Exam 2018/2019

January 9, 2019, from 09:00 to 12:00
Documents allowed, Internet not allowed
Do what you can, and do not worry

$(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$ is a filtered probability space, with complete and right continuous filtration.
$B = (B_t)_{t\geq 0}$ is a d-dimensional Brownian motion issued from the origin, $d \geq 1$.
If Z is a semi-martingale, we denote by $\langle Z\rangle$ the increasing process of its local martingale part.
If $Z = Z_0 + M + V$, do not confuse $\langle Z\rangle = \langle M\rangle$ with the finite variation part V of Z.

Exercise 1 (Nature of an integral)**.** Set $d = 1$. Let us consider the following integral, for $t \geq 0$,

$$I_t = \int_0^t B_s \mathrm{d}s.$$

1. Is it a Lebesgue–Stieltjes integral? A Wiener integral? An Itô integral? Justify your answer
2. Show that $\mathrm{d}(tB_t) = B_t \mathrm{d}t + t\mathrm{d}B_t$
3. Deduce from the preceding question that $I_t = \int_0^t (t-s)\mathrm{d}B_s$ for all $t \geq 0$
4. Deduce from the preceding question that $I_t \sim \mathcal{N}(0, \frac{1}{3}t^3)$ for all $t \geq 0$
5. For all $t \geq 0$, $n \geq 1$, $0 \leq k \leq n$, let us define $t_k = \frac{k}{n}t$. Show that

$$\sum_{k=0}^{n-1} B_{t_k}(t_{k+1} - t_k) = \frac{t}{n}\sum_{j=0}^{n-2}(n-j-1)(B_{t_{j+1}} - B_{t_j}).$$

6. Deduce from the preceding question another proof that $I_t \sim \mathcal{N}(0, \frac{1}{3}t^3)$ for all $t \geq 0$
7. Is the process $(I_t)_{t\geq 0}$ a martingale?

Elements of solution for Exercise 1.

1. Since the integrator is of finite variation and the integrand is bounded and measurable (actually continuous), it is a Lebesgue–Stieltjes integral, and in particular an Itô integral with respect to a semi-martingale without martingale part. However it is not a Wiener integral.
2. The Itô formula for $f(x, y) = xy$ and $X_t = (t, B_t)$ gives

$$tB_t = 0 + \int_0^t B_s \mathrm{d}s + \int_0^t s\mathrm{d}B_s,$$

3. From the preceding question (actually, it is an integration by parts)

$$\int_0^t B_s \mathrm{d}s = tB_t - \int_0^t s\mathrm{d}B_s = \int_0^t (t-s)\mathrm{d}B_s.$$

4. The integral in the right hand side is a Wiener integral. Thus it is Gaussian with mean zero and variance equal to the squared L^2 norm of the integrand:

$$\mathbb{E}\int_0^t B_s \mathrm{d}s = 0 \quad \text{and} \quad \mathbb{E}\Big(\Big(\int_0^t B_s \mathrm{d}s\Big)^2\Big) = \int_0^t (t-s)^2 \mathrm{d}s = \frac{t^3}{3}.$$

5. With $t_k = \frac{k}{n}t$ for all $0 \leq k \leq n$, we have

$$S_n = \sum_{k=0}^{n-1} B_{t_k}(t_{k+1} - t_k) = \frac{t}{n}\sum_{k=0}^{n-1} B_{t_k} = \frac{t}{n}\sum_{k=0}^{n-1}\sum_{j=0}^{k-1}(B_{t_{j+1}} - B_{t_j}) = \frac{t}{n}\sum_{j=0}^{n-2}(n-j-1)(B_{t_{j+1}} - B_{t_j}).$$

6. Fix $t \geq 0$. Since I_t is a Lebesgue–Stieltjes integral with continuous integrand, we have $\lim_{n\to\infty} S_n = I_t$ almost surely and thus in law. For all n and j, since $B_{t_{j+1}} - B_{t_j}$ are independent and Gaussian, we get that $S_n \sim \mathcal{N}(\mathbb{E}(S_n), \mathbb{E}(S_n^2) - \mathbb{E}(S_n)^2)$. But the convergence in law of Gaussians is equivalent to the convergence of the first two moments. Now it remains to note that we have $\mathbb{E}(S_n) = 0$ and

$$\mathbb{E}(S_n^2) = \frac{t^2}{n^2}\sum_{j=0}^{n-2}(n-j-1)^2\mathbb{E}((B_{t_{j+1}} - B_{t_j})^2) = \frac{t^3}{n^3}\sum_{j=0}^{n-1} j^2 = t^3\sum_{j=1}^{n-1}\Big(\frac{j}{n}\Big)^2\frac{1}{n} \underset{n\to\infty}{\longrightarrow} t^3\int_0^1 x^2\mathrm{d}x = \frac{t^3}{3}.$$

7. Beware that the integrand in $\int_0^t (t-s)\mathrm{d}B_s$ depends on t. The process $(-\int_0^t s\mathrm{d}B_s)_{t\geq 0}$ is a martingale, however the process $(\int_0^t t\mathrm{d}B_s)_{t\geq 0} = (tB_t)_{t\geq 0}$ is not a martingale: for all $0 \leq s \leq t$,

$$\mathbb{E}((t+s)B_{t+s} \mid \mathcal{F}_t) = (t+s)B_s \neq sB_s.$$

Exercise 2 (Study of a special process)**.** Set $d = 2$. For all $t \geq 0$, we write $B_t = (X_t, Y_t)$ and

$$A_t = \int_0^t X_s\mathrm{d}Y_s - \int_0^t Y_s\mathrm{d}X_s.$$

1. Show that $\langle A\rangle = \int_0^\bullet (X_s^2 + Y_s^2)\mathrm{d}s$ and that the process A is a square integrable martingale
2. From now on let $\lambda > 0$. Show that for all $t \geq 0$,

$$\mathbb{E}\mathrm{e}^{\mathrm{i}\lambda A_t} = \mathbb{E}\cos(\lambda A_t).$$

3. From now on, let $f : \mathbb{R}_+ \to \mathbb{R}$ be $\mathcal{C}^2$, and let us define the continuous semi-martingales

$$(Z_t)_{t\geq 0} = (\cos(\lambda A_t))_{t\geq 0} \quad\text{and}\quad (W_t)_{t\geq 0} = \Big(-\frac{f'(t)}{2}(X_t^2 + Y_t^2) + f(t)\Big)_{t\geq 0}.$$

Show that for all $t \geq 0$,

$$Z_t = 1 - \lambda\int_0^t \sin(\lambda A_s)\mathrm{d}A_s - \frac{\lambda^2}{2}\int_0^t (X_s^2 + Y_s^2)Z_s\mathrm{d}s.$$

and

$$W_t = f(0) - \int_0^t f'(s)X_s\mathrm{d}X_s - \int_0^t f'(s)Y_s\mathrm{d}Y_s - \frac{1}{2}\int_0^t f''(s)(X_s^2 + Y_s^2)\mathrm{d}s,$$

and deduce that

$$\langle Z, W\rangle = 0.$$

4. Show that if f solves $f'' = f'^2 - \lambda^2$ then $Z\mathrm{e}^W$ is a continuous local martingale and

$$Z_t\mathrm{e}^{W_t} = \mathrm{e}^{f(0)} - \lambda\int_0^t \sin(\lambda A_s)\mathrm{e}^{W_s}\mathrm{d}A_s - \int_0^t f'(s)Z_s\mathrm{e}^{W_s}X_s\mathrm{d}X_s - \int_0^t f'(s)Z_s\mathrm{e}^{W_s}Y_s\mathrm{d}Y_s.$$

5. Let $r > 0$. By using $f(t) = -\log\cosh(\lambda(r-t))$ deduce from the previous question that

$$\mathbb{E}\mathrm{e}^{\mathrm{i}\lambda A_r} = \frac{1}{\cosh(\lambda r)}.$$

Elements of solution for Exercise 2. For all $t \geq 0$, A_t is the algebraic area between planar Brownian motion and its chord, and the process A is the *Lévy area*. This exercise is a slightly more detailed version of [1, Exercise 5.30 pages 144–145]. Its goal is to compute the characteristic function or Fourier transform of A_t.

1. Since $\langle X\rangle_t = \langle Y\rangle_t = t$ and $\langle X,Y\rangle_t = 0$, we get

$$\begin{aligned}\langle A\rangle_t = \langle A,A\rangle_t &= \Big\langle \int_0^\bullet X_s\mathrm{d}Y_s\Big\rangle_t + \Big\langle \int_0^\bullet Y_s\mathrm{d}X_s\Big\rangle_t + 2\Big\langle \int_0^\bullet X_s\mathrm{d}Y_s, \int_0^\bullet Y_s\mathrm{d}X_s\Big\rangle_t\\ &= \int_0^t X_s^2\mathrm{d}\langle X\rangle_s + \int_0^t Y_s^2\mathrm{d}\langle Y\rangle_s + \int_0^t X_sY_s\mathrm{d}\langle X,Y\rangle_s\\ &= \int_0^t (X_s^2+Y_s^2)\mathrm{d}s.\end{aligned}$$

It follows by the Fubini–Tonelli theorem that

$$\mathbb{E}\langle A\rangle_t = \int_0^t \mathbb{E}(X_s^2+Y_s^2)\mathrm{d}s = \int_0^t 2s\mathrm{d}s = t^2 < \infty$$

and thus, by a famous martingale criterion, the process A is a square integrable martingale.

Alternatively, since for all $t\geq 0$, $\mathbb{E}\int_0^t X_s^2\mathrm{d}\langle Y\rangle_s = \int_0^t \mathbb{E}(X_s^2)\mathrm{d}s < \infty$ the process $\int_0^\bullet X_s\mathrm{d}Y_s$ and by symmetry the process $\int_0^\bullet Y_s\mathrm{d}X_s$ are both square integrable martingales, and thus the process A is also a square integrable martingales as being the difference of two square integrable martingales.

2. For all $\lambda\in\mathbb{R}$, $t\geq 0$, $\mathbb{E}(\mathrm{e}^{\mathrm{i}\lambda A_t}) = \mathbb{E}(\cos(\lambda A_t)) + \mathrm{i}\mathbb{E}(\sin(\lambda A_t))$. Since $(X,Y)\overset{\mathrm{d}}{=}(Y,X)$, we get, for all $t\geq 0$,

$$-A_t = \int_0^t Y_s\mathrm{d}X_s - \int_0^t X_s\mathrm{d}Y_s \overset{\mathrm{d}}{=} \int_0^t X_s\mathrm{d}Y_s - \int_0^t Y_s\mathrm{d}X_s = A_t,$$

and thus the characteristic function or Fourier transform of A_t is real.

3. The canonical decompositions are given by the Itô formula. Namely, for Z,

$$\begin{aligned}Z_t &= 1 - \lambda\int_0^t \sin(\lambda A_s)\mathrm{d}A_s - \frac{\lambda^2}{2}\int_0^t \cos(\lambda A_s)\mathrm{d}\langle A\rangle_s\\ &= 1 - \lambda\int_0^t \sin(\lambda A_s)\mathrm{d}A_s - \frac{\lambda^2}{2}\int_0^t (X_s^2+Y_s^2)Z_s\mathrm{d}s.\end{aligned}$$

Similarly, for W, by the Itô formula for the function $g(x,y,t) = -\frac{f'(t)}{2}(x^2+y^2)+f(t)$ and the vector of semi-martingale $S_t = (X_t,Y_t,t)$ with martingale part (X_t,Y_t),

$$\begin{aligned}W_t &= g(0,0,0) + \int_0^t \partial_1 g(S_s)\mathrm{d}X_s + \int_0^t \partial_2 g(S_s)\mathrm{d}Y_s + \int_0^t \partial_3 g(S_s)\mathrm{d}s + \frac{1}{2}\int_0^t (\partial_x^2 g + \partial_y^2 g)(S_s)\mathrm{d}s\\ &= f(0) - \int_0^t f'(s)X_s\mathrm{d}X_s - \int_0^t f'(s)Y_s\mathrm{d}Y_s + \int_0^t\Big(-\frac{f''(s)}{2}(X_s^2+Y_s^2)+f'(s)\Big)\mathrm{d}s - \int_0^t f'(s)\mathrm{d}s\\ &= f(0) - \int_0^t f'(s)X_s\mathrm{d}X_s - \int_0^t f'(s)Y_s\mathrm{d}Y_s - \frac{1}{2}\int_0^t f''(s)(X_s^2+Y_s^2)\mathrm{d}s.\end{aligned}$$

The computation of $\langle Z,W\rangle$ involves only the local martingale parts, namely

$$\begin{aligned}\langle Z,W\rangle_t &= \lambda\Big\langle \int_0^\bullet \sin(\lambda A_s)\mathrm{d}A_s, \int_0^\bullet f'(s)X_s\mathrm{d}X_s + \int_0^\bullet f'(s)Y_s\mathrm{d}Y_s\Big\rangle_t\\ &= \lambda\int_0^t f'(s)\sin(\lambda A_s)X_s\mathrm{d}\langle A,X\rangle_s + \lambda\int_0^t f'(s)\sin(\lambda A_s)Y_s\mathrm{d}\langle A,Y\rangle_s.\end{aligned}$$

Now since $\langle A,X\rangle_t = -\displaystyle\int_0^t Y_s\mathrm{d}s$ and $\langle A,Y\rangle_t = \displaystyle\int_0^t X_s\mathrm{d}s$, we get

$$\langle Z,W\rangle_t = \lambda\int_0^t f'(s)(-X_sY_s+X_sY_s)\sin(\lambda A_s)\mathrm{d}s = 0.$$

4. The Itô formula gives (we benefit from the fact that $\langle Z,W\rangle = 0$ from the previous question)

$$Z_t\mathrm{e}^{W_t} = \mathrm{e}^{f(0)} + \int_0^t \mathrm{e}^{W_s}\mathrm{d}Z_s + \int_0^t Z_s\mathrm{e}^{W_s}\mathrm{d}W_s + \frac{1}{2}\int_0^t Z_s\mathrm{e}^{W_s}\mathrm{d}\langle W\rangle_s.$$

By collecting the finite variation parts from $\mathrm{d}Z$ and $\mathrm{d}W$ from a previous question we get

$$-\frac{\lambda^2}{2}\int_0^t (X_s^2+Y_s^2)Z_s\mathrm{e}^{W_s}\mathrm{d}s-\frac{1}{2}\int_0^t f''(s)(X_s^2+Y_s^2)Z_s\mathrm{e}^{W_s}\mathrm{d}s+\frac{1}{2}\int_0^t Z_s\mathrm{e}^{W_s}\mathrm{d}\langle W\rangle_s.$$

Now from a previous question

$$\langle W\rangle_t=\Big\langle\int_0^\bullet f'(s)X_s\mathrm{d}X_s+\int_0^\bullet f'(s)Y_s\mathrm{d}Y_s\Big\rangle_t=\int_0^t f'^2(s)(X_s^2+Y_s^2)\mathrm{d}s.$$

It follows that the finite variation part of $Z\mathrm{e}^W$ vanishes when $f''=f'^2-\lambda^2$.

5. With $f(t)=-\log\cosh(\lambda(r-t))$, we have

$$f'(t)=\lambda\frac{\sinh(\lambda(r-t))}{\cosh(\lambda(r-t))}=\lambda\tanh(\lambda(r-t))$$

and

$$f''(t)=-\frac{\lambda^2}{\cosh(\lambda(r-t))^2}=-\lambda^2(1-\tanh(\lambda(r-t))^2)=-\lambda^2+f'^2(t).$$

It follows from the previous question that $Z\mathrm{e}^W$ is a continuous local martingale. Note that $f(r)=f'(r)=0$ and $W_r=0$, and by using previous questions,

$$\mathbb{E}\mathrm{e}^{\mathrm{i}\lambda A_r}=\mathbb{E}\cos(\lambda A_r)=\mathbb{E}Z_r=\mathbb{E}(Z_r\mathrm{e}^{W_r}).$$

On the other hand, since $f(0)=-\log\cosh(\lambda r)$, $Z_0=1$, $W_0=f(0)$, we get

$$\mathbb{E}(Z_0\mathrm{e}^{W_0})=\mathrm{e}^{f(0)}=\frac{1}{\cosh(\lambda r)}.$$

It remains to show that the local martingale $Z\mathrm{e}^W$ is a martingale on the time interval $[0,r]$. From the previous question, since f, cos, and sin are bounded, it suffices to show that

$$\mathbb{E}\int_0^t \mathrm{e}^{2W_s}\mathrm{d}\langle A\rangle_s<\infty\quad\text{and}\quad\mathbb{E}\int_0^t \mathrm{e}^{2W_s}(X_s^2+Y_s^2)\mathrm{d}s<\infty.$$

But the first condition follows from the second thanks to the formula for $\langle A\rangle$ provided by a previous question. On the other hand, if $t\in[0,r]$ then $f'(t)\geq 0$ and thus $W_s\leq f(t)$ for all $s\in[0,t]$, which implies that the second condition is satisfied by using $\mathbb{E}(X_s^2+Y_s^2)=2s$.

Exercise 3 (Criterion for a stochastic differential equation)**.** Set $d=1$. Let σ,b be two functions $\mathbb{R}\to\mathbb{R}$ such that for some finite constant $C<\infty$ and for all $x,y\in\mathbb{R}$,

$$|\sigma(x)-\sigma(y)|\leq C\sqrt{x-y}\quad\text{and}\quad|b(x)-b(y)|\leq C|x-y|$$

The goal of this exercise is to prove pathwise uniqueness for the stochastic differential equation

$$\mathrm{d}X_t=\sigma(X_t)\mathrm{d}B_t+b(X_t)\mathrm{d}t. \tag{SDE}$$

A solution X is a continuous semi-martingale with canonical decomposition $X=X_0+M+V$ with $X_0\in\mathrm{L}^2$, local martingale part $M=\int_0^\bullet\sigma(X_s)\mathrm{d}B_s$, and finite variation part $V=\int_0^\bullet b(X_s)\mathrm{d}s$. Note that the continuity of σ,X,b gives that almost surely, for all $t\geq 0$, $s\mapsto\sigma(X_s)+b(X_s)$ is locally bounded.

1. Let Z be a continuous semi-martingale such that $\langle Z\rangle=\int_0^\bullet\varphi_s\mathrm{d}s$ for a progressive process φ such that $0\leq\varphi\leq C|Z|$ for some constant $C<\infty$. Prove that for all $t\geq 0$ and all $a>0$,

$$\mathbb{E}\int_0^t\frac{\mathbf{1}_{0<|Z_s|\leq a}}{|Z_s|}\mathrm{d}\langle Z\rangle_s\leq Ct.$$

2. Deduce from the preceding question that for all $t \geq 0$,

$$\lim_{n\to\infty} n\mathbb{E}\int_0^t \mathbf{1}_{\{0<|Z_s|\leq \frac{1}{n}\}}\mathrm{d}\langle Z\rangle_s = 0.$$

3. For all $n \geq 1$, $x \in \mathbb{R}$, let us define $g_n(x) = 2n(1+nx)\mathbf{1}_{x\in[-\frac{1}{n},0)} + 2n\mathbf{1}_{x=0} + 2n(1-nx)\mathbf{1}_{x\in(0,\frac{1}{n}]}$.
Let $f_n : \mathbb{R} \to \mathbb{R}$ be the twice differentiable function such that $f_n'' = g_n$ and $f_n(0) = f_n'(0) = 0$.
Show that for all $x \in \mathbb{R}$, the following properties hold true:

   (a) $f_n'(x) \in [-1,1]$ and $\lim_{n\to\infty} f_n'(x) = \mathrm{sign}(x) = \mathbf{1}_{x>0} - \mathbf{1}_{x<0}$

   (b) $|f_n(x)| \leq |x|$ and $\lim_{n\to\infty} f_n(x) = |x|$.

4. By using Itô formula, prove that for all continuous semi-martingale $Z = (Z_t)_{t\geq 0}$, all $t \geq 0$,

$$\int_0^t \mathbf{1}_{Z_s=0}\mathrm{d}\langle Z\rangle_s = 0.$$

5. From now on, let X and X' be two solutions of (SDE) on $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$ and with respect to the Brownian motion B. Show that for all $t \geq 0$,

$$\langle X - X'\rangle_t = \int_0^t (\sigma(X_s) - \sigma(X_s'))^2\mathrm{d}s.$$

6. By using the assumption on σ, deduce from the preceding questions that for all $t \geq 0$,

$$\lim_{n\to\infty} \mathbb{E}\int_0^t g_n(X_s - X_s')\mathrm{d}\langle X - X'\rangle_s = 0.$$

7. Set $Z = X - X'$. From now on, let T be a stopping time such that the semi-martingale $(Z_{t\wedge T})_{t\geq 0}$ is bounded. By using notably the assumption on σ, prove that for all $t \geq 0$, $n \geq 1$,

$$\mathbb{E}(f_n(Z_{t\wedge T})) = \mathbb{E}(f_n(Z_0)) + \mathbb{E}\int_0^{t\wedge T} f_n'(Z_s)(b(X_s) - b(X_s'))\mathrm{d}s + \frac{1}{2}\mathbb{E}\int_0^{t\wedge T} f_n''(Z_s)\mathrm{d}\langle Z\rangle_s.$$

8. Deduce from the preceding questions and the assumption on b that for all $t \geq 0$,

$$\mathbb{E}(|X_{t\wedge T} - X_{t\wedge T}'|) = \mathbb{E}(|X_0 - X_0'|) + \mathbb{E}\int_0^{t\wedge T} (b(X_s) - b(X_s'))\mathrm{sign}(X_s - X_s')\mathrm{d}s.$$

9. By using the Grönwall lemma, deduce that if $X_0 = X_0'$ then $X_t = X_t'$ for all $t \geq 0$.

Elements of solution for Exercise 3. The result is known as the Yamada–Watanabe criterion. This is a slightly more detailed version of [1, Exercise 8.14 pages 231–232].

1. We have, using the properties of Z and φ,

$$\int_0^t \frac{\mathbf{1}_{0<|Z_s|\leq a}}{|Z_s|}\mathrm{d}\langle Z\rangle_s = \int_0^t \frac{\mathbf{1}_{0<|Z_s|\leq a}}{|Z_s|}\varphi_s\mathrm{d}s \leq \int_0^t C\mathrm{d}s = Ct.$$

2. For all $n \geq 1$, we have $n\mathbf{1}_{0<|Z_s|\leq\frac{1}{n}} \leq \frac{\mathbf{1}_{0<|Z_s|\leq\frac{1}{n}}}{|Z_s|} \leq \frac{\mathbf{1}_{0<|Z_s|\leq 1}}{|Z_s|}$, which is integrable on $[0,t]$ by the preceding question used with $a = 1$, and thus the desired result follows then by dominated convergence.

3. The function g_n is $= 0$ on $(-\infty, -\frac{1}{n})$, then increases from 0 to 2 on $[-\frac{1}{n}, 0]$, then decreases from 2 to 0 on $[0, \frac{1}{n}]$, then stays at 0 on $[\frac{1}{n}, +\infty)$. Since $\int_{-\infty}^0 g_n(y)\mathrm{d}y = 1$, we have, for all $x \in \mathbb{R}$,

$$f_n'(x) = \int_0^x g_n(u)\mathrm{d}u, \quad \text{in such a way that } f_n'(0) = 0 \text{ and } f_n'' = g_n.$$

The function f_n' is $=-1$ on $(-\infty,-\frac{1}{n}]$, $=0$ at 0, and $=1$ on $[\frac{1}{n},+\infty)$. Also for all $x\in\mathbb{R}$,

$$\lim_{n\to\infty} f_n'(x) = \mathbf{1}_{x>0}-\mathbf{1}_{x<0} =: \mathrm{sign}(x).$$

Next, for all $x\in\mathbb{R}$, we have

$$f_n(x)=\int_0^x f_n'(u)\mathrm{d}u \quad \text{in such a way that } f_n(0)=0 \text{ and } f_n''=g_n.$$

Since $g_n\geq 0$, we have that f_n' is non-decreasing, and thus f_n' takes actually its values in $[-1,1]$, and is in particular bounded. It follows by dominated convergence that for all $x\in\mathbb{R}$,

$$\lim_{n\to\infty} f_n(x)=\int_0^x \lim_{n\to\infty} f_n'(u)\mathrm{d}u=\int_0^x \mathrm{sign}(u)\mathrm{d}u=|x|.$$

Finally, for all $x\in\mathbb{R}$, $|f_n(x)|\leq\int_0^{|x|}\mathrm{d}u=|x|$.

4. The Itô formula for function f_n of question 3 and semi-martingale Z gives, for all $t\geq 0$,

$$f_n(Z_t)=f_n(Z_0)+\int_0^t f_n'(Z_s)\mathrm{d}Z_s+\frac{1}{2}\int_0^t f_n''(Z_s)\mathrm{d}\langle Z\rangle_s.$$

Since $|\frac{f_n''}{2n}|\leq 1$ and $\lim_{n\to\infty}\frac{f_n''(x)}{2n}=\mathbf{1}_{x=0}$ for all $x\in\mathbb{R}$, by dominated convergence,

$$\lim_{n\to\infty}\frac{1}{2n}\int_0^t f_n''(Z_s)\mathrm{d}\langle Z\rangle_s=\int_0^t \mathbf{1}_{Z_s=0}\mathrm{d}\langle Z\rangle_s \quad \text{a.s.}$$

On the other hand, since by question 3, $\lim_{n\to\infty}\frac{f_n(x)}{2n}=0$ for all $x\in\mathbb{R}$, it follows that a.s.

$$\lim_{n\to\infty}\frac{f_n(Z_t)}{2n}=0 \quad \text{and} \quad \lim_{n\to\infty}\frac{f_n(Z_0)}{2n}=0.$$

Finally since by question 3, $\frac{1}{2n}|f_n'|\leq\frac{1}{2n}\leq 1$ and $\lim_{n\to\infty}\frac{1}{2n}|f_n'(x)|=0$ for all $x\in\mathbb{R}$, dominated convergence for stochastic integrals gives

$$\frac{1}{2n}\int_0^t f_n'(Z_s)\mathrm{d}Z_s \xrightarrow[n\to\infty]{\mathbb{P}} 0.$$

5. Since X and X' are both solutions on the same space and for the same Brownian motion, we have, for all X_0 and X_0' and for all $t\geq 0$,

$$X_t-X_t'=X_0-X_0'+\int_0^t(\sigma(X_s)-\sigma(X_s'))\mathrm{d}B_s+\int_0^t(b(X_s)-b(X_s'))\mathrm{d}s.$$

The righ hand side gives the canonical decomposition of the semi-martingale $X-X'$. In this decomposition, the first integral is the local martingale part, and

$$\langle X-X'\rangle=\int_0^\bullet(\sigma(X_s)-\sigma(X_s'))^2\mathrm{d}s.$$

6. By assumption on σ, the process $\varphi=(\sigma(X)-\sigma(X'))^2$ satisfies $0\leq\varphi\leq C|X-X'|$. We can then use question 5 and question 2 with $Z=X-X'$ to get, for all $t\geq 0$,

$$\lim_{n\to\infty} n\mathbb{E}\int_0^t \mathbf{1}_{0<|Z_s|\leq\frac{1}{n}}\mathrm{d}\langle Z\rangle_s=0.$$

If g_n is as in question 3, then for all $x\in\mathbb{R}$, $0\leq g_n(x)\leq 2n\mathbf{1}_{0<|x|\leq\frac{1}{n}}+2n\mathbf{1}_{x=0}$. Thus, for all $t\geq 0$,

$$0\leq\mathbb{E}\int_0^t g_n(Z_s)\mathrm{d}\langle Z\rangle_s\leq 2n\mathbb{E}\int_0^t\mathbf{1}_{0<|Z_s|\leq\frac{1}{n}}\mathrm{d}\langle Z\rangle_s+2n\mathbb{E}\int_0^t\mathbf{1}_{Z_s=0}\mathrm{d}\langle Z\rangle_s\xrightarrow[n\to\infty]{}0,$$

where we have used question 2 and question 4.

7. The Itô formula for the $\mathscr{C}^2$ function f_n and the continuous semi-martingale Z^T gives

$$f_n(Z_t^T) = f_n(Z_0^T) + \int_0^{t\wedge T} f_n'(Z_s)\mathrm{d}Z_s + \frac{1}{2}\int_0^{t\wedge T} f_n''(Z_s)\mathrm{d}\langle Z\rangle_s,$$

and since $\mathrm{d}Z_s = (\sigma(X_s) - \sigma(X_s'))\mathrm{d}B_s + (b(X_s) - b(X_s'))\mathrm{d}s$, we get

$$\int_0^{t\wedge T} f'(Z_s)\mathrm{d}Z_s = \int_0^t f'(Z_s^T)(\sigma(X_s) - \sigma(X_s'))\mathbf{1}_{s\le T}\mathrm{d}B_s + \int_0^{t\wedge T} f'(Z_s)(b(X_s) - b(X_s'))\mathrm{d}s.$$

Now, by the assumptions on σ and T, we get

$$|\sigma(X_s) - \sigma(X_s')|\mathbf{1}_{s\le T} \le C\sqrt{|Z_s^T|} \le C'.$$

This boundedness, together with the one of f_n', imply that the first integral in the right hand side above (the $\mathrm{d}B_s$ one) is a martingale. Since this martingale is issued from the origin, its expectation vanishes for all times. On the other hand, since f_n is continuous and Z^T is bounded, the random variables $f_n(Z_t^T)$ and $f_n(Z_0^T)$ are integrable. All in all, we obtain

$$\mathbb{E}(f_n(Z_t^T)) = \mathbb{E}(f_n(Z_0^T)) + \mathbb{E}\int_0^{t\wedge T} f_n'(Z_s)(b(X_s) - b(X_s'))\mathrm{d}s + \frac{1}{2}\mathbb{E}\int_0^{t\wedge T} f_n''(Z_s)\mathrm{d}\langle Z\rangle_s.$$

8. Since $f_n'' = g_n \ge 0$, we get, by using question 6, that

$$0 \le \mathbb{E}\int_0^{t\wedge T} f_n''(Z_s)\mathrm{d}\langle Z\rangle_s \le \mathbb{E}\int_0^t g_n(Z_s)\mathrm{d}\langle Z\rangle_s \xrightarrow[n\to\infty]{} 0.$$

On the other hand, by the assumption on b and the boundedness of Z^T, we have, on $\{s \le T\}$,

$$|b(X_s) - b(X_s')|^2 \le C^2|X_s - X_s'|^2 = C^2|Z_s^T| \le C'.$$

But since f_n' is bounded (takes its values in $[-1,1]$), we get, by dominated convergence

$$\lim_{n\to\infty}\int_0^{t\wedge T} f_n'(Z_s)(b(X_s) - b(X_s'))\mathrm{d}s = \int_0^{t\wedge T} \mathrm{sign}(Z_s)(b(X_s) - b(X_s'))\mathrm{d}s.$$

Finally, since Z_t^T is bounded, and since from question 3, for all $x \in \mathbb{R}$, $|f_n(x)| \le |x|$ and $\lim_{n\to\infty} f_n(x) = |x|$, we get, by dominated convergence, $\lim_{n\to\infty}\mathbb{E}(f_n(Z_t^T)) = \mathbb{E}(|Z_t^T|)$. Finally

$$\mathbb{E}(|X_{t\wedge T} - X_{t\wedge T}'|) = \mathbb{E}(|X_0 - X_0'|) + \mathbb{E}\int_0^{t\wedge T}(b(X_s) - b(X_s'))\mathrm{sign}(X_s - X_s')\mathrm{d}s.$$

9. From the preceding question, we get, by using the assumption on b,

$$\alpha(t) = \mathbb{E}(|X_{t\wedge T} - X_{t\wedge T}'|) \le \mathbb{E}(|X_0 - X_0'|) + C\mathbb{E}\int_0^t |X_{s\wedge T} - X_{s\wedge T}'|\mathrm{d}s = \alpha(0) + C\int_0^t \alpha(s)\mathrm{d}s.$$

By the Grönwall lemma, we obtain $\alpha(t) \le \alpha(0)\mathrm{e}^{Ct}$ for all $t \ge 0$. It follows that if $\alpha(0) = 0$ then $\alpha(t) = 0$ for all $t \ge 0$. This means that if $X_0 = X_0'$ then $X_{t\wedge T} = X_{t\wedge T}$ for all $t \ge 0$. By writing this for $t \in \mathbb{Q}_+$, and by taking $T = T_m$ such that $\lim_{m\to\infty} T_m = +\infty$ almost surely, we get that $X_t = X_t'$ for all $t \in \mathbb{Q}_+$, and thus for all $t \ge 0$ since X and X' are continuous.

References

[1] Jean-François Le Gall. *Brownian motion, martingales, and stochastic calculus*, volume 274 of *Graduate Texts in Mathematics*. Springer, french edition, 2016.