Classical beta ensembles at high temperature

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Beta ensembles in the real line

- Joint probability density function of the eigenvalues

$$(\lambda_1, \dots, \lambda_N) \propto \prod_{i<j} |\lambda_i - \lambda_j|^{\beta} \prod_{l=1}^{N} w(\lambda_l),$$

where $w \colon \mathbb{R} \to [0, \infty)$ is a weight.

- Classical weights

$$w(\lambda) = \begin{cases} e^{-\frac{\lambda^2}{2}}, & \text{Gaussian weight,} \\ \lambda^{\alpha} e^{-\lambda} \mathbf{1}_{(0,\infty)}(\lambda), & \text{Laguerre weight } (\alpha > -1), \\ \lambda^{a}(1-\lambda)^{b} \mathbf{1}_{(0,1)}(\lambda), & \text{Jacobi weight } (a, b > -1). \end{cases}$$

- Gaussian weight: $\beta = 1, 2, 4 \leftrightarrow$ GOE, GUE, GSE
- Laguerre weight: $\beta = 1, 2 \leftrightarrow$ Wishart ensemble, Laguerre ensemble
- Jacobi weight: $\beta = 1, 2 \leftrightarrow$ multivariate analysis of variance (MANOVA), or double Wishart

Beta ensembles in the real line

- One dimensional Coulomb gas at the inverse temperature β

$$(\lambda_1, \dots, \lambda_N) \propto \exp\Big(\frac{\beta}{2}\sum_{i\neq j}\log|\lambda_j - \lambda_i| + \sum_l \log w(\lambda_l)\Big).$$

- For fixed β, under a suitable scaling (by $(N\beta/2)^{-1}$ for Gaussian beta ensembles), the empirical distribution

$$L_N = \frac{1}{N}\sum_{i=1}^N \delta_{\lambda_i}$$

converges to a limit distribution (Gaussian: semi-circle distribution, Laguerre: Marchenko–Pastur distributions, Jacobi: Kesten–Mckey distributions). These facts can be shown

- by analyzing the joint density (Johansson (1998)),
- via random tridiagonal matrix models + combinatoric arguments (Dumitriu & Edelman (2006), Dumitriu & Paquette (2012)),
- via random tridiagonal matrix models + spectral measures (T. (2016)).

- For fixed β, under a suitable scaling, the empirical distribution $L_N = \frac{1}{N}\sum_{i=1}^N \delta_{\lambda_i}$ converges to a limit distribution. These laws still hold when $N\beta \to \infty$.
- What happens in the regime $N\beta \to const \in (0, \infty)$? Known results:
  - + Gaussian: Allez, Bouchaud & Guionnet (2012), T. & Shirai (2015)
  - + Laguerre: Allez, Bouchaud, Majumdar & Vivo (2013)
  - + The works of Allez et al. are based on joint density
- This talk introduces an approach by T. & Shirai (2015) based on random tridiagonal matrix models and duality relation of beta ensembles, which is applicable to beta Laguerre and Jacobi ensembles as well.

Gaussian beta ensembles

- (Scaled) Gaussian beta ensembles

$$(\lambda_1, \dots, \lambda_N) \propto \prod_{i<j} |\lambda_i - \lambda_j|^{\beta} \exp(-\frac{\beta N}{4} \sum_{j=1}^{n} \lambda_j^2).$$

- Tridiagonal matrix model (Dumitriu & Edelman (2002))

$$H_n^{(\beta)} = \frac{1}{\sqrt{\beta N}} \begin{pmatrix} \mathcal{N}(0,2) & \chi_{(N-1)\beta} & & \\ \chi_{(N-1)\beta} & \mathcal{N}(0,2) & \chi_{(N-2)\beta} & \\ & \ddots & \ddots & \ddots \\ & & \chi_{\beta} & \mathcal{N}(0,2) \end{pmatrix}.$$

Here $\chi_k^2 = \underbrace{\mathcal{N}(0,1)^2 + \cdots + \mathcal{N}(0,1)^2}_{k}, k = 1, 2, \dots$. (Trotter (1984) introduced tridiagonal model for GUE.)

Jacobi matrices

μ: nontrivial prob. meas. on $\mathbb{R}$ s.t. $\int |x|^k d\mu(x) < \infty, k = 0, 1, \dots$.

- $\{1, x, x^2, \dots\}$ are independent in $L^2(\mathbb{R}, \mu)$.
- Define $\{P_n(x)\}_{n=0}^{\infty}$ as $\begin{cases} P_n(x) = x^n + \text{lower order}, \\ P_n \perp x^j, \quad j = 0, \dots, n-1. \end{cases}$
- $p_n := P_n / \|P_n\|_{L^2}$.

Theorem

(i) $xp_n(x) = b_{n+1}p_{n+1}(x) + a_{n+1}p_n(x) + b_n p_{n-1}(x), \quad n = 0, 1, \dots,$

where $b_{n+1} = \frac{\|P_n\|}{\|P_{n+1}\|}, a_{n+1} = \frac{\langle P_n, xP_n \rangle}{\|P_n\|^2}, P_{-1} \equiv 0.$

(ii) *Multiplication by x in the orthonormal set $\{p_j\}$ has the matrix*

$$J = \begin{pmatrix} a_1 & b_1 & & \\ b_1 & a_2 & b_2 & \\ & \ddots & \ddots & \ddots \end{pmatrix}, \quad Jp = xp, p = (p_0, p_1, \dots,)^t.$$

Spectral measures of Jacobi matrices

- Given a Jacobi matrix J, finite or infinite

$$J = \begin{pmatrix} a_1 & b_1 & & \\ b_1 & a_2 & b_2 & \\ & \ddots & \ddots & \ddots \end{pmatrix}, \quad a_i \in \mathbb{R}, b_i > 0.$$

- There is a measure μ on $\mathbb{R}$, called spectral measure of (J, e_1), s.t.

$$\langle \mu, x^k \rangle = (J^k e_1, e_1) = J^k(1,1), k = 0, 1, \dots$$

- Uniqueness is equivalent to the essential seft-adjointness of J on $\ell^2(\mathbb{N})$.

Some examples of Jacobi matrices

- Semicircle distribution

$$sc(x) = \frac{1}{2\pi}\sqrt{4 - x^2}\mathbf{1}_{[-2,2]}(x), \leftrightarrow J = \begin{pmatrix} 0 & 1 & & \\ 1 & 0 & 1 & \\ & \ddots & \ddots & \ddots \end{pmatrix}.$$

- Standard Gaussian distribution

$$\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \leftrightarrow J = \begin{pmatrix} 0 & \sqrt{1} & & & \\ \sqrt{1} & 0 & \sqrt{2} & & \\ & \sqrt{2} & 0 & \sqrt{3} & \\ & & \ddots & \ddots & \ddots \end{pmatrix}.$$

- Laguerre weights $\frac{1}{\Gamma(\alpha+1)} x^\alpha e^{-x} \mathbf{1}_{(0,\infty)}(x), (\alpha > -1)$

$$J = \begin{pmatrix} \sqrt{\alpha+1} & & & \\ \sqrt{1} & \sqrt{\alpha+2} & & \\ & \sqrt{2} & \sqrt{\alpha+3} & \\ & & \ddots & \ddots \end{pmatrix} \begin{pmatrix} \sqrt{\alpha+1} & \sqrt{1} & & \\ & \sqrt{\alpha+2} & \sqrt{2} & \\ & & \sqrt{\alpha+3} & \sqrt{3} \\ & & & \ddots & \ddots \end{pmatrix}.$$

Finite Jacobi matrices

- μ: trivial prob. meas.,i.e.,

$$\mu = \sum_{j=1}^{N} q_j^2 \delta_{\lambda_j}, \quad \begin{cases} \{\lambda_j\} : \text{distinct}, \\ \sum q_j^2 = 1, q_j > 0. \end{cases}$$

- $\{x^j\}_{j=0}^{N-1}$: independent in $L^2(\mathbb{R}, \mu)$. Define $P_0, \dots, P_{N-1}$. $p_n := P_n / \|P_n\|$;
-

$$J = \begin{pmatrix} a_1 & b_1 & & \\ b_1 & a_2 & b_2 & \\ & \ddots & \ddots & \ddots \\ & & b_{N-1} & a_N \end{pmatrix}$$

- $\{\lambda_j\}_{j=1}^{N}$: the eigenvalues of J, $\{v_j\}_{j=1}^{N}$: the corresponding normalized eigenvectors. Then

$$\mu = \sum_{j=1}^{N} |v_j(1)|^2 \delta_{\lambda_j} = \sum_{j=1}^{N} q_j^2 \delta_{\lambda_j}.$$

GβE, Jacobi/tridiagonal matrix model (Dumitriu & Edelman 2002)

- $H_N^{(\beta)} := \frac{1}{\sqrt{\beta N}} \begin{pmatrix} \mathcal{N}(0,2) & \chi_{(N-1)\beta} & & \\ \chi_{(N-1)\beta} & \mathcal{N}(0,2) & \chi_{(N-2)\beta} & \\ & \ddots & \ddots & \ddots \\ & & \chi_\beta & \mathcal{N}(0,2) \end{pmatrix}$
- The eigenvalues of $H_N^{(\beta)}$ have (scaled) GβE distribution,i.e.,

$$(\lambda_1, \dots, \lambda_N) \propto \prod_{i<j} |\lambda_i - \lambda_j|^\beta \exp(-\frac{\beta N}{4} \sum_{j=1}^N \lambda_j^2).$$

- $H_N^{(\beta)}$ is $1-1$ correspondence with the spectral measure

$$\mu_N^{(\beta)} = \sum_{j=1}^N q_j^2 \delta_{\lambda_j},$$

$(q_1, \dots, q_N)$ is distributed as $(\chi_\beta, \dots, \chi_\beta)$ normalized to unit length, independent of $(\lambda_1, \dots, \lambda_N)$.

Limiting behaviours of GβE

- The spectral measure and the empirical distribution have the same mean

$$\mathbf{E}\Big[\int f d\mu_N^{(\beta)}\Big] = \mathbf{E}\Big[\sum q_j^2 f(\lambda_j)\Big] = \sum \mathbf{E}[q_j^2]\mathbf{E}[f(\lambda_j)]$$
$$= \frac{1}{N}\sum \mathbf{E}[f(\lambda_j)] = \mathbf{E}\Big[\int f dL_N\Big].$$

- The limiting behavior of spectral measures follows directly from those of the entries

$$\frac{1}{\sqrt{\beta N}}\begin{pmatrix} \mathcal{N}(0,2) & \chi_{(N-1)\beta} & & \\ \chi_{(N-1)\beta} & \mathcal{N}(0,2) & \chi_{(N-2)\beta} & \\ & \ddots & \ddots & \ddots \\ & & \chi_\beta & \mathcal{N}(0,2) \end{pmatrix} \to \begin{cases} \begin{pmatrix} 0 & 1 & & \\ 1 & 0 & 1 & \\ & \ddots & \ddots & \ddots \end{pmatrix} & \text{as } N\beta \to \infty, \\ \frac{1}{\sqrt{2c}}\begin{pmatrix} \mathcal{N}(0,2) & \chi_{2c} & & \\ \chi_{2c} & \mathcal{N}(0,2) & \chi_{2c} & \\ & \ddots & \ddots & \ddots \end{pmatrix} & \text{as } N\beta \to 2c. \end{cases}$$

This shows the convergence of the spectral measures, and hence, the convergence of the mean of spectral measures.

Gaussian beta ensembles at zero temperature and duality

- Gaussian beta ensembles at zero temperature

$$\frac{1}{\sqrt{N\beta}}\begin{pmatrix} \mathcal{N}(0,2) & \chi_{(N-1)\beta} & & \\ \chi_{(N-1)\beta} & \mathcal{N}(0,2) & \chi_{(N-2)\beta} & \\ & \ddots & \ddots & \ddots \\ & & \chi_{\beta} & \mathcal{N}(0,2) \end{pmatrix} \xrightarrow{\beta\to\infty} \frac{1}{\sqrt{N}}\begin{pmatrix} 0 & \sqrt{N-1} & & \\ \sqrt{N-1} & 0 & \sqrt{N-2} & \\ & \ddots & \ddots & \ddots \\ & & \sqrt{1} & 0 \end{pmatrix}$$

- Duality. Let $m_p(N,\kappa) = \mathbf{E}\Big[\int x^{2p} dL_n\Big] = \mathbf{E}\Big[\int x^{2p} d\mu_N^{(\beta)}\Big], \kappa = \frac{\beta}{2}$. Then

$$m_p(N,\kappa) = (-\kappa)^p m_p(-\kappa N, \kappa^{-1}).$$

- As $N\beta \to 2c$, or $N\kappa \to c$, $m_p(N,\kappa) \to m_p(-c, @\infty)$. Then the Jacobi matrix of the limiting distribution becomes

$$\frac{1}{\sqrt{c}}\begin{pmatrix} 0 & \sqrt{c+1} & & \\ \sqrt{c+1} & 0 & \sqrt{c+2} & \\ & \sqrt{c+2} & 0 & \sqrt{c+3} \\ & \ddots & \ddots & \ddots \end{pmatrix} \leftrightarrow \text{associated Hermite polynomials}$$

Conclusion

- Beta ensembles with classical weights

$$(\lambda_1, \dots, \lambda_N) \propto \prod_{i<j} |\lambda_i - \lambda_j|^{\beta} \prod_{l=1}^{N} w(\lambda_l), \quad w(\lambda) = \begin{cases} e^{-\frac{\lambda^2}{2}}, \\ \lambda^{\alpha} e^{-\lambda} \mathbf{1}_{(0,\infty)}(\lambda), \\ \lambda^a (1-\lambda)^b \mathbf{1}_{(0,1)}(\lambda). \end{cases}$$

In the regime that $N\beta \to 2c$, the empirical distribution converges to the probability measure of c-associated orthogonal polynomials. Theory of associated orthogonal polynomials helps to derive the explicit formula for the limiting distribution: Askey & Wimp (1984), Ismail, Letessier & Valent (1988), Ismail & Masson (1991).

* For Laguerre weights $\frac{1}{\Gamma(\alpha+1)} x^{\alpha} e^{-x} \mathbf{1}_{(0,\infty)}(x), (\alpha > -1)$, the associated version

$$J_c = \begin{pmatrix} \sqrt{c+\alpha+1} & & & \\ \sqrt{c+1} & \sqrt{c+\alpha+2} & & \\ & \sqrt{c+2} & \sqrt{c+\alpha+3} & \\ & & \ddots & \ddots \end{pmatrix} \begin{pmatrix} \sqrt{c+\alpha+1} & \sqrt{c+1} & & \\ & \sqrt{c+\alpha+2} & \sqrt{c+2} & \\ & & \sqrt{c+\alpha+3} & \sqrt{c+3} \\ & & & \ddots & \ddots \end{pmatrix}.$$

Conclusion

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$$(\lambda_1, \dots, \lambda_N) \propto \prod_{i<j} |\lambda_i - \lambda_j|^{\beta} \prod_{l=1}^{N} w(\lambda_l), \quad w(\lambda) = \begin{cases} e^{-\frac{\lambda^2}{2}}, \\ \lambda^{\alpha} e^{-\lambda} \mathbf{1}_{(0,\infty)}(\lambda), \\ \lambda^{a}(1-\lambda)^{b} \mathbf{1}_{(0,1)}(\lambda). \end{cases}$$

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? How about other weights?

- Gaussian beta ensembles in the regime $N\beta \to 2c$: the local statistics converges to a homogeneous Poisson point process (Benaych-Georges & Péché (2015), Nakano & T. (2016)).
- Gaussian beta ensembles in the regime $N\beta = o((\log N)^{-1})$: the largest eigenvalue converges to the Gumbel distribution (Pakzad (2018)).

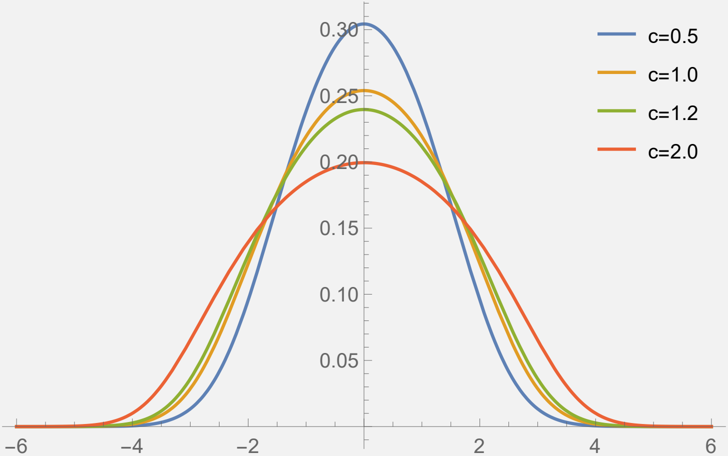

Figure: Density of the spectral measure of $\begin{pmatrix} 0 & \sqrt{c+1} & & \\ \sqrt{c+1} & 0 & \sqrt{c+2} & \\ & \sqrt{c+2} & 0 & \sqrt{c+3} \\ & \ddots & \ddots & \ddots \end{pmatrix}$

Thank you very much for your attention!