Exercise 1 (Nature of an integral). Set \( d = 1 \). Let us consider the following integral, for \( t \geq 0 \),
\[
I_t = \int_0^t B_s \, ds.
\]


2. Show that \( d(tB_t) = B_t \, dt + t \, dB_t \);

3. Deduce from the preceding question that \( I_t = \int_0^t (t-s) \, dB_s \) for all \( t \geq 0 \);

4. Deduce from the preceding question that \( I_t \sim \mathcal{N}(0, \frac{1}{3} t^3) \) for all \( t \geq 0 \);

5. For all \( t \geq 0, n \geq 1, 0 \leq k \leq n \), let us define \( t_k = \frac{k}{n} t \). Show that
\[
\sum_{k=0}^{n-1} B_{t_k}(t_{k+1} - t_k) = \frac{t}{n} \sum_{j=0}^{n-2} (n-j-1)(B_{t_{j+1}} - B_{t_j}).
\]

6. Deduce from the preceding question another proof that \( I_t \sim \mathcal{N}(0, \frac{1}{3} t^3) \) for all \( t \geq 0 \);

7. Is the process \((I_t)_{t \geq 0}\) a martingale?

Elements of solution for Exercise 1.

1. Since the integrator is of finite variation and the integrand is bounded and measurable (actually continuous), it is a Lebesgue–Stieltjes integral, and in particular an Itô integral with respect to a semi-martingale without martingale part. However it is not a Wiener integral.

2. The Itô formula for \( f(x, y) = xy \) and \( X_t = (t, B_t) \) gives
\[
tB_t = 0 + \int_0^t B_s \, ds + \int_0^t s \, dB_s,
\]

3. From the preceding question (actually, it is an integration by parts)
\[
\int_0^t B_s \, ds = tB_t - \int_0^t s \, dB_s = \int_0^t (t-s) \, dB_s.
\]

4. The integral in the right hand side is a Wiener integral. Thus it is Gaussian with mean zero and variance equal to the squared \( L^2 \) norm of the integrand:
\[
\mathbb{E} \left( \int_0^t B_s \, ds \right) = 0 \quad \text{and} \quad \mathbb{E} \left\{ \left( \int_0^t B_s \, ds \right)^2 \right\} = \int_0^t (t-s)^2 \, ds = \frac{t^3}{3}.
\]

5. With \( t_k = \frac{k}{n} t \) for all \( 0 \leq k \leq n \), we have
\[
S_n = \sum_{k=0}^{n-1} B_{t_k}(t_{k+1} - t_k) = \frac{t}{n} \sum_{k=0}^{n-1} B_{t_k} = \frac{t}{n} \sum_{k=0}^{n-2} \sum_{j=0}^{k-1} (B_{t_{j+1}} - B_{t_j}) = \frac{t}{n} \sum_{j=0}^{n-2} (n-j-1)(B_{t_{j+1}} - B_{t_j}).
\]
6. Fix \( t \geq 0 \). Since \( I_t \) is a Lebesgue–Stieltjes integral with continuous integrand, we have \( \lim_{n \to -\infty} S_n = I_t \) almost surely and thus in law. For all \( n \) and \( j \), since \( B_{t+j} - B_t \) are independent and Gaussian, we get that \( S_n \sim \mathcal{N}(\mathbb{E}(S_n), \mathbb{E}(S_n^2) - \mathbb{E}(S_n)^2) \). But the convergence in law of Gaussians is equivalent to the convergence of the first two moments. Now it remains to note that we have \( \mathbb{E}(S_n) = 0 \) and

\[
\mathbb{E}(S_n^2) = \sum_{j=0}^{n-2} (n-j-1)^2 \mathbb{E}((B_{t+j} - B_t)^2) = \sum_{j=0}^{n-1} j^2 = \sum_{j=1}^{n-1} \frac{j^2}{2} = \frac{n^3}{3} \mathbb{E}(\frac{1}{n}) \to \frac{1}{3} \int_0^1 x^2 \, dx = \frac{1}{3}.
\]

7. Beware that the integrand in \( f_0(t-s) \, dB_s \) depends on \( t \). The process \( (- f_0(t) \, dB_t)_{t \geq 0} \) is a martingale, however the process \( (f_0(t) \, dB_t)_{t \geq 0} = (t B_t)_{t \geq 0} \) is not a martingale: for all \( 0 \leq s \leq t \),

\[
\mathbb{E}((t+s) B_{t+s} \mid \mathcal{F}_t) = (t+s) B_s \neq s B_s.
\]

**Exercise 2** (Study of a special process). Set \( d = 2 \). For all \( t \geq 0 \), we write \( B_t = (X_t, Y_t) \) and

\[
A_t = \int_0^t X_s \, dY_s - \int_0^t Y_s \, dX_s.
\]

1. Show that \( \langle A \rangle = \int_0^t (X_s^2 + Y_s^2) \, ds \) and that the process \( A \) is a square integrable martingale;

2. From now on let \( \lambda > 0 \). Show that for all \( t \geq 0 \),

\[
\mathbb{E} e^{\lambda A_t} = \mathbb{E} \cos(\lambda A_t).
\]

3. From now on, let \( f : \mathbb{R}+ \to \mathbb{R} \) be \( \mathcal{C}^2 \), and let us define the continuous semi-martingales

\[
(Z_t)_{t \geq 0} = (\cos(\lambda A_t))_{t \geq 0} \quad \text{and} \quad (W_t)_{t \geq 0} = \left( - \frac{f'(t)}{2} (X_t^2 + Y_t^2) + f(t) \right)_{t \geq 0}.
\]

Show that for all \( t \geq 0 \),

\[
Z_t = 1 - \lambda \int_0^t \sin(\lambda A_s) dA_s - \frac{\lambda^2}{2} \int_0^t (X_s^2 + Y_s^2) Z_s \, ds.
\]

and

\[
W_t = f(0) - \int_0^t f'(s) X_s \, dY_s - \int_0^t f'(s) Y_s \, dX_s - \frac{1}{2} \int_0^t f''(s) (X_s^2 + Y_s^2) \, ds,
\]

and deduce that

\[
\langle Z, W \rangle = 0.
\]

4. Show that if \( f \) solves \( f'' = f'^2 - \lambda^2 \) then \( Z e^{W} \) is a continuous local martingale and

\[
Z e^{W_t} = e^{f(0)} - \lambda \int_0^t \sin(\lambda A_s) e^{W_s} \, dA_s - \int_0^t f'(s) Z_s e^{W_s} X_s \, dX_s - \int_0^t f'(s) Z_s e^{W_s} Y_s \, dY_s.
\]

5. Let \( r > 0 \). By using \( f(t) = - \log \cosh(\lambda (r-t)) \) deduce from the previous question that

\[
\mathbb{E} e^{\lambda A_t} = \frac{1}{\cosh(\lambda r)}.
\]

**Elements of solution for Exercise 2.** For all \( t \geq 0 \), \( A_t \) is the algebraic area between planar Brownian motion and its chord, and the process \( A \) is the Lévy area. This exercise is a slightly more detailed version of [1, Exercise 5.30 pages 144–145]. Its goal is to compute the characteristic function or Fourier transform of \( A_t \).
1. Since \( \langle X \rangle_t = \langle Y \rangle_t = t \) and \( \langle X, Y \rangle_t = 0 \), we get

\[
\langle A \rangle_t = \langle A, A \rangle_t = \left\langle \int_0^t X_s \, dY_s \right\rangle_t + \left\langle \int_0^t Y_s \, dX_s \right\rangle_t + 2 \left\langle \int_0^t X_s \, dY_s, \int_0^t Y_s \, dX_s \right\rangle_t
\]

\[= \int_0^t X_s^2 \, dY_s + \int_0^t Y_s^2 \, dX_s + \int_0^t X_s \, dY_s \int_0^t Y_s \, dX_s
\]

\[= \int_0^t (X_s^2 + Y_s^2) \, ds.
\]

It follows by the Fubini–Tonelli theorem that

\[
\mathbb{E}(A)_t = \int_0^t \mathbb{E}(X_s^2 + Y_s^2) \, ds = \int_0^t 2sd = t^2 < \infty
\]

and thus, by a famous martingale criterion, the process \( A \) is a square integrable martingale.

Alternatively, since for all \( t \geq 0 \), \( \mathbb{E} \int_0^t X_s^2 \, dY_s = \mathbb{E} \int_0^t Y_s^2 \, dX_s < \infty \) the process \( \int_0^t X_s \, dY_s \) and by symmetry the process \( \int_0^t Y_s \, dX_s \) are both square integrable martingales, and thus the process \( A \) is also a square integrable martingale as being the difference of two square integrable martingales.

2. For all \( \lambda \in \mathbb{R}, t \geq 0 \), \( \mathbb{E}(e^{i\lambda A_t}) = \mathbb{E}(\cos(\lambda A_t)) + i \mathbb{E}(\sin(\lambda A_t)) \). Since \( \langle X, Y \rangle \equiv \langle Y, X \rangle \), we get, for all \( t \geq 0 \),

\[
-A_t = \int_0^t Y_s \, dX_s - \int_0^t X_s \, dY_s = \int_0^t X_s \, dY_s - \int_0^t Y_s \, dX_s = A_t,
\]

and thus the characteristic function or Fourier transform of \( A_t \) is real.

3. The canonical decompositions are given by the Itô formula. Namely, for \( Z \),

\[
Z_t = 1 - \lambda \int_0^t \sin(\lambda A_s) \, dA_s - \frac{1}{2} \int_0^t \cos(\lambda A_s) \, d\langle A \rangle_s
\]

\[= 1 - \lambda \int_0^t \sin(\lambda A_s) \, dA_s - \frac{1}{2} \int_0^t (X_s^2 + Y_s^2) \, Z_s \, ds.
\]

Similarly, for \( W \), by the Itô formula for the function \( g(x, y, t) = \frac{-E(t)}{2}(x^2 + y^2) + f(t) \) and the vector of semi-martingale \( S_t = (X_t, Y_t) \) with martingale part \( (X_t, Y_t) \),

\[
W_t = g(0, 0, 0) + \int_0^t \partial_1 g(S_s) \, dX_s + \int_0^t \partial_2 g(S_s) \, dY_s + \int_0^t \partial_3 g(S_s) \, ds + \frac{1}{2} \int_0^t (\partial^2_{11} g + \partial^2_{12} g) (S_s) \, ds
\]

\[= f(0) - \int_0^t f'(s) X_s \, dX_s - \int_0^t f'(s) Y_s \, dY_s + \int_0^t \left( - \frac{E^*(0)}{2}(X_s^2 + Y_s^2) + f'(s) \right) \, ds - \int_0^t f''(s) \, ds
\]

\[= f(0) - \int_0^t f'(s) X_s \, dX_s - \int_0^t f'(s) Y_s \, dY_s - \frac{1}{2} \int_0^t f''(s)(X_s^2 + Y_s^2) \, ds.
\]

The computation of \( \langle Z, W \rangle \) involves only the local martingale parts, namely

\[
\langle Z, W \rangle_t = \lambda \left\langle \int_0^t \sin(\lambda A_s) \, dA_s, \int_0^t f'(s) X_s \, dX_s, \int_0^t f'(s) Y_s \, dY_s \right\rangle_t
\]

\[= \lambda \int_0^t f'(s) \sin(\lambda A_s) X_s \, d(A, X)_s + \lambda \int_0^t f'(s) \sin(\lambda A_s) Y_s \, d(A, Y)_s.
\]

Now since \( \langle A, X \rangle_t = -\int_0^t Y_s \, ds \) and \( \langle A, Y \rangle_t = \int_0^t X_s \, ds \), we get

\[
\langle Z, W \rangle_t = \lambda \int_0^t f'(s)(-X_s Y_s + X_s Y_s) \sin(\lambda A_s) \, ds = 0.
\]

4. The Itô formula gives (we benefit from the fact that \( \langle Z, W \rangle = 0 \) from the previous question)

\[
Z_t e^{W_t} = e^{f(0)} + \int_0^t e^{W_s} \, dZ_s + \int_0^t Z_s e^{W_s} \, dW_s + \frac{1}{2} \int_0^t Z_s e^{W_s} \, d\langle W \rangle_s.
\]
By collecting the finite variation parts from dZ and dW from a previous question we get
\[-\frac{\lambda^2}{2} \int_0^t (X_s^2 + Y_s^2) Z_s \, dW_s - \frac{1}{2} \int_0^t \int_0^t f''(s)(X_s^2 + Y_s^2) Z_s \, dW_s \, ds + \frac{1}{2} \int_0^t Z_s \, dW_s \, d(W)_s.\]

Now from a previous question
\[\langle W \rangle_t = \left( \int_0^t f'(s) X_s \, dX_s + \int_0^t f'(s) Y_s \, dY_s \right)_t = \int_0^t f''(s)(X_s^2 + Y_s^2) \, ds.\]
It follows that the finite variation part of Ze^W vanishes when f'' = f'^2 - \lambda^2.

5. With f(t) = -\log \cosh(\lambda(r - t)), we have
\[f'(t) = \frac{\lambda \sinh(\lambda(r - t))}{\cosh(\lambda(r - t))} = \lambda \tanh(\lambda(r - t))\]
and
\[f''(t) = -\frac{\lambda^2}{\cosh(\lambda(r - t))^2} = -\lambda^2 (1 - \tanh^2(\lambda(r - t))) = -\lambda^2 + \lambda^2 f''(t).\]

It follows from the previous question that Ze^W is a continuous local martingale. Note that f(r) = f'(r) = 0 and W_r = 0, and by using previous questions,
\[E e^{\lambda A_r} = E \cos(\lambda A_r) = E Z_r = E(Z_r e^{W_r}).\]
On the other hand, since f(0) = -\log \cosh(\lambda r), Z_0 = 1, W_0 = f(0), we get
\[E(Z_0 e^{W_0}) = e^{f(0)} = \frac{1}{\cosh(\lambda r)}.\]

It remains to show that the local martingale Ze^W is a martingale on the time interval [0, r]. From the previous question, since f, cos, and sin are bounded, it suffices to show that
\[E \int_0^t e^{2W_s} \, d(A)_s < \infty \quad \text{and} \quad E \int_0^t e^{2W_s} (X_s^2 + Y_s^2) \, ds < \infty.\]
But the first condition follows from the second thanks to the formula for \langle A \rangle provided by a previous question. On the other hand, if t ∈ [0, r] then f'(t) ≥ 0 and thus W_s ≤ f(t) for all s ∈ [0, t], which implies that the second condition is satisfied by using E(X_s^2 + Y_s^2) = 2s.

**Exercise 3** (Criterion for a stochastic differential equation). Set d = 1. Let σ, b be two functions \( \mathbb{R} \rightarrow \mathbb{R} \) such that for some finite constant C < ∞ and for all x, y ∈ \( \mathbb{R} \),
\[|\sigma(x) - \sigma(y)| \leq C|x - y| \quad \text{and} \quad |b(x) - b(y)| \leq C|x - y|\]
The goal of this exercise is to prove pathwise uniqueness for the stochastic differential equation
\[dX_t = \sigma(X_t) \, dB_t + b(X_t) \, dt.\]  
A solution X is a continuous semi-martingale with canonical decomposition \( X = X_0 + M + V \) with \( X_0 \in L^2 \), local martingale part \( M = \int_0^t \sigma(X_s) \, dB_s \), and finite variation part \( V = \int_0^t b(X_s) \, ds \). Note that the continuity of \( \sigma, X, b \) gives that almost surely, for all t ≥ 0, s → \( \sigma(X_s) + b(X_s) \) is locally bounded.

1. Let Z be a continuous semi-martingale such that \( \langle Z \rangle = \int_0^t \varphi \, ds \) for a progressive process \( \varphi \) such that 0 ≤ \( \varphi \) ≤ C|Z| for some constant C < ∞. Prove that for all t ≥ 0 and all a > 0,
\[E \int_0^t \frac{1_{|Z_s| > a}}{|Z_s|} \, d\langle Z \rangle_s \leq Ct.\]
2. Deduce from the preceding question that for all \( t \geq 0 \),

\[
\lim_{n \to \infty} nE \int_0^t 1_{|Z_s| \leq \frac{1}{n}} d\langle Z \rangle_s = 0.
\]

3. For all \( n \geq 1 \), \( x \in \mathbb{R} \), let us define \( g_n(x) = 2n(1+nx)1_{x \in [-\frac{1}{n}, 0)} + 2n1_{x=0} + 2n(1-nx)1_{x \in (0, \frac{1}{n})} \).

Let \( f_n : \mathbb{R} \to \mathbb{R} \) be the twice differentiable function such that \( f''_n = g_n \) and \( f_n(0) = f''_n(0) = 0 \).

Show that for all \( x \in \mathbb{R} \), the following properties hold true:

(a) \( f_n'(x) \in [-1, 1] \) and \( \lim_{n \to \infty} f_n'(x) = \text{sign}(x) = 1_{x>0} - 1_{x<0} \);

(b) \( |f_n(x)| \leq |x| \) and \( \lim_{n \to \infty} f_n(x) = |x| \).

4. By using Itô formula, prove that for all continuous semi-martingale \( Z = (Z_t)_{t \geq 0} \), all \( t \geq 0 \),

\[
\int_0^t 1_{Z_s = 0} d\langle Z \rangle_s = 0.
\]

5. From now on, let \( X \) and \( X' \) be two solutions of (SDE) on \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}) \) and with respect to the Brownian motion \( B \). Show that for all \( t \geq 0 \),

\[
\langle X - X' \rangle_t = \int_0^t (\sigma(X_s) - \sigma(X'_s))^2 ds.
\]

6. By using the assumption on \( \sigma \), deduce from the preceding questions that for all \( t \geq 0 \),

\[
\lim_{n \to \infty} \mathbb{E} \int_0^t g_n(X_s - X'_s) d\langle X - X' \rangle_s = 0.
\]

7. Set \( Z = X - X' \). From now on, let \( T \) be a stopping time such that the semi-martingale \((Z_{t \wedge T})_{t \geq 0} \) is bounded. By using notably the assumption on \( \sigma \), prove that for all \( t \geq 0 \), \( n \geq 1 \),

\[
\mathbb{E}(f_n(Z_{t \wedge T})) = \mathbb{E}(f_n(Z_0)) + \mathbb{E} \int_0^{t \wedge T} f'_n(Z_s)(b(X_s) - b(X'_s))ds + \frac{1}{2} \mathbb{E} \int_0^{t \wedge T} f''_n(Z_s)d\langle Z \rangle_s.
\]

8. Deduce from the preceding questions and the assumption on \( b \) that for all \( t \geq 0 \),

\[
\mathbb{E}(|X_{t \wedge T} - X'_{t \wedge T}|) = \mathbb{E}(|X_0 - X'_0|) + \mathbb{E} \int_0^{t \wedge T} (b(X_s) - b(X'_s))\text{sign}(X_s - X'_s)ds.
\]

9. By using the Grönwall lemma, deduce that if \( X_0 = X'_0 \) then \( X_t = X'_t \) for all \( t \geq 0 \).

Elements of solution for Exercise 3. The result is known as the Yamada–Watanabe criterion. This is a slightly more detailed version of [1, Exercise 8.14 pages 231–232].

1. We have, using the properties of \( Z \) and \( \varphi \),

\[
\int_0^t 1_{0 < |Z_s| \leq a} |Z_s| d\langle Z \rangle_s = \int_0^t \frac{1_{0 < |Z_s| \leq a}}{|Z_s|} \varphi_s ds \leq \int_0^t Cds = Ct.
\]

2. For all \( n \geq 1 \), we have \( n1_{0 < |Z_s| \leq \frac{1}{n}} \leq \frac{1_{0 < |Z_s| \leq \frac{1}{n}}}{|Z_s|} \leq \frac{1_{0 < |Z_s| \leq 1}}{|Z_s|} \), which is integrable on \([0, t] \) by the preceding question used with \( a = 1 \), and thus the desired result follows then by dominated convergence.

3. The function \( g_n \) is \( 0 \) on \((-\infty, -\frac{1}{n}) \), then increases from \( 0 \) to \( 2 \) on \([-\frac{1}{n}, 0) \), then decreases from \( 2 \) to \( 0 \) on \([0, \frac{1}{n}) \), then stays at \( 0 \) on \([\frac{1}{n}, +\infty) \). Since \( \int_{-\infty}^0 g_n(y)dy = 1 \), we have, for all \( x \in \mathbb{R} \),

\[
f'_n(x) = \int_0^x g_n(u)du, \quad \text{in such a way that } f'_n(0) = 0 \text{ and } f''_n = g_n.
\]
The function \( f_n' \) is \(-1\) on \((-\infty, -\frac{1}{n}]\), \(0\) at \(0\), and \(1\) on \([\frac{1}{n}, +\infty)\). Also for all \( x \in \mathbb{R} \),

\[
\lim_{n \to \infty} f_n'(x) = \mathbf{1}_{x > 0} - \mathbf{1}_{x < 0} =: \text{sign}(x).
\]

Next, for all \( x \in \mathbb{R} \), we have

\[
f_n(x) = \int_0^x f_n'(u)\,du \quad \text{in such a way that } f_n(0) = 0 \text{ and } f_n'' = g_n.
\]

Since \( g_n \geq 0 \), we have that \( f_n' \) is non-decreasing, and thus \( f_n' \) takes actually its values in \([-1, 1]\), and is in particular bounded. It follows by dominated convergence that for all \( x \in \mathbb{R} \),

\[
\lim_{n \to \infty} f_n(x) = \int_0^x \lim_{n \to \infty} f_n'(u)\,du = \int_0^x \text{sign}(u)\,du = |x|.
\]

Finally, for all \( x \in \mathbb{R} \), \( |f_n(x)| \leq f_0|x|\,du = |x| \).

4. The Itô formula for function \( f_n \) of question 3 and semi-martingale \( Z \) gives, for all \( t \geq 0 \),

\[
f_n(Z_t) = f_n(Z_0) + \int_0^t f_n'(Z_s)\,dZ_s + \frac{1}{2} \int_0^t f_n''(Z_s)\,d[Z_s].
\]

Since \( |f_n''(x)| \leq 1 \) and \( \lim_{n \to \infty} \frac{f_n''(x)}{2n} = 1_{x=0} \) for all \( x \in \mathbb{R} \), by dominated convergence,

\[
\lim_{n \to \infty} \frac{1}{2n} \int_0^t f_n''(Z_s)\,d[Z_s] = \int_0^t 1_{Z_s=0}\,d[Z_s] \quad \text{a.s.}
\]

On the other hand, since by question 3, \( \lim_{n \to \infty} \frac{f_n(x)}{2n} = 0 \) for all \( x \in \mathbb{R} \), it follows that a.s.

\[
\lim_{n \to \infty} \frac{f_n(Z_t)}{2n} = 0 \quad \text{and} \quad \lim_{n \to \infty} \frac{f_n(Z_0)}{2n} = 0.
\]

Finally since by question 3, \( \frac{1}{2n} |f_n'| \leq \frac{1}{2n} \leq 1 \) and \( \lim_{n \to \infty} \frac{1}{2n} |f_n'(x)| = 0 \) for all \( x \in \mathbb{R} \), dominated convergence for stochastic integrals gives

\[
\frac{1}{2n} \int_0^t f_n'(Z_s)\,dZ_s \xrightarrow{n \to \infty} 0.
\]

5. Since \( X \) and \( X' \) are both solutions on the same space and for the same Brownian motion, we have, for all \( t \geq 0 \),

\[
X_t - X'_t = X_0 - X'_0 + \int_0^t (\sigma(X_s) - \sigma(X'_s))\,dB_s + \int_0^t (b(X_s) - b(X'_s))\,ds.
\]

The right hand side gives the canonical decomposition of the semi-martingale \( X - X' \). In this decomposition, the first integral is the local martingale part, and

\[
\langle X - X' \rangle = \int_0^t (\sigma(X_s) - \sigma(X'_s))^2\,ds.
\]

6. By assumption on \( \sigma \), the process \( \varphi = (\sigma(X) - \sigma(X'))^2 \) satisfies \( 0 \leq \varphi \leq C|X - X'| \). We can then use question 5 and question 2 with \( Z = X - X' \) to get, for all \( t \geq 0 \),

\[
\lim_{n \to \infty} n\mathbb{E} \int_0^t 1_{|Z_s| \leq \frac{1}{n}}\,d[Z_s] = 0.
\]

If \( g_n \) is as in question 3, then for all \( x \in \mathbb{R} \), \( 0 \leq g_n(x) \leq 2n1_{0 < |x| < \frac{1}{n}} + 2n1_{x=0} \). Thus, for all \( t \geq 0 \),

\[
0 \leq \mathbb{E} \int_0^t g_n(Z_s)\,d[Z_s] \leq 2n\mathbb{E} \int_0^t 1_{0 < |Z_s| < \frac{1}{n}}\,d[Z_s] + 2n\mathbb{E} \int_0^t 1_{Z_s=0}\,d[Z_s] \xrightarrow{n \to \infty} 0,
\]

where we have used question 2 and question 4.
7. The Itô formula for the \( \mathcal{C}^2 \) function \( f_n \) and the continuous semi-martingale \( Z^T \) gives

\[
f_n(Z^T_t) = f_n(Z^T_0) + \int_0^{t \wedge T} f'_n(Z_s) dB_s + \int_0^{t \wedge T} \frac{1}{2} f''_n(Z_s) dB_s^2,
\]

and since \( dB_s = (\sigma(X_s) - \sigma(X'_s)) dB_s + (b(X_s) - b(X'_s)) ds \), we get

\[
\int_0^{t \wedge T} f'_n(Z_s) dB_s = \int_0^{t} f'_n(Z_s) (\sigma(X_s) - \sigma(X'_s)) 1_{s \leq T} dB_s + \int_0^{t \wedge T} f'_n(Z_s)(b(X_s) - b(X'_s)) ds.
\]

Now, by the assumptions on \( \sigma \) and \( T \), we get

\[
|\sigma(X_s) - \sigma(X'_s)| 1_{s \leq T} \leq C |Z^T_s| \leq C'.
\]

This boundedness, together with the one of \( f'_n \), imply that the first integral in the right hand side above (the \( dB_s \) one) is a martingale. Since this martingale is issued from the origin, its expectation vanishes for all times. On the other hand, since \( f_n \) is continuous and \( Z^T \) is bounded, the random variables \( f_n(Z^T_0) \) and \( f_n(Z^T_0) \) are integrable. All in all, we obtain

\[
\mathbb{E}(f_n(Z^T_0)) = \mathbb{E}(f_n(Z_0^T)) + \mathbb{E} \int_0^{t \wedge T} f'_n(Z_s)(b(X_s) - b(X'_s)) ds + \frac{1}{2} \mathbb{E} \int_0^{t \wedge T} f''_n(Z_s) dB_s^2.
\]

8. Since \( f''_n = g_n \geq 0 \), we get, by using question 6, that

\[
0 \leq \mathbb{E} \int_0^{t \wedge T} f''_n(Z_s) dB_s^2 \leq \mathbb{E} \int_0^t g_n(Z_s) ds \rightarrow 0, \quad n \to \infty.
\]

On the other hand, by the assumption on \( b \) and the boundedness of \( Z^T \), we have, on \( \{s \leq T\} \),

\[
|b(X_s) - b(X'_s)|^2 \leq C^2 |X_s - X'_s|^2 = C^2 |Z^T_s|^2 \leq C'.
\]

But since \( f'_n \) is bounded (takes its values in \([-1, 1]\)), we get, by dominated convergence

\[
\lim_{n \to \infty} \int_0^{t \wedge T} f'_n(Z_s)(b(X_s) - b(X'_s)) ds = \int_0^{t \wedge T} \text{sign}(Z_s)(b(X_s) - b(X'_s)) ds.
\]

Finally, since \( Z^T_0 \) is bounded, and since from question 3, for all \( x \in \mathbb{R}, |f_n(x)| \leq |x| \) and \( \lim_{n \to \infty} f_n(x) = |x| \), we get, by dominated convergence, \( \lim_{n \to \infty} \mathbb{E}(f_n(Z^T_0)) = \mathbb{E}(|Z^T_0|) \). Finally

\[
\mathbb{E}(|X^T_0 - X'_0|) = \mathbb{E}(|X_0 - X'_0|) + \mathbb{E} \int_0^{t \wedge T} (b(X_s) - b(X'_s)) \text{sign}(X_s - X'_s) ds.
\]

9. From the preceding question, we get, by using the assumption on \( b \),

\[
\alpha(t) = \mathbb{E}(|X^T_0 - X'_0|) \leq \mathbb{E}(|X_0 - X'_0|) + C \int_0^t |X^T_s - X'_s| ds = \alpha(0) + C \int_0^t \alpha(s) ds.
\]

By the Grönwall lemma, we obtain \( \alpha(t) \leq \alpha(0) e^{Ct} \) for all \( t \geq 0 \). It follows that if \( \alpha(0) = 0 \) then \( \alpha(t) = 0 \) for all \( t \geq 0 \). This means that if \( X_0 = X'_0 \) then \( X^T_t = X'_t \) for all \( t \geq 0 \). By writing this for \( t \in \mathbb{Q}_+ \), and by taking \( T = T_m \) such that \( \lim_{m \to \infty} T_m = +\infty \) almost surely, we get that \( X_t = X'_t \) for all \( t \in \mathbb{Q}_+ \), and thus for all \( t \geq 0 \) since \( X \) and \( X' \) are continuous.

References