of \( Z^T \), we have, on \( \{s \leq T\} \),
\[
|b(X_s) - b(X'_s)|^2 \leq C^2 |X_s - X'_s|^2 = C^2 |Z^T_s| \leq C'.
\]

But since \( f'_n \) is bounded (takes its values in \([-1, 1]\)), we get, by dominated convergence
\[
\lim_{n \to \infty} \int_0^T f'_n(Z^T_s) (b(X_s) - b(X'_s)) \, ds = \int_0^T \text{sign}(Z^T_s) (b(X_s) - b(X'_s)) \, ds.
\]

Finally, since \( Z^T_t \) is bounded, and since from question 3, for all \( x \in \mathbb{R}, \{|f_n(x)| \leq |x|\} \) and \( \lim_{n \to \infty} f_n(x) = |x| \), we get, by dominated convergence, \( \lim_{n \to \infty} E(f_n(Z^T_t)) = E(|Z^T_t|) \). Finally
\[
E(|X_{T \wedge T'} - X'_{T \wedge T'}|) = E(|X_0 - X'_0|) + \int_0^{T \wedge T'} (b(X_s) - b(X'_s)) \text{sign}(X_s - X'_s) \, ds.
\]

9. From the preceding question, we get, by using the assumption on \( b \),
\[
\alpha(t) = E(|X_T - X'_T|) \leq E(|X_0 - X'_0|) + CE \int_0^t |X_{s \wedge T} - X'_{s \wedge T}| \, ds = a(0) + C \int_0^t \alpha(s) \, ds.
\]

By the Grönwall lemma, we obtain \( \alpha(t) \leq a(0)e^{Ct} \) for all \( t \geq 0 \). It follows that if \( a(0) = 0 \) then \( \alpha(t) = 0 \) for all \( t \geq 0 \). This means that if \( X_0 = X'_0 \) then \( X_{t \wedge T} = X'_{t \wedge T} \) for all \( t \geq 0 \). By writing this for \( t \in \mathbb{Q}_+ \), and by taking \( T = T_m \) such that \( \lim_{m \to \infty} T_m = +\infty \) almost surely, we get that \( X_t = X'_t \) for all \( t \in \mathbb{Q}_+ \), and thus for all \( t \geq 0 \) since \( X \) and \( X' \) are continuous.

References