Exercise 1

Elements of solution for Exercise 1.

3. Deduce from the preceding question that

4. Deduce from the preceding question that

5. With \( t_k = \frac{k}{n} t \) for all \( 0 \leq k \leq n \), we have

\[
S_n = \sum_{k=0}^{n-1} B_{t_k} (t_{k+1} - t_k) = \frac{t}{n} \sum_{k=0}^{n-1} B_{t_k} = \frac{t}{n} \sum_{k=0}^{n-1} \sum_{j=0}^{k-1} (B_{t_{j+1}} - B_{t_j}) = \frac{t}{n} \sum_{j=0}^{n-2} (n-j-1)(B_{t_{j+1}} - B_{t_j}).
\]
6. Fix \( t \geq 0 \). Since \( I_t \) is a Lebesgue – Stieltjes integral with continuous integrand, we have \( \lim_{n \to -\infty} S_n = I_t \) almost surely and thus in law. For all \( n \) and \( j \), since \( B_{t_{j+1}} - B_{t_j} \) are independent and Gaussian, we get that \( S_n \sim \mathcal{N}(\mathbb{E}(S_n), \mathbb{E}(S_n^2) - \mathbb{E}(S_n)^2) \). But the convergence in law of Gaussians is equivalent to the convergence of the first two moments. Now it remains to note that we have \( \mathbb{E}(S_n) = 0 \) and

\[
\mathbb{E}(S_n^2) = \frac{t^2}{n^2} \sum_{j=0}^{n-2} (n - j - 1)^2 \mathbb{E}((B_{t_{j+1}} - B_{t_j})^2) = \frac{t^3}{n^3} \sum_{j=0}^{n-1} j^2 = \frac{1}{n} \sum_{j=1}^{n-1} \frac{j^2}{3} \to \frac{1}{3} \int_0^t x^2 \, dx = \frac{t^3}{3}.
\]

7. Beware that the integrand in \( \int_0^t (t-s) \, dB_s \) depends on \( t \). The process \( (-\int_0^t s \, dB_s)_{t \geq 0} \) is a martingale, however the process \( (\int_0^t t \, dB_s)_{t \geq 0} = (tB_t)_{t \geq 0} \) is not a martingale: for all \( 0 \leq s \leq t \),

\[
\mathbb{E}((t+s)B_{t+s} \mid \mathcal{F}_t) = (t+s)B_t \neq sB_s.
\]

**Exercise 2** (Study of a special process). Set \( d = 2 \). For all \( t \geq 0 \), we write \( B_t = (X_t, Y_t) \) and

\[
A_t = \int_0^t X_s \, dY_s - \int_0^t Y_s \, dX_s.
\]

1. Show that \( \langle A \rangle = \int_0^t (X_s^2 + Y_s^2) \, ds \) and that the process \( A \) is a square integrable martingale

2. From now on let \( \lambda > 0 \). Show that for all \( t \geq 0 \),

\[
\mathbb{E}e^{\lambda A_t} = \mathbb{E}\cos(\lambda A_t).
\]

3. From now on, let \( f : \mathbb{R}_+ \to \mathbb{R} \) be \( \mathcal{C}^2 \), and let us define the continuous semi-martingales

\[
(Z_t)_{t \geq 0} = (\cos(\lambda A_t))_{t \geq 0} \quad \text{and} \quad (W_t)_{t \geq 0} = \left(-\frac{f'(t)}{2} (X_t^2 + Y_t^2) + f(t)\right)_{t \geq 0}.
\]

Show that for all \( t \geq 0 \),

\[
Z_t = 1 - \lambda \int_0^t \sin(\lambda A_s) \, dA_s - \frac{\lambda^2}{2} \int_0^t (X_s^2 + Y_s^2) \, Z_s \, ds,
\]

and

\[
W_t = f(0) - \int_0^t f'(s) X_s \, dY_s - \int_0^t f'(s) Y_s \, dX_s - \frac{1}{2} \int_0^t f''(s) (X_s^2 + Y_s^2) \, ds,
\]

and deduce that

\[
\langle Z, W \rangle = 0.
\]

4. Show that if \( f \) solves \( f'' = f' - \lambda^2 \) then \( Ze^W \) is a continuous local martingale and

\[
Ze^W = e^{f(0)} - \lambda \int_0^t \sin(\lambda A_s) e^{W_s} \, dA_s - \int_0^t f'(s) Z_s e^{W_s} X_s \, dX_s - \int_0^t f'(s) Z_s e^{W_s} Y_s \, dY_s.
\]

5. Let \( r > 0 \). By using \( f(t) = -\log \cosh(\lambda(r-t)) \) deduce from the previous question that

\[
\mathbb{E}e^{\lambda A_t} = \frac{1}{\cosh(\lambda r)}.
\]

**Elements of solution for Exercise 2.** For all \( t \geq 0 \), \( A_t \) is the algebraic area between planar Brownian motion and its chord, and the process \( A \) is the Lévy area. This exercise is a slightly more detailed version of [1, Exercise 5.30 pages 144–145]. Its goal is to compute the characteristic function or Fourier transform of \( A_t \).
1. Since \( \langle X \rangle_t = \langle Y \rangle_t = t \) and \( \langle X, Y \rangle_t = 0 \), we get

\[
\langle A \rangle_t = \langle A, A \rangle_t = \left( \int_0^t X_s dY_s \right)_t + \left( \int_0^t Y_s dX_s \right)_t + 2 \left( \int_0^t X_s dY_s, \int_0^t Y_s dX_s \right)_t \\
= \int_0^t X_s^2 d\langle X \rangle_s + \int_0^t Y_s^2 d\langle Y \rangle_s + \int_0^t X_s Y_s d\langle X, Y \rangle_s \\
= \int_0^t (X_s^2 + Y_s^2) ds.
\]

It follows by the Fubini–Tonelli theorem that

\[
\mathbb{E}(A)_t = \int_0^t \mathbb{E}(X_s^2 + Y_s^2) ds = \int_0^t 2 s ds = t^2 < \infty
\]

and thus, by a famous martingale criterion, the process \( A \) is a square integrable martingale.

Alternatively, since for all \( t \geq 0 \), \( \mathbb{E} \int_0^t X_s^2 d\langle Y \rangle_s = \mathbb{E} \int_0^t Y_s^2 d\langle X \rangle_s < \infty \) the process \( \int_0^t X_s dY_s \) and by symmetry the process \( \int_0^t Y_s dX_s \) are both square integrable martingales, and thus the process \( A \) is also a square integrable martingale as being the difference of two square integrable martingales.

2. For all \( \lambda \in \mathbb{R}, t \geq 0 \), \( \mathbb{E}(e^{\lambda A_t}) = \mathbb{E}(\cos(\lambda A_t)) + i \mathbb{E}(\sin(\lambda A_t)) \). Since \( \langle X, Y \rangle_t = \langle (X, Y) \rangle_t \), we get, for all \( t \geq 0 \),

\[
-A_t = \int_0^t Y_s dX_s - \int_0^t X_s dY_s = \int_0^t X_s dY_s - \int_0^t Y_s dX_s = A_t,
\]

and thus the characteristic function or Fourier transform of \( A_t \) is real.

3. The canonical decompositions are given by the Itô formula. Namely, for \( Z \),

\[
Z_t = 1 - \lambda \int_0^t \sin(\lambda A_s) dA_s - \frac{\lambda^2}{2} \int_0^t \cos(\lambda A_s) d\langle A \rangle_s \\
= 1 - \lambda \int_0^t \sin(\lambda A_s) dA_s - \frac{\lambda^2}{2} \int_0^t (X_s^2 + Y_s^2) Z_s ds.
\]

Similarly, for \( W \), by the Itô formula for the function \( g(x, y, t) = -\frac{E(\partial f)(x^2 + y^2) + f(t)}{2} \) and the vector of semi-martingale \( S_t = (X_t, Y_t, t) \) with martingale part \( (X_t, Y_t) \),

\[
W_t = g(0, 0, 0) + \int_0^t \partial_1 g(S_s) dX_s + \int_0^t \partial_2 g(S_s) dY_s + \int_0^t \partial_3 g(S_s) ds + \frac{1}{2} \int_0^t \left( \partial^2_{11} g + \partial^2_{22} g \right)(S_s) ds \\
= f(0) - \int_0^t f'(s) X_s dX_s - \int_0^t f'(s) Y_s dY_s + \int_0^t \left( -\frac{E(f')(s)(X_s^2 + Y_s^2) + f'(s)}{2} \right) ds - \int_0^t f'(s) ds \\
= f(0) - \int_0^t f'(s) X_s dX_s - \int_0^t f'(s) Y_s dY_s - \frac{1}{2} \int_0^t f''(s)(X_s^2 + Y_s^2) ds.
\]

The computation of \( \langle Z, W \rangle \) involves only the local martingale parts, namely

\[
\langle Z, W \rangle_t = \lambda \left( \int_0^t \sin(\lambda A_s) dA_s, \int_0^t f'(s) X_s dX_s + \int_0^t f'(s) Y_s dY_s \right)_t \\
= \lambda \int_0^t f'(s) \sin(\lambda A_s) X_s d\langle A, X \rangle_s + \lambda \int_0^t f'(s) \sin(\lambda A_s) Y_s d\langle A, Y \rangle_s.
\]

Now since \( \langle A, X \rangle_t = -\int_0^t Y_s ds \) and \( \langle A, Y \rangle_t = \int_0^t X_s ds \), we get

\[
\langle Z, W \rangle_t = \lambda \int_0^t f'(s) (-X_s Y_s + X_s Y_s) \sin(\lambda A_s) ds = 0.
\]

4. The Itô formula gives (we benefit from the fact that \( \langle Z, W \rangle = 0 \) from the previous question)

\[
Z_t e^W_t = e^{f(0)} + \int_0^t e^W_s dZ_s + \int_0^t Z_s e^W_s dW_s + \frac{1}{2} \int_0^t Z_s e^W_s d\langle W \rangle_s.
\]
By collecting the finite variation parts from \( dZ \) and \( dW \) from a previous question we get
\[
-\frac{\lambda^2}{2} \int_0^t (X_s^2 + Y_s^2) Z_s e^{W_s} \, ds - \frac{1}{2} \int_0^t f''(s)(X_s^2 + Y_s^2) Z_s e^{W_s} \, ds + \frac{1}{2} \int_0^t Z_s e^{W_s} \, d(W)_s.
\]

Now from a previous question
\[
\langle W \rangle_t = \left\langle \int_0^t f'(s) X_s \, dX_s + \int_0^t f'(s) Y_s \, dY_s \right\rangle_t = \int_0^t f''(s)(X_s^2 + Y_s^2) \, ds.
\]

It follows that the finite variation part of \( Ze^{W} \) vanishes when \( f'' = f'^2 - \lambda^2 \).

5. With \( f(t) = -\log\cosh(\lambda(r-t)) \), we have
\[
f'(t) = \lambda \frac{\sinh(\lambda(r-t))}{\cosh(\lambda(r-t))} = \lambda \tanh(\lambda(r-t))
\]
and
\[
f''(t) = -\frac{\lambda^2}{\cosh(\lambda(r-t))} = -\lambda^2 (1 - \tanh(\lambda(r-t))^2) = -\lambda^2 + f'^2(t).
\]

It follows from the previous question that \( Ze^{W} \) is a continuous local martingale. Note that \( f(r) = f'(r) = 0 \) and \( W_r = 0 \), and by using previous questions,
\[
\mathbb{E}e^{tA_r} = \mathbb{E}\cos(\lambda A_r) = \mathbb{E}Z_r = \mathbb{E}(Ze^{W_r}).
\]

On the other hand, since \( f(0) = -\log\cosh(\lambda r) \), \( Z_0 = 1 \), \( W_0 = f(0) \), we get
\[
\mathbb{E}(Ze^{W_0}) = e^{f(0)} = \frac{1}{\cosh(\lambda r)}.
\]

It remains to show that the local martingale \( Ze^{W} \) is a martingale on the time interval \([0, r]\). From the previous question, since \( f \), \( \cos \), and \( \sin \) are bounded, it suffices to show that
\[
\mathbb{E} \int_0^t e^{2W_s} \, d(A)_s < \infty \quad \text{and} \quad \mathbb{E} \int_0^t e^{2W_s}(X_s^2 + Y_s^2) \, ds < \infty.
\]

But the first condition follows from the second thanks to the formula for \( \langle A \rangle \) provided by a previous question. On the other hand, if \( t \in [0, r] \) then \( f'(t) \geq 0 \) and thus \( W_s \leq f(t) \) for all \( s \in [0, t] \), which implies that the second condition is satisfied by using \( \mathbb{E}(X_s^2 + Y_s^2) = 2s \).

**Exercise 3** (Criterion for a stochastic differential equation). Set \( d = 1 \). Let \( \sigma, b \) be two functions \( \mathbb{R} \to \mathbb{R} \) such that for some finite constant \( C < \infty \) and for all \( x, y \in \mathbb{R}, \)
\[
|\sigma(x) - \sigma(y)| \leq C \sqrt{x - y} \quad \text{and} \quad |b(x) - b(y)| \leq C|x - y|.
\]

The goal of this exercise is to prove pathwise uniqueness for the stochastic differential equation
\[
dx_t = \sigma(X_t) \, dB_t + b(X_t) \, dt. \quad \text{(SDE)}
\]

A solution \( X \) is a continuous semi-martingale with canonical decomposition \( X = X_0 + M + V \) with \( X_0 \in L^2 \), local martingale part \( M = \int_0^t \sigma(X_s) \, dB_s \), and finite variation part \( V = \int_0^t b(X_s) \, ds \). Note that the continuity of \( \sigma, X, b \) gives that almost surely, for all \( t \geq 0, s \to \sigma(X_s) + b(X_s) \) is locally bounded.

1. Let \( Z \) be a continuous semi-martingale such that \( \langle Z \rangle = \int_0^t \varphi \, ds \) for a progressive process \( \varphi \) such that \( 0 \leq \varphi \leq C|Z| \) for some constant \( C < \infty \). Prove that for all \( t \geq 0 \) and all \( a > 0, \)
\[
\mathbb{E} \int_0^t 1_{|0 < Z_s \leq a|} \, d|Z|_s \leq Ct.
\]
2. Deduce from the preceding question that for all \( t \geq 0 \),
\[
\lim_{n \to \infty} n \mathbb{E} \int_0^t 1_{|Z_s| \leq \frac{1}{n}} \, d\langle Z \rangle_s = 0.
\]

3. For all \( n \geq 1 \), \( x \in \mathbb{R} \), let us define \( g_n(x) = 2n(1 + nx)1_{x \in [-\frac{1}{n}, 0]} + 2n1_{x=0} + 2n(1 - nx)1_{x \in (0, \frac{1}{n})} \).
Let \( f_n : \mathbb{R} \to \mathbb{R} \) be the twice differentiable function such that \( f''_n = g_n \) and \( f_n(0) = f'_n(0) = 0 \).
Show that for all \( x \in \mathbb{R} \), the following properties hold true:

(a) \( f'_n(x) \in [-1, 1] \) and \( \lim_{n \to \infty} f'_n(x) = \text{sign}(x) = 1_{x>0} - 1_{x<0} \)
(b) \( |f_n(x)| \leq |x| \) and \( \lim_{n \to \infty} f_n(x) = |x| \).

4. By using Itô formula, prove that for all continuous semi-martingale \( Z = (Z_t)_{t \geq 0} \), all \( t \geq 0 \),
\[
\int_0^t 1_{Z_s = 0} \, d\langle Z \rangle_s = 0.
\]

5. From now on, let \( X \) and \( X' \) be two solutions of (SDE) on \( (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}) \) and with respect to the Brownian motion \( B \). Show that for all \( t \geq 0 \),
\[
\langle X - X' \rangle_t = \int_0^t (\sigma(X_s) - \sigma(X'_s))^2 \, ds.
\]

6. By using the assumption on \( \sigma \), deduce from the preceding questions that for all \( t \geq 0 \),
\[
\lim_{n \to \infty} \mathbb{E} \int_0^t g_n(X_s - X'_s) \, d\langle X - X' \rangle_s = 0.
\]

7. Set \( Z = X - X' \). From now on, let \( T \) be a stopping time such that the semi-martingale \( (Z_{t \wedge T})_{t \geq 0} \) is bounded. By using notably the assumption on \( \sigma \), prove that for all \( t \geq 0 \), \( n \geq 1 \),
\[
\mathbb{E}(f_n(Z_{t \wedge T})) = \mathbb{E}(f_n(Z_0)) + \mathbb{E} \int_0^{t \wedge T} f'_n(Z_s)(b(X_s) - b(X'_s)) \, ds + \frac{1}{2} \mathbb{E} \int_0^{t \wedge T} f''_n(Z_s) \, d\langle Z \rangle_s.
\]

8. Deduce from the preceding questions and the assumption on \( b \) that for all \( t \geq 0 \),
\[
\mathbb{E}(|X_{t \wedge T} - X'_{t \wedge T}|) = \mathbb{E}(|X_0 - X'_0|) + \mathbb{E} \int_0^{t \wedge T} (b(X_s) - b(X'_s)) \text{sign}(X_s - X'_s) \, ds.
\]

9. By using the Grönwall lemma, deduce that if \( X_0 = X'_0 \), then \( X_t = X'_t \) for all \( t \geq 0 \).

**Elements of solution for Exercise 3.** The result is known as the Yamada–Watanabe criterion. This is a slightly more detailed version of [1, Exercise 8.14 pages 231–232].

1. We have, using the properties of \( Z \) and \( \varphi \),
\[
\int_0^t 1_{0 < |Z_s| \leq a} \, d\langle Z \rangle_s = \int_0^t \frac{1_{0 < |Z_s| \leq a}}{|Z_s|} \varphi_s \, ds \leq \int_0^t C \, ds = Ct.
\]

2. For all \( n \geq 1 \), we have \( n1_{0 < |Z_s| \leq \frac{1}{n}} \leq \frac{1_{0 < |Z_s| \leq \frac{1}{n}}}{|Z_s|} \leq 1_{0 < |Z_s| \leq 1} \), which is integrable on \([0, t] \) by the preceding question used with \( a = 1 \), and thus the desired result follows then by dominated convergence.

3. The function \( g_n \) is \( 0 \) on \((\infty, -\frac{1}{n}] \), then increases from \( 0 \) to \( 2 \) on \([-\frac{1}{n}, 0] \), then decreases from \( 2 \) to \( 0 \) on \([0, \frac{1}{n}] \), then stays at \( 0 \) on \([\frac{1}{n}, \infty) \). Since \( \int_{-\infty}^0 g_n(y) \, dy = 1 \), we have, for all \( x \in \mathbb{R} \),
\[
f'_n(x) = \int_0^x g_n(u) \, du, \quad \text{in such a way that } f'_n(0) = 0 \text{ and } f''_n = g_n.
\]
The function $f_n'$ is $-1$ on $(-\infty, -\frac{1}{n}]$, $0$ at $0$, and $1$ on $[\frac{1}{n}, +\infty)$. Also for all $x \in \mathbb{R}$,

$$\lim_{n \to \infty} f_n'(x) = 1_{x > 0} - 1_{x < 0} =: \text{sign}(x).$$

Next, for all $x \in \mathbb{R}$, we have

$$f_n(x) = \int_0^x f_n'(u)\,du \quad \text{in such a way that } f_n(0) = 0 \text{ and } f_n'' = g_n.$$

Since $g_n \geq 0$, we have that $f_n'$ is non-decreasing, and thus $f_n'$ takes actually its values in $[-1, 1]$, and is in particular bounded. It follows by dominated convergence that for all $x \in \mathbb{R}$,

$$\lim_{n \to \infty} f_n(x) = \int_0^x \lim_{n \to \infty} f_n'(u)\,du = \int_0^x \text{sign}(u)\,du = |x|.$$

Finally, for all $x \in \mathbb{R}$, $|f_n(x)| \leq f_0|x|\,du = |x|$.

4. The Itô formula for function $f_n$ of question 3 and semi-martingale $Z$ gives, for all $t \geq 0$,

$$f_n(Z_t) = f_n(Z_0) + \int_0^t f_n'(Z_s)\,dZ_s + \frac{1}{2} \int_0^t f_n''(Z_s)\,d[Z_s].$$

Since $|f_n''| \leq 1$ and $\lim_{n \to \infty} \frac{f_n''(x)}{2n} = 1_{x=0}$ for all $x \in \mathbb{R}$, by dominated convergence,

$$\lim_{n \to \infty} \frac{1}{2n} \int_0^t f_n''(Z_s)\,d[Z_s] = \int_0^t 1_{Z_s=0}\,d[Z_s] \quad \text{a.s.}$$

On the other hand, since by question 3, $\lim_{n \to \infty} \frac{f_n(x)}{2n} = 0$ for all $x \in \mathbb{R}$, it follows that a.s.

$$\lim_{n \to \infty} \frac{f_n(Z_t)}{2n} = 0 \quad \text{and} \quad \lim_{n \to \infty} \frac{f_n(Z_0)}{2n} = 0.$$

Finally since by question 3, $\frac{1}{2n} |f_n'(x)| \leq \frac{1}{2n} \leq 1$ and $\lim_{n \to \infty} \frac{1}{2n} |f_n'(x)| = 0$ for all $x \in \mathbb{R}$, dominated convergence for stochastic integrals gives

$$\frac{1}{2n} \int_0^t f_n'(Z_s)\,dZ_s \xrightarrow{p}{n \to \infty} 0.$$

5. Since $X$ and $X'$ are both solutions on the same space and for the same Brownian motion, we have, for all $X_0$ and $X'_0$ and for all $t \geq 0$,

$$X_t - X'_t = X_0 - X'_0 + \int_0^t (\sigma(X_s) - \sigma(X'_s))\,dB_s + \int_0^t (b(X_s) - b(X'_s))\,ds.$$

The righthand side gives the canonical decomposition of the semi-martingale $X - X'$. In this decomposition, the first integral is the local martingale part, and

$$\langle X - X' \rangle = \int_0^t (\sigma(X_s) - \sigma(X'_s))^2\,ds.$$

6. By assumption on $\sigma$, the process $\varphi = (\sigma(X) - \sigma(X'))^2$ satisfies $0 \leq \varphi \leq C|X - X'|$. We can then use question 5 and question 2 with $Z = X - X'$ to get, for all $t \geq 0$,

$$\lim_{n \to \infty} nE\int_0^t 1_{0<|Z_s|\leq \frac{1}{n}}\,d[Z_s] = 0.$$

If $g_n$ is as in question 3, then for all $x \in \mathbb{R}$, $0 \leq g_n(x) \leq 2n1_{0<|x|\leq \frac{1}{n}} + 2n1_{x=0}$. Thus, for all $t \geq 0$,

$$0 \leq E\int_0^t g_n(Z_s)\,d[Z_s] \leq 2nE\int_0^t 1_{0<|Z_s|\leq \frac{1}{n}}\,d[Z_s] + 2nE\int_0^t 1_{Z_s=0}\,d[Z_s] \xrightarrow{n \to \infty} 0,$$

where we have used question 2 and question 4.
7. The Itô formula for the \( \mathcal{C}^2 \) function \( f_n \) and the continuous semi-martingale \( Z_T \) gives

\[
f_n(Z_T^n) = f_n(Z_0^n) + \int_0^{T \wedge T} f_n'(Z_s) dB_s + \frac{1}{2} \int_0^{T \wedge T} f_n''(Z_s) d\langle Z \rangle_s,
\]

and since \( dZ_s = (\sigma(X_s) - \sigma(X'_s)) dB_s + (b(X_s) - b(X'_s)) ds \), we get

\[
\int_0^{T \wedge T} f'(Z_s) dZ_s = \int_0^{T \wedge T} f'(Z_s)(\sigma(X_s) - \sigma(X'_s)) 1_{s \leq T} dB_s + \int_0^{T \wedge T} f'(Z_s)(b(X_s) - b(X'_s)) ds.
\]

Now, by the assumptions on \( \sigma \) and \( T \), we get

\[
|\sigma(X_s) - \sigma(X'_s)| 1_{s \leq T} \leq C \sqrt{|Z_s^T|} \leq C'.
\]

This boundedness, together with the one of \( f_n'' \), imply that the first integral in the right hand side above (the \( dB_s \) one) is a martingale. Since this martingale is issued from the origin, its expectation vanishes for all times. On the other hand, since \( f_n \) is continuous and \( Z_T \) is bounded, the random variables \( f_n(Z_T^n) \) and \( f_n(Z_T^0) \) are integrable. All in all, we obtain

\[
\mathbb{E}(f_n(Z_T^n)) = \mathbb{E}(f_n(Z_T^0)) + \mathbb{E} \left[ \int_0^{T \wedge T} f_n'(Z_s)(b(X_s) - b(X'_s)) ds + \frac{1}{2} \int_0^{T \wedge T} f_n''(Z_s) d\langle Z \rangle_s \right].
\]

8. Since \( f_n'' = g_n \geq 0 \), we get, by using question 6, that

\[
0 \leq \mathbb{E} \left[ \int_0^{T \wedge T} f_n''(Z_s) d\langle Z \rangle_s \right] \leq \mathbb{E} \left[ \int_0^{T \wedge T} g_n(Z_s) d\langle Z \rangle_s \right] \xrightarrow{n \to \infty} 0.
\]

On the other hand, by the assumption on \( b \) and the boundedness of \( Z_T \), we have, on \( \{s \leq T\} \),

\[
|b(X_s) - b(X'_s)|^2 \leq C^2|X_s - X'_s|^2 = C^2|Z_s^T| \leq C'.
\]

But since \( f_n' \) is bounded (takes its values in \([-1, 1]\)), we get, by dominated convergence

\[
\lim_{n \to \infty} \int_0^{T \wedge T} f_n'(Z_s)(b(X_s) - b(X'_s)) ds = \int_0^{T \wedge T} \text{sign}(Z_s)(b(X_s) - b(X'_s)) ds.
\]

Finally, since \( Z_T^0 \) is bounded, and since from question 3, for all \( x \in \mathbb{R} \), \( |f_n(x)| \leq |x| \) and \( \lim_{n \to \infty} f_n(x) = |x| \), we get, by dominated convergence, \( \lim_{n \to \infty} \mathbb{E}(f_n(Z_T^n)) = \mathbb{E}(|Z_T^0|) \). Finally

\[
\mathbb{E}(|X_{T \wedge T} - X'_{T \wedge T}|) = \mathbb{E}(|X_0 - X'_0|) + \mathbb{E} \left[ \int_0^{T \wedge T} (b(X_s) - b(X'_s)) \text{sign}(X_s - X'_s) ds \right].
\]

9. From the preceding question, we get, by using the assumption on \( b \),

\[
\alpha(t) = \mathbb{E}(|X_t - X'_t|) \leq \mathbb{E}(|X_0 - X'_0|) + C \mathbb{E} \left[ \int_0^t |X_s - X'_s| ds \right] = \alpha(0) + C \int_0^t \alpha(s) ds.
\]

By the Grönwall lemma, we obtain \( \alpha(t) \leq \alpha(0) e^{Ct} \) for all \( t \geq 0 \). It follows that if \( \alpha(0) = 0 \) then \( \alpha(t) = 0 \) for all \( t \geq 0 \). This means that if \( X_0 = X'_0 \) then \( X_{t \wedge T} = X'_{t \wedge T} \) for all \( t \geq 0 \). By writing this for \( t \in Q_+ \), and by taking \( T = T_m \) such that \( \lim_{m \to \infty} T_m = +\infty \) almost surely, we get that \( X_t = X'_t \) for all \( t \in Q_+ \), and thus for all \( t \geq 0 \) since \( X \) and \( X' \) are continuous.

References