Unbounded largest eigenvalues of sample covariance matrices: Asymptotics, fluctuations and applications to long memory stationary processes

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based on a joint work with F. Merlevède and J. Najim

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$$\mu^{T_N} := \frac{1}{N} \sum_{i=1}^{N} \delta_{\lambda_i(T_N)} \xrightarrow{\mathcal{D}} \mu \quad \text{with} \quad \sup \operatorname{supp} \mu = \infty\,?$$

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- This question was raised in the study of *long memory stationary process*. If a process $(\mathcal{X}_t)_{t \in \mathbb{Z}}$ satisfies
$$\mathbb{E}\mathcal{X}_t = 0, \quad \operatorname{Cov}(\mathcal{X}_{t+h}, \mathcal{X}_t) = \gamma(h), \quad \forall t, h \in \mathbb{Z}$$
with the autocovariance function γ satisfying
$$\sum_{h \in \mathbb{Z}} |\gamma(h)| = \infty.$$
Then $(\mathcal{X}_t)_{t \in \mathbb{Z}}$ is a (centered) long memory stationary process.

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where $\vec{X}_i$ are i.i.d obersations of $\begin{pmatrix}\mathcal{X}_1 & \cdots & \mathcal{X}_N\end{pmatrix}^\top$ drawn from a centered long memory stationary process $(\mathcal{X}_t)_{t\in\mathbb{Z}}$.

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- These questions are tightly related to the asymptotic properties of the population covariance matrix, which is the following Toeplitz matrix:

$$T_N := \mathrm{Cov} \begin{pmatrix} \mathcal{X}_1 \\ \vdots \\ \mathcal{X}_N \end{pmatrix} = (\gamma(i-j))_{i,j=1}^{N},$$

with $\mu^{T_N} \xrightarrow{\mathcal{D}} \mu$ and $\sup \operatorname{supp} \mu = \infty$ as natural properties due to the long memory of the process.

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- After these, the asymptotics and joint fluctuations of any p (a fixed integer) top eigenvalues of

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  are studied.
- In the general model S_N, the fluctuations depend not only on the entry distribution but also on the eigenvectors of T_N. But for some Toeplitz T_N, the universality holds.