Fluctuations of stationary KPZ models and multiple integral

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(Based on collaborations with T. Imamura, M. Mucciconi)

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Reference: arXiv: 1701.05991

1. Introduction: Tracy-Widom distributions

GUE (Gaussian unitary ensemble). For H:N-dim hermitian matrix

$$P(H)dH \propto e^{-\mathrm{Tr}H^2} dH$$

GUE Tracy-Widom distribution for the largest e.v. $x_{\max}$

$$\lim_{N\to\infty} \mathbb{P}\left[\frac{x_{\max} - \sqrt{2N}}{2^{-1/2}N^{-1/6}} \le s\right] = F_2(s) = \det(1 - P_s K_2 P_s)_{L^2(\mathbb{R})}$$

where P_s: projection onto $[s, \infty)$ and K_2 is the Airy kernel

$$K_2(x,y) = \int_0^\infty \mathrm{d}\lambda \mathrm{Ai}(x+\lambda)\mathrm{Ai}(y+\lambda)$$

The joint eigenvalue (x_i) density is (with $\Delta(x)$ Vandelmonde)

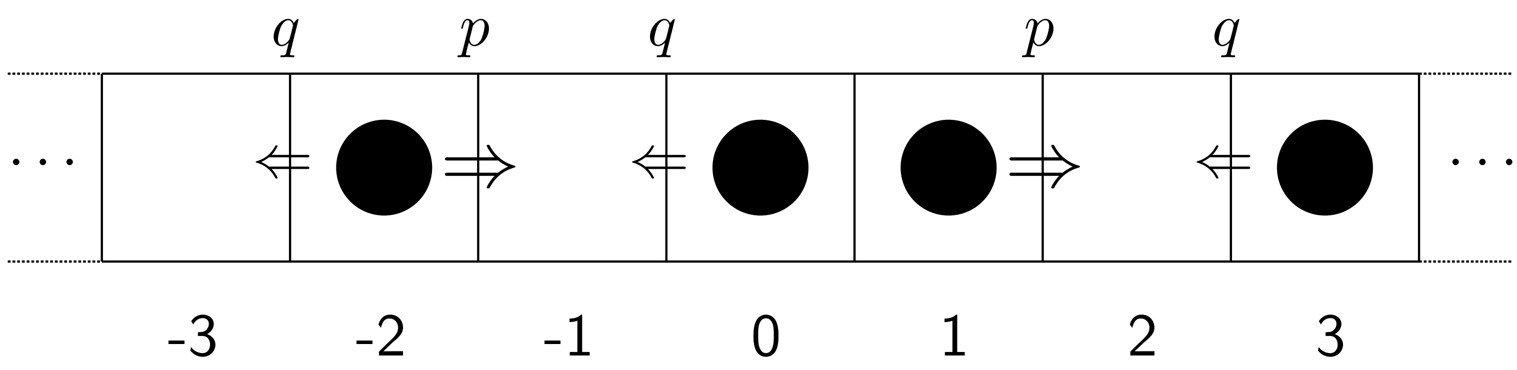
$$\frac{1}{Z}\Delta(x)^2 \prod_i e^{-x_i^2}$$

Determinantal process

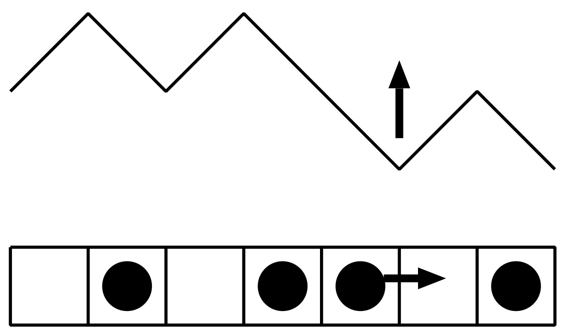
- The point process (random point field) whose correlation functions are written in the form of determinants are called a determinantal process.
- Once we have a measure in the form of a product of two determinants (in many cases related to non-intersecting paths), there is an associated determinantal process and the Fredholm determinant appears naturally.
- Eigenvalues of the GUE is determinantal.

ASEP

ASEP = asymmetric simple exclusion process



- TASEP(Totally ASEP, $q = 0, p = 1$). Bernoulli is stationary.
- $N(x,t)$: Integrated current at $(x, x+1)$ up to time t
  $\Leftrightarrow$ height for surface growth
- ASEP is in the KPZ universality class
  for surface growth

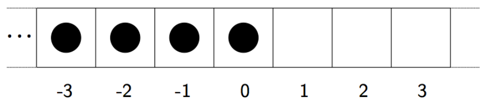


TASEP current fluctuations

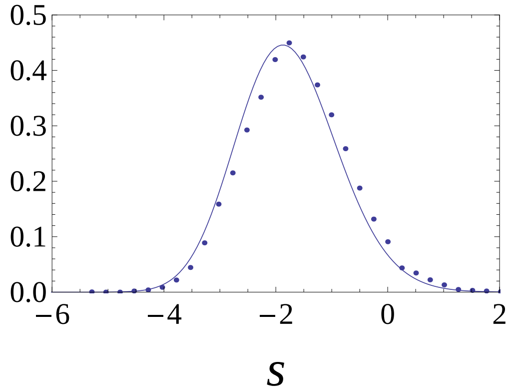
Theorem. (2000 Johansson)

For the step i.c. (only all negative sites are occupied at $t = 0$)

$$\lim_{t\to\infty} \mathbb{P}\left[\frac{N(0,t) - t/4}{-2^{-4/3}t^{1/3}} \le s\right] = F_2(s)$$



where $F_2(s)$ is the GUE Tracy-Widom distribution.



TASEP is related to the Schur measure

$$\frac{1}{Z} s_\lambda(a) s_\lambda(b)$$

s_λ can be written as a single det ($\Rightarrow$ determinantal process).

Stationary case was also studied (2006 Ferrari-Spohn based on 2004 Imamura-TS). The limit distribution is Baik-Rains dist. The stationary TASEP and Schur measure don't coincide for finite t.

Baik-Rains distribution

$$F_\omega(s) := \frac{\partial}{\partial s}\nu_\omega(s)$$

where

$$\nu_\omega(s) = F_2(s)\left(s - \omega^2 - \sum_{\substack{i,j=1\\(i,j)\neq(1,1)}}^{2} \int_s^\infty d\xi \mathcal{B}_\omega^{(i)}(\xi)\mathcal{B}_{-\omega}^{(j)}(\xi) - \int_s^\infty d\xi (\rho_{\mathcal{A}}\mathcal{A}\mathcal{B}_\omega)(\xi)\mathcal{B}_{-\omega}(\xi)\right)$$

where $F_2 = \det(1-\mathcal{A})$ is the GUE Tracy-Widom distribution,

$$\mathcal{A}(\xi,\zeta) = K_2(\xi,\zeta)1_{\geq s}, \quad \rho_{\mathcal{A}} = (1-\mathcal{A})^{-1},$$

$$\mathcal{B}_\omega^{(1)}(\xi) = e^{\omega^3/3-\omega\xi}, \quad \mathcal{B}_\omega^{(2)}(\xi) = -\int_0^\infty dz e^{\omega z}\mathrm{Ai}(\xi+z),$$

$$\mathcal{B}_\omega(\xi) = \mathcal{B}_\omega^{(1)}(\xi) + \mathcal{B}_\omega^{(2)}(\xi).$$

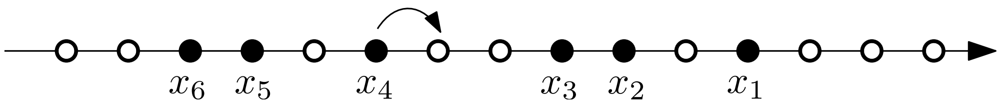
2000 Baik-Rains for PNG model (a different representation).

q-TASEP

A version of TASEP. $0 < q < 1$. $x_i(t)$: the ith particle position.

Hopping rate is $1 - q^{\mathrm{gap}}$ with "gap" $= x_{i-1} - x_i - 1$.

Step i.c. $(x_i(0) = -i, i \geq 1)$



Borodin-Corwin found a connection to Macdonald measure (or q-Whittaker) of the form,

$$\frac{1}{Z} P_\lambda(a) Q_\lambda(b).$$

$P_\lambda(a), Q_\lambda(b)$ are Macdonald functions for which a single det formula is not known. But using their properties one can find a formula for q-moment $\langle q^{k(x_N(t)+N)} \rangle$ and the q-Laplace transform $\langle \frac{1}{(\zeta q^{x_N(t)+N}; q)_\infty} \rangle$ is written as a Fredholm determinant, from which one can show Tracy-Widom limit in the long time limit.

Stationary: $\langle q^{k(x_N(t)+N)} \rangle$ diverges and determinant is invisible.

Our approach: Basic idea and two formulas

- Instead of relying on the moments, we study the distribution of a particle position directly.
- We first reduce the problem to that of N-particle q-TASEP with two sets of parameters $\{a_i\}, \{\alpha_i\}$. We still use (a two-sided version of) the q-Whittaker measure.
- To study a particle position, we rewrite the Cauchy identity.
- We use two formulas: Ramanujan's sum formula and Cauchy determinant for theta function (next slide).

Ramanujan's sum formula and Cauchy determinant for theta function

Ramanujan's sum formula

Theorem. For $|q| < 1, |b/a| < |z| < 1$,

$$\sum_{n\in\mathbb{Z}} \frac{(bq^n;q)_\infty}{(aq^n;q)_\infty} z^n = \frac{(az;q)_\infty(\frac{q}{az};q)_\infty(q;q)_\infty(\frac{b}{a};q)_\infty}{(a;q)_\infty(\frac{q}{a};q)_\infty(z;q)_\infty(\frac{b}{az};q)_\infty}$$

We introduce a modified Jacobi theta function

$$\theta(z) = (z, q/z; q)_\infty.$$

Also $\tilde{\theta}(1/z) = \frac{1}{\sqrt{z}}\tilde{\theta}(z)$ which satisfies $\tilde{\theta}(1/z) = \tilde{\theta}(z)$.

Cauchy determinant

Let $[x]$ satisfy $[-x] = -[x]$ and the Riemann relation

$$\begin{aligned}&[x+y][x-y][u+v][u-v] \\ = {}&[x+u][x-u][y+v][y-v] - [x+v][x-v][y+u][y-u]\end{aligned}$$

$[x]$ satisfying the above two relations is necessarily in the form $e^{ax^2+b}f(cx)$ where $f(x)$ is either $f(x) = x$, $\sin \pi x$ or $\sigma(x)$, the Weierstrass sigma function. $\tilde{\theta}(q^x)$ is an example of $[x]$.

Theorem. (1882 Frobenius) For $[x]$ above, the Cauchy determinant formula holds. With $B = \sum_i b_i, C = \sum_i c_i$,

$$\frac{[\lambda + B - C] \prod_{i<j} [b_i - b_j][c_j - c_i]}{[\lambda] \prod_{i,j} [b_i - c_j]} = \det \left(\frac{[\lambda + b_i - c_j]}{[\lambda][b_i - c_j]} \right)$$

Result: Fredholm det for the q-Laplace transform

Theorem. For N particle q-TASEP with parameters $\{a_i\}, \{\alpha_i\}$,

$$\left\langle \frac{1}{(\zeta q^{x_N(t)+N}; q)_\infty} \right\rangle = \det(1 - fK)_{L^2(\mathbb{Z})},$$

where $\zeta \neq q^n, n \in \mathbb{Z}, \langle \cdots \rangle$ is the expectation and

$$f(n) = \frac{1}{1 - q^n/\zeta}, \quad K(n,m) = \sum_{l=0}^{N-1} \phi_l(m)\psi_l(n)$$

$$\phi_l(n) = \sqrt{a_{l+1} - \alpha_{l+1}} \int_D dv \frac{e^{-vt}}{v^{n+N}} \frac{1}{v - a_{l+1}} \prod_{j=1}^{l} \frac{v - \alpha_j}{v - a_j} \prod_k \frac{(q\alpha_k/v; q)_\infty}{(qv/a_k; q)_\infty}$$

$$\psi_l(n) = \sqrt{a_{l+1} - \alpha_{l+1}} \int_{C_r} dz \frac{e^{zt} z^{n+N}}{z - \alpha_{l+1}} \prod_{j=1}^{l} \frac{z - a_j}{z - \alpha_j} \prod_k \frac{(qz/a_k; q)_\infty}{(q\alpha_k/z; q)_\infty}$$

Here C_r, D is around $\{0, \alpha_i q^j\}, \{a_i\}$ respectively.

Result: Long time limit for stationary q-TASEP

Thm. For the stationary q-TASEP, with the parameter $\alpha = q^\theta, \theta > 0$, by which one can control the density, we have

$$\lim_{N\to\infty} \mathbb{P}(x_N(\kappa N) > (\eta - 1)N - \gamma N^{1/3} s) = F_{\omega=0}(s), \quad \forall s \in \mathbb{R},$$

where κ, η, γ are given by

$$\kappa = \sum_{n=0}^{\infty} \frac{q^n}{(1-q^{\theta+n})^2}, \quad \eta = \sum_{n=0}^{\infty} \frac{q^{2\theta+2n}}{(1-q^{\theta+n})^2},$$

$$\gamma = \left(\sum_{n=0}^{\infty} \frac{q^{2\theta+2n}}{(1-q^{\theta+n})^3} \right)^{1/3}$$

Some comments

- The stationary ASEP was independently studied by Aggarwal. The approach is different. He uses analytic continuation at the level of higher spin six vertex model and takes a limiting procedure.
- A big advantage of our approach is that the same calculation can be directly applied to various cases in a parallel way.
- One can generalize our approach to study higher spin model (See the poster by Matteo Mucciconi).
- An important step is the mutliple integral formula.

Mutliple integral formula

In a derivation of our formula, we find a mutliple integral formula,

$$\langle\frac{1}{(\zeta q^{\lambda_N};q)_\infty}\rangle = \frac{(q;q)_\infty^N}{N!}\int_{\mathbb{T}^N}\prod_{i=1}^N \frac{dz_i}{z_i}\frac{\theta(\frac{\zeta A}{Z})}{\theta(\zeta)}\frac{\prod_{i\neq j}(z_i/z_j;q)_\infty}{\prod_{i,j}(a_i/z_j;q)_\infty}\prod_i \frac{\prod_j(\alpha_i/a_j;q)_\infty e^{z_i t}}{\prod_j(\alpha_i/z_j;q)_\infty e^{a_i t}}$$

- Various cases can be studied in a parallel fashion. For example the stationary TASEP can be studied simply by setting $q=0$ in the formulas and in our approach one can study the stationary TASEP directly for finite t (without approximation as in previous works).

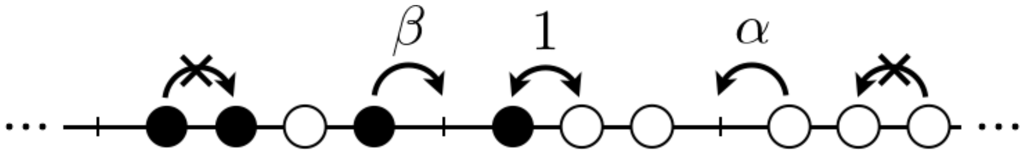
- Instead of trying to find a determinantal process, one may try to find a quantity which can be written in this type of multiple integral.

- $q=0$ case was also useful for studying a two species model.

An analysis of a two species exclusion process

Chen, de Gier, Hiki, Sasamoto, arXiv:1803.06829

Arndt-Heinzel-Rittenberg (AHR) model



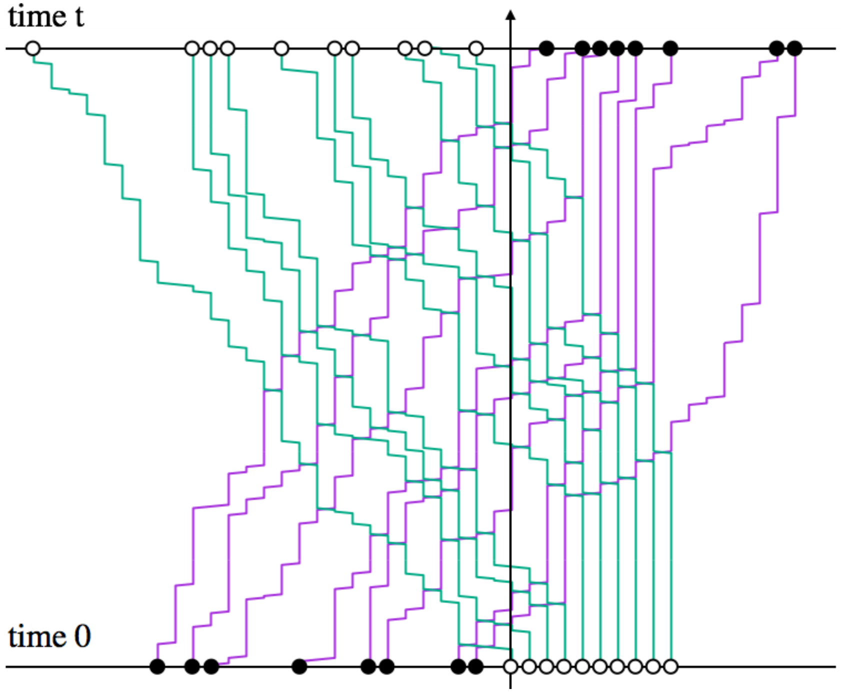
$\alpha + \beta = 1$

Nonlinear fluctuating hydrodynamics predicts that long time fluctuations of its "normal modes" are described by Tracy-Widom type distributions.

We have given a confirmation by exact calculation using Bethe ansatz. This is also the first result for multi-species model at the level of distribution.

$\rho - 1$ Step i.c.

Infinite $+$ particles ($\bullet$) with density ρ on the left and infinite $-$ particles ($\circ$) packed on the right.



Multiple-integral formula for current distribution

A step i.c. in which there are N + particles on the left with density ρ and M − particles are packed on the right.

When $\alpha = \beta = \frac{1}{2}$, for the currents $N_\pm(t)$ at the origin,

$$P_{N,M}(N_+(t) = N, N_-(t) = M) = \frac{1}{N!M!} \oint \prod_{j=1}^{N} \frac{\mathrm{d}z_j}{2\pi \mathrm{i}} \prod_{k=1}^{M} \frac{\mathrm{d}w_k}{2\pi \mathrm{i}} \; \mathrm{e}^{\Lambda t}$$

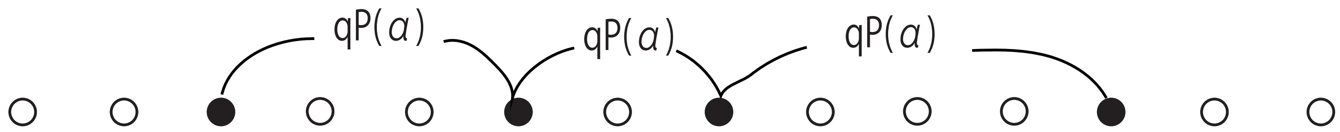
$$\frac{\rho^N \prod_{1 \le i < j \le N} (z_i - z_j)^2 \prod_{1 \le k < l \le M} (w_l - w_k)^2}{\prod_{j=1}^{N} (z_j - 1)^N (1 - (1-\rho) z_j) \prod_{k=1}^{M} (w_k - 1)^M \prod_{j=1}^{N} \prod_{k=1}^{M} \left(\frac{1}{2}(z_j + w_k) \right)}$$

with $\Lambda = \sum_{j=1}^{N} \frac{1}{2}(1/z_j - 1) + \sum_{k=1}^{M} \frac{1}{2}(1/w_k - 1)$.

2. Analysis of q-TASEP: 2.1 Stationary measure

- For $0 \le \alpha < 1$, "gaps" between the neighboring particles, $x_{i-1} - x_i - 1$, are independent and

$$\mathbb{P}[\text{a gap} = n] = (\alpha; q)_\infty \frac{\alpha^n}{(q;q)_n} \quad (q\text{-Poisson})$$

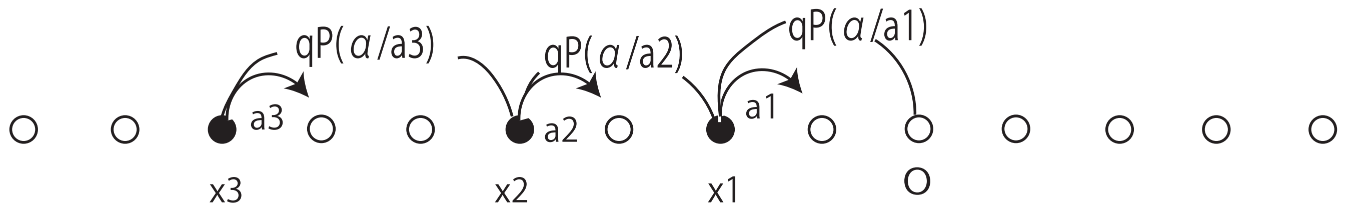


- $\rho = \rho(\alpha)$ and average current $j(\rho)$ are calculated explicitly.

- We can assume that there is a particle 1 at the origin initially at $t = 0$. We are interested in the distribution of the Nth particle position $x_N^{(0)}(t)$.

2.2 Reduction to N particle q-TASEP with a_i

The problem is reduced to the N-particle q-TASEP with hopping rates, $a_i(1-q^{\text{gap}}), 1 \le i \le N$, with the initial condition that the position of the first particle $x_1(0)$ and the gaps of the particles, $x_{i-1}(0) - x_i(0) - 1, 2 \le i \le N$, are independent and distributed as q-Poisson with parameter $\alpha/a_i, 1 \le i \le N$ with $a_i > \alpha$.

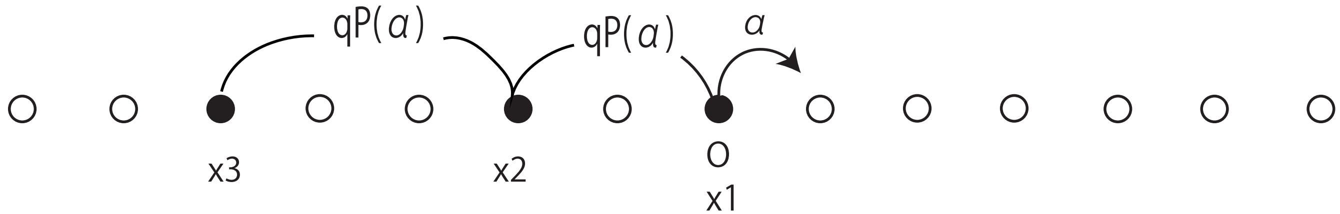


- Note $x_N(t) = x_N^{(0)}(t) - \chi - 1$, $\chi \sim q\mathrm{Po}(a_1/\alpha)$.

- To study $x_N^{(0)}(t)$ for stationary case, we set $a_1 = a, a_i = 1, 2 \le i \le N$ and then take $a \to \alpha$ limit.

Arguments to see the reduction

- When $a_i \equiv 1$, in the stationary measure with parameter α, the hopping rate of each particle is α. The right half can be replaced by a particle with hopping rate α (Burke theorem).



- In TASEP particles can not affect the particles ahead and hence it is enough to consider N particle q-TASEP with the first particle with hopping rate α starting at the origin and the gaps are independent and q-Poisson distributed with parameter α.

- We generalize the particle hopping rates to $a_i(1-q^{\mathsf{gap}})$ $i \geq 1$. $a_1 = \alpha, a_i = 1, i \geq 2$ corresponds to the stationary q-TASEP. The gaps are independent and distributed as q-Poisson with parameter $\alpha/a_i, i \geq 2$ with $a_i > \alpha$.

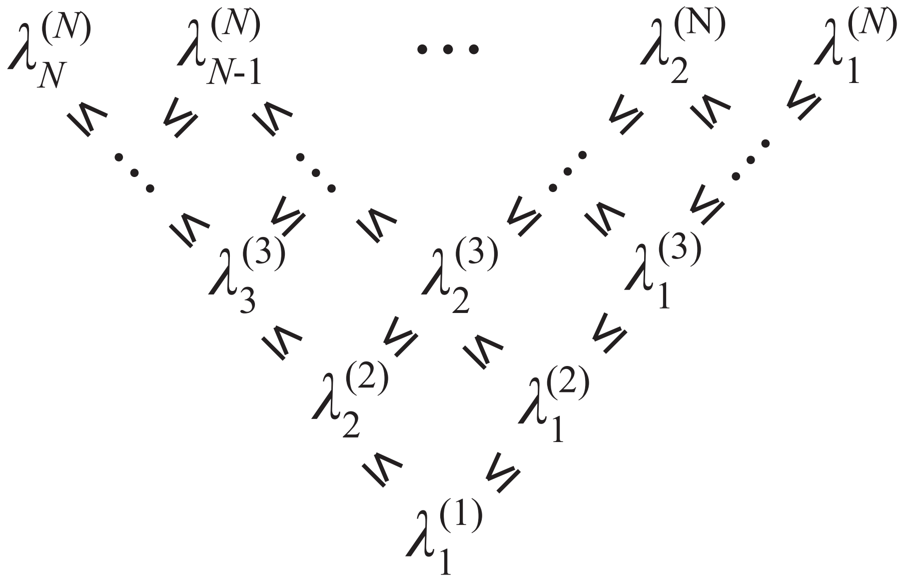
- Algebraically it is useful to study the case in which the position of the first particle is also random. The gaps are independent and distributed as q-Poisson with parameter $\alpha/a_i, i \geq 1$ with $a_i > \alpha$. Note $a_1 \to \alpha$ is singular.

- The Nth particle positions $X_N(t)$ and $X_N^{(0)}(t)$ are simply related as $X_N(t) = X_N^{(0)}(t) - \chi - 1$, $\chi \sim q\mathrm{Po}(a_1/\alpha)$.

- To summarize. One can study the stationary fluctuation by setting $a_1 = a, a_i = 1, 2 \leq i \leq N$ (N-particle q-TASEP with 2 parameters a, α) and then taking $a \to \alpha$ limit.

2.3 Dynamics on Gelfand-Tsetlin cone

Stochastic dynamics of $\lambda_j^{(k)} \in \mathbb{Z}, 1 \leq j \leq k \leq N$, satisfying



Hopping rate of $\lambda_j^{(k)}$:

$$r_{jk} = a_k \frac{(1 - q^{\mu_{j-1}^{(k-1)} - \mu_j^{(k)}})(1 - q^{\mu_j^{(k)} - \mu_{j+1}^{(k)} + 1})}{1 - q^{\mu_j^{(k)} - \mu_j^{(k-1)} + 1}}$$

Dynamics of $x_j(t) := \lambda_j^{(j)}(t) - N$ is q-TASEP with a_j.

2.4 Two-sided q-Whittaker process

The skew q-Whittaker function (with 1 variable)

$$P_{\lambda/\mu}(a) = \prod_{i=1}^{n} a^{\lambda_i} \cdot \prod_{i=1}^{n-1} \frac{a^{-\mu_i}(q;q)_{\lambda_i-\lambda_{i+1}}}{(q;q)_{\lambda_i-\mu_i}(q;q)_{\mu_i-\lambda_{i+1}}}$$

q-Whittaker function with N variables

$$P_\lambda(a) = \sum_{\substack{\lambda_i^{(k)}, 1\le i\le k\le N-1 \\ \lambda_{i+1}^{(k+1)} \le \lambda_i^{(k)} \le \lambda_i^{(k+1)}}} \prod_{j=1}^{N} P_{\lambda^{(j)}/\lambda^{(j-1)}}(a_j)$$

where the sum is over GT with $\lambda = \lambda^{(N)}$ and $a = (a_1, \ldots, a_N)$.

Another function with N variables $\alpha = (\alpha_1, \ldots, \alpha_N)$.

$$Q_\lambda(\alpha, t) = \prod_{i=1}^{N-1} (q^{\lambda_i - \lambda_{i+1}+1}; q)_\infty \int_{\mathbb{T}^N} \prod_{i=1}^{N} \frac{dz_i}{z_i} \cdot P_\lambda\left(\frac{1}{z}\right) \Pi(z; \alpha, t)\, m_N^q(z)$$

where

$$\Pi(a; \alpha, t) = \prod_{i,j=1}^{N} \frac{1}{(\alpha_i/a_j; q)_\infty} \cdot \prod_{j=1}^{N} e^{a_j t}$$

$$m_N^q(z) = \frac{1}{(2\pi i)^N N!} \prod_{1 \le i < j \le N} (z_i/z_j; q)_\infty (z_j/z_i; q)_\infty$$

Two-sided q-Whittaker process

Definition. For two sets of N parameters a, α, set

$$P_t(\underline{\lambda}_N) := \frac{\prod_{j=1}^N P_{\lambda^{(j)}/\lambda^{(j-1)}}(a_j) \cdot Q_{\lambda^{(N)}}(\alpha, t)}{\Pi(a; \alpha, t)}$$

Proposition.

$P_t(\underline{\lambda}_N)$ satisfies the Kolmogorov forward equation for the Markov dynamics introduced before on GT cone.

One can also check the half-stationary initial condition on the q-TASEP marginal $\lambda_i^{(i)}$ when $\alpha_1 = \alpha, \alpha_i = 0, 2 \leq i \leq N$.

To summarize, if we can study the q-Whittaker process, we can study the N-particle q-TASEP with two parameters a, α.

2.5 Two-sided q-Whittaker measure

$x_N(t)(=\lambda_N^{(N)}-N)$ can be studied by focusing on $\lambda^{(N)}(t)$.

Marginal for $\lambda^{(N)}(t)$ is given by two-sided q-Whittaker measure:

$$\mathbb{P}[\lambda^{(N)}(t)=\lambda]=\frac{P_\lambda(a)Q_\lambda(\alpha,t)}{\Pi(a;\alpha,t)}$$

Let us recall the Cauchy identity

$$\sum_{\lambda\in\mathcal{P}_N} P_\lambda(x)Q_\lambda(y)=\prod_{ij=1}^{N}\frac{1}{(x_iy_j;q)_\infty}$$

where $Q_\lambda(y)$ is the ordinary q-Whittaker function.

Nth particle position

By writing $P_\lambda(x) = X^{\lambda_N} R_\ell(x)$, $\ell_j = \lambda_j - \lambda_{j+1}$ the Cauchy identity can be rewritten as

$$\sum_{\ell_1,\cdots,\ell_{N-1}=0}^{\infty} R_\ell(x) R_\ell(y) \prod_{j=1}^{N-1} \frac{1}{(q;q)_{\ell_j}} = \frac{(XY;q)_\infty}{\prod_{ij=1}^{N}(x_i y_j;q)_\infty}$$

with $X = X_1 \cdots x_N, Y = y_1 \cdots y_N$. Using this we have

$\mathbb{P}[\lambda_N^{(N)}(t) = l]$

$$= (q;q)_\infty^{N-1} \int_{\mathbb{T}^N} \prod_{j=1}^{N} \frac{dz_j}{z_j} \cdot \left(\frac{A}{Z}\right)^l m_N^q(z) \frac{\Pi(z;\alpha,t)}{\Pi(a;\alpha,t)} \cdot \frac{(A/Z;q)_\infty}{\prod_{ij=1}^{N}(a_i/z_j;q)_\infty}$$

where $A = \prod_{i=1}^{N} a_i$ and $Z = \prod_{i=1}^{N} z_i$.

2.6 Multiple integral formula for q-Laplace transform

By definition of the expectation value,

$$\langle \frac{1}{(\zeta q^{\lambda_N};q)_\infty} \rangle = \sum_{l\in\mathbb{Z}} \frac{1}{(\zeta q^l;q)_\infty} \int_{\mathbb{T}^N} \prod_{i=1}^N \frac{dz_i}{z_i} \left(\frac{A}{Z}\right)^l m_N(z) \frac{\Pi(z;\alpha,t)}{\Pi(a;q,t)} \frac{(q;q)_\infty^{N-1}(A/Z;q)_\infty}{\prod_{i,j}(a_i/z_j;q)_\infty}$$

Using the Ramanujan's formula with $a=\zeta, b=0, z=A/Z$,

$$\sum_{l\in\mathbb{Z}} \frac{1}{(\zeta q^l;q)_\infty} \left(\frac{A}{Z}\right)^l = \frac{(\frac{\zeta A}{Z};q)_\infty(\frac{qZ}{\zeta A};q)_\infty(q;q)_\infty}{(\zeta,q)_\infty(\frac{q}{\zeta};q)_\infty(\frac{A}{Z};q)_\infty} = \frac{\theta(\frac{\zeta A}{Z})(q;q)_\infty}{\theta(\zeta)(\frac{A}{Z};q)_\infty},$$

we find

$$\langle \frac{1}{(\zeta q^{\lambda_N};q)_\infty} \rangle = \frac{(q;q)_\infty^N}{N!} \int_{\mathbb{T}^N} \prod_{i=1}^N \frac{dz_i}{z_i} \frac{\theta(\frac{\zeta A}{Z})}{\theta(\zeta)} \frac{\prod_{i\neq j}(z_i/z_j;q)_\infty}{\prod_{i,j}(a_i/z_j;q)_\infty} \frac{\Pi(z;\alpha,t)}{\Pi(a;q,t)}.$$

2.7 Fredholm determinant for the q-Laplace transform

Theorem. For $\zeta \neq q^n, n \in \mathbb{Z}$

$$\left\langle \frac{1}{(\zeta q^{x_N(t)+N}; q)_\infty} \right\rangle = \det(1 - fK)_{L^2(\mathbb{Z})}$$

where $\langle \cdots \rangle$ is the expectation and

$$f(n) = \frac{1}{1 - q^n/\zeta}, \quad K(n, m) = \sum_{l=0}^{N-1} \phi_l(m)\psi_l(n)$$

$$\phi_l(n) = \sqrt{a_{l+1} - \alpha_{l+1}} \int_D dv \frac{e^{-vt}}{v^{n+N}} \frac{1}{v - a_{l+1}} \prod_{j=1}^{l} \frac{v - \alpha_j}{v - a_j} \prod_k \frac{(q\alpha_k/v; q)_\infty}{(qv/a_k; q)_\infty}$$

$$\psi_l(n) = \sqrt{a_{l+1} - \alpha_{l+1}} \int_{C_r} dz \frac{e^{zt} z^{n+N}}{z - \alpha_{l+1}} \prod_{j=1}^{l} \frac{z - a_j}{z - \alpha_j} \prod_k \frac{(qz/a_k; q)_\infty}{(q\alpha_k/z; q)_\infty}$$

Here C_r, D is around $\{0, \alpha_i q^j\}, \{a_i\}$ respectively.

Proof

After some calculations form the multiple integral formula, we find

$$\langle \frac{1}{(\zeta q^{\lambda_N}; q)_\infty} \rangle = \frac{(q;q)_\infty^N}{N!} \int_{\mathbb{T}^N} \prod_{i=1}^{N} \frac{dz_i}{z_i} \frac{\prod_{i<j}(a_i - a_j) \prod_{i<j}(z_i - z_j)}{\prod_{i,j}(a_i - z_j)}$$

$$\times \frac{\prod_{i<j} \tilde{\theta}(a_i/a_j) \prod_{i<j} \tilde{\theta}(z_i/z_j)}{\prod_{i,j} \tilde{\theta}(a_i/z_j)}$$

$$\times \frac{\tilde{\theta}(\frac{\zeta A}{Z})}{\tilde{\theta}(\zeta)} \prod_{i} \frac{a_i \prod_k (z_i/a_k; q)_\infty g(z_i; \alpha, t)}{\prod_{k \neq i}(a_i/a_k; q)_\infty g(a_i; \alpha, t)}$$

where

$$g(z; \alpha, t) = \frac{e^{zt}}{\prod_j (\alpha_j/z; q)_\infty}.$$

By the Cauchy determinant formula,

$$\langle \frac{1}{(\zeta q^{\lambda_N}; q)_\infty} \rangle = \frac{1}{N!} \int_{\mathbb{T}^N} \prod_{i=1}^{N} \frac{dz_i}{z_i} \det(\frac{a_i}{a_i - z_j}) \det(\frac{\tilde{\theta}(\zeta a_i / z_j)}{\tilde{\theta}(\zeta) \tilde{\theta}(a_i / z_j)})$$
$$\times \prod_i \frac{\prod_k (z_i / a_k; q)_\infty g(z_i; \alpha, t)(q;q)_\infty}{\prod_{k \neq i} (a_i / a_k; q)_\infty g(a_i; \alpha, t)}$$

[Using the Cauchy-Binet identity]

$$= \det \left(\int_{\mathbb{T}} \frac{dz}{z} \frac{a_i}{a_i - z} \frac{\theta(\zeta a_i / z)}{\theta(\zeta) \theta(a_i / z)} \frac{(q;q)_\infty \prod_k (z / a_k; q)_\infty g(z; \alpha, t)}{\prod_{k \neq i} (a_i / a_k; q)_\infty g(a_i; \alpha, t)} \right)$$

By making the contour smaller and taking the pole at $z = a_i$

$$= \det \left(\delta_{ij} - \int_{C_r} \frac{dz}{z} \frac{a_i}{a_i - z} \frac{\theta(\zeta a_i/z)}{\theta(\zeta)\theta(a_i/z)} \frac{(q;q)_\infty \prod_k (z/a_k;q)_\infty g(z;\alpha,t)}{\prod_{k\neq i}(a_i/a_k;q)_\infty g(a_i;\alpha,t)} \right)$$

Here using the Ramanujan's formula again with

$a = 1/\zeta, b = q/\zeta, z \to z/a_j,$

$$\sum_{n\in\mathbb{Z}} \frac{1}{1-q^n/\zeta} \left(\frac{z}{a_j}\right)^n = \frac{(\frac{z}{\zeta a_j})_\infty (\frac{q\zeta a_j}{z};q)_\infty (q;q)_\infty^2}{(1/\zeta;q)_\infty (q\zeta;q)_\infty (z/a_j;q)_\infty (qa_j/z;q)_\infty}$$

$$= \frac{\theta(\frac{z}{\zeta a_j})}{\theta(1/\zeta)\theta(z/a_j)} (q;q)_\infty^2,$$

we see

$$\langle \frac{1}{(\zeta q^{\lambda_N}; q)_\infty} \rangle = \det \left(\delta_{ij} - \sum_{n \in \mathbb{Z}} \frac{1}{1 - q^n/\zeta} \int_{C_r} \frac{dz}{z} \frac{a_i}{a_i - z} \right.$$

$$\left. \times \ \frac{z^n \prod_k (z/a_k; q)_\infty g(z; \alpha, t)}{a_j^n (q; q)_\infty \prod_{k \neq i} (a_i/a_k; q)_\infty g(a_i; \alpha, t)} \right)$$

$$= \det(\delta_{ij} - \sum_{n \in \mathbb{Z}} A(i, n) B(n, j))$$

with

$$A(i, n) = \frac{1}{1 - q^n/\zeta} \int_{C_r} \frac{dz}{z} \frac{a_i}{a_i - z} z^n \prod_k (z/a_k; q)_\infty g(z; \alpha, t)$$

$$B(n, j) = \frac{1}{(q; q)_\infty (a_i/a_k; q)_\infty g(a_i; \alpha, t)}$$

Here use $\det(1-AB) = \det(1-BA)$. We see

$$
\begin{aligned}
(BA)(m,n) &= \sum_{i=1}^{N} B(m,i)A(i,n) \\
&= \sum_{i=1}^{N} \frac{1}{a_i^m (q;q)_\infty (a_i/a_k;q)_\infty g(a_i;\alpha,t)} \frac{1}{1-q^n/\zeta} \\
&\quad \times \int_{C_r} \frac{dz}{z} \frac{a_i}{a_i - z} z^n \prod_k (z/a_k;q)_\infty g(z;\alpha,t) \\
&= \frac{-1}{1-q^n/\zeta} \int_D dv \int_{C_r} \frac{dz}{z} \frac{1}{v-z} \frac{z^n \prod_k (z/a_k;q)_\infty g(z;\alpha,t)}{v^n \prod_k (v/a_k;q)_\infty g(v;\alpha,t)}
\end{aligned}
$$

where the contour D is around $\{a_i\}$. Here

$$\begin{aligned}
&\frac{\prod_k (z/a_k;q)_\infty g(z;\alpha,t)}{\prod_k (v/a_k;q)_\infty g(v;\alpha,t)} \\
&= \frac{\prod_k (qz/a_k;q)_\infty (qv/\alpha_k;q)_\infty e^{zt}}{\prod_k (qv/a_k;q)_\infty (qz/\alpha_k;q)_\infty e^{vt}} \frac{(z-a_k)(v-\alpha_k)}{(v-a_k)(z-\alpha_k)} \left(\frac{z}{v}\right)^N
\end{aligned}$$

Hence

$$\begin{aligned}
\langle \frac{1}{(\zeta q^{\lambda_N};q)_\infty} \rangle &= \frac{1}{1-q^n/\zeta} \int_D dv \int_{C_r} \frac{dz}{z} \frac{z^{n+N} e^{zt} \prod_k (qz/a_k;q)_\infty (qv/a_k;q)_\infty}{v^{n+N} e^{vt} \prod_k (qv/a_k;q)_\infty (qz/\alpha_k;q)_\infty} \\
&\times \left(\frac{1}{z-v} \prod_k \frac{(z-a_k)(v-\alpha_k)}{(v-a_k)(z-\alpha_k)} - 1 \right)
\end{aligned}$$

By using

$$\frac{1}{z-v}\prod_{k}\frac{(z-a_k)(v-\alpha_k)}{(v-a_k)(z-\alpha_k)}-1$$

$$=\sum_{l=0}^{N-1}\frac{a_{l+1}-\alpha_{l+1}}{(z-\alpha_{l+1})(v-a_{l+1})}\prod_{j=1}^{l}\frac{(z-a_j)(v-\alpha_j)}{(z-\alpha_j)(v-a_j)}$$

we arrive at the desired Fredholm determinant expression.

2.8 Stationary limit: X_N and $X_N^{(0)}$

For the two parameter q-TASEP with

$$\alpha_j = \alpha,\ \alpha_k = 0, k \neq j, \quad a_1 = a, a_2 = \cdots = a_N = 1, \quad 0 < \alpha < a < 1,$$

$$G(\zeta) = \left\langle \frac{1}{(\zeta q^{X_N(t)+N}; q)_\infty} \right\rangle, \quad G_0(\zeta) = \left\langle \frac{1}{(\zeta q^{X_N^{(0)}(t)+N-1}; q)_\infty} \right\rangle$$

are related by

$$G(\zeta) = (\alpha/a; q)_\infty \sum_{m=0}^{\infty} \frac{(\alpha/a)^m}{(q;q)_m} G_0(\zeta q^{-m})$$

or

$$G_0(\zeta) = \frac{1}{(\alpha/a; q)_\infty} \sum_{k=0}^{\infty} \frac{(-1)^k q^{k(k-1)/2} (\alpha/a)^k}{(q;q)_k} G(\zeta q^{-k})$$

Long time limit for stationary q-TASEP

By taking $a \to \alpha$ limit carefully and performing asymptotic analysis, one finally arrives at

Thm. For the stationary q-TASEP, with the parameter $\alpha = q^\theta(1+\omega/(\gamma N^{1/3})), \theta > 0, \omega \in \mathbb{R}$, we have, for $\forall s \in \mathbb{R}$,

$$\lim_{N\to\infty} \mathbb{P}(x_N(\kappa N) > (\eta - 1)N - \gamma N^{1/3} s) = F_\omega(s)$$

where κ, η, γ are given by

$$\kappa = \sum_{n=0}^{\infty} \frac{q^n}{(1-q^{\theta+n})^2}, \quad \eta = \sum_{n=0}^{\infty} \frac{q^{2\theta+2n}}{(1-q^{\theta+n})^2},$$

$$\gamma = \left(\sum_{n=0}^{\infty} \frac{q^{2\theta+2n}}{(1-q^{\theta+n})^3} \right)^{1/3}$$

Summary

- We have explained how to study the stationary KPZ models, in particular for the case of q-TASEP.
- In our approach, instead of using the so-far standard method of q-moments, which diverge for random initial conditions, we use (a two-sided version of) the q-Whittaker process and directly study the distribution of a particle position.
- Two technically essential ingredients were the Ramanujan's summation formula, Cauchy determinant for theta functions.
- Our approach can be applied to more general case of higher spin vertex model (see the poster by Mucciconi).
- Multiple integral appearing in our analysis unifies various cases and have many applications.