Density of states and level statistics for 1-d Schrödinger operators

Trinh Kahn Duy[0] Shinnichi Kotani[1] Fumihiko Nakano[2]

[0]Touhoku University

[1]Osaka University

[2]Gakushuin University

2017 年 2 月 20 日

Known Facts on RSO

$(\Omega, \mathcal{F}, \mathbf{P})$: probability space

$$(H_\omega \phi)(x) := \sum_{|y-x|=1} \phi(y) + V_\omega(x)\phi(x), \quad \omega \in \Omega, \quad \phi \in \ell^2(\mathbf{Z}^d)$$

$\{V_\omega(x)\}_{x \in \mathbf{Z}^d}$: i.i.d. with "good" distribution μ.

Known Facts on RSO

$(\Omega, \mathcal{F}, \mathbf{P})$: probability space

$$(H_\omega \phi)(x) := \sum_{|y-x|=1} \phi(y) + V_\omega(x)\phi(x), \quad \omega \in \Omega, \quad \phi \in \ell^2(\mathbf{Z}^d)$$

$\{V_\omega(x)\}_{x \in \mathbf{Z}^d}$: i.i.d. with "good" distribution μ.

(1) Spectrum

$$\sigma(H_\omega) = \Sigma := [-2d, 2d] + \text{ supp } \mu, \quad a.s.$$

Known Facts on RSO

$(\Omega, \mathcal{F}, \mathbf{P})$: probability space

$$(H_\omega \phi)(x) := \sum_{|y-x|=1} \phi(y) + V_\omega(x)\phi(x), \quad \omega \in \Omega, \quad \phi \in \ell^2(\mathbf{Z}^d)$$

$\{V_\omega(x)\}_{x \in \mathbf{Z}^d}$: i.i.d. with "good" distribution μ.

(1) Spectrum

$$\sigma(H_\omega) = \Sigma := [-2d, 2d] + \text{ supp } \mu, \quad a.s.$$

(2) Anderson localization : $\exists I (\subset \Sigma)$ s.t. $\sigma(H_\omega) \cap I$ is a.s. pp with exponentially decaying e.f.'s.

IDS and Level Statistics 1

(1) Integrated Density of States (Macroscopic Limit) :
Let $H_L := H|_{[0,L]^d}$ with D-bc.

IDS and Level Statistics 1

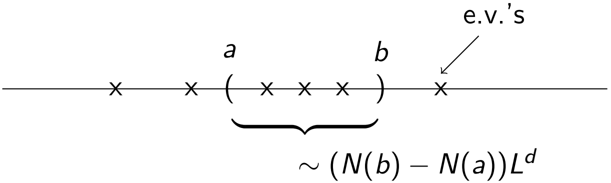
(1) Integrated Density of States (Macroscopic Limit) :
Let $H_L := H|_{[0,L]^d}$ with D-bc. Then $\exists N(\cdot)$ s.t.

$$\sharp\{\text{ e.v.'s of } H_L \le E\} = N(E) \cdot L^d (1 + o(1)), \quad L \to \infty$$

IDS and Level Statistics 1

(1) Integrated Density of States (Macroscopic Limit) :
Let $H_L := H|_{[0,L]^d}$ with D-bc. Then $\exists N(\cdot)$ s.t.

$$\sharp\{\text{ e.v.'s of } H_L \leq E\} = N(E) \cdot L^d(1 + o(1)), \quad L \to \infty$$



IDS and Level Statistics 2

(2) Level Statistics (Microscopic Limit, Minami 1996) :

Let E_0 : "localized region", $n(E_0) := \frac{d}{dE} N(E_0)$. Then

IDS and Level Statistics 2

(2) Level Statistics (Microscopic Limit, Minami 1996) :

Let E_0 : "localized region", $n(E_0) := \frac{d}{dE} N(E_0)$. Then

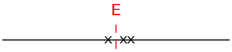
$$\sharp \left\{ \text{e.v.'s of } H_L \text{ in } E_0 + \frac{1}{L^d}[a, b] \right\} \xrightarrow{d} \text{Poisson}(n(E_0)(b-a))$$

IDS and Level Statistics 2

(2) Level Statistics (Microscopic Limit, Minami 1996) :

Let E_0 : "localized region", $n(E_0) := \frac{d}{dE} N(E_0)$. Then

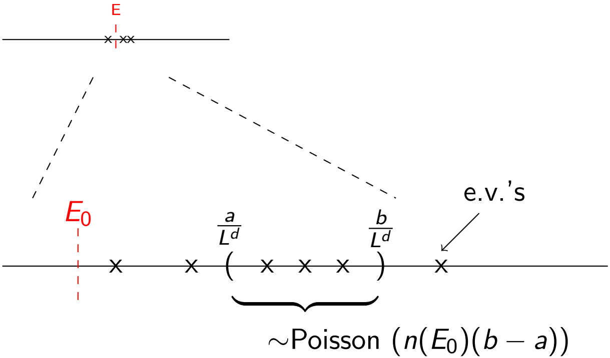
$$\sharp \left\{ \text{e.v.'s of } H_L \text{ in } E_0 + \frac{1}{L^d}[a,b] \right\} \xrightarrow{d} \text{Poisson}(n(E_0)(b-a))$$



IDS and Level Statistics 2

(2) Level Statistics (Microscopic Limit, Minami 1996) :
Let E_0 : "localized region", $n(E_0) := \frac{d}{dE} N(E_0)$. Then

$$\sharp \left\{ \text{e.v.'s of } H_L \text{ in } E_0 + \frac{1}{L^d}[a, b] \right\} \xrightarrow{d} \text{Poisson}(n(E_0)(b-a))$$



An Extension (Killip-N, N, 2007)

(1) (Macroscopic Limit) Let

ϕ_k : e.f. corr. to $E_k(L)$, $x_k := \langle x \rangle_{\phi_k} \in \mathbf{R}^d$: loc. center of ϕ_k.

An Extension (Killip-N, N, 2007)

(1) (Macroscopic Limit) Let

ϕ_k : e.f. corr. to $E_k(L)$, $x_k := \langle x \rangle_{\phi_k} \in \mathbf{R}^d$: loc. center of ϕ_k.

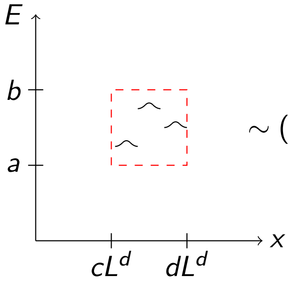
Then

$$\bar{\xi}_L := \frac{1}{L^d} \sum_k \delta_{(E_k(L),\, x_k/L^d)} \overset{v}{\to} dN \otimes dx, \quad a.s.$$

An Extension (Killip-N, N, 2007)

(1) (Macroscopic Limit) Let
ϕ_k : e.f. corr. to $E_k(L)$, $x_k := \langle x \rangle_{\phi_k} \in \mathbf{R}^d$: loc. center of ϕ_k.
Then

$$\bar{\xi}_L := \frac{1}{L^d} \sum_k \delta_{(E_k(L),\, x_k/L^d)} \stackrel{v}{\to} dN \otimes dx, \quad a.s.$$



$\sim (N(b) - N(a))(d - c)L^d$

An Extension (Killip-N, N, 2007)

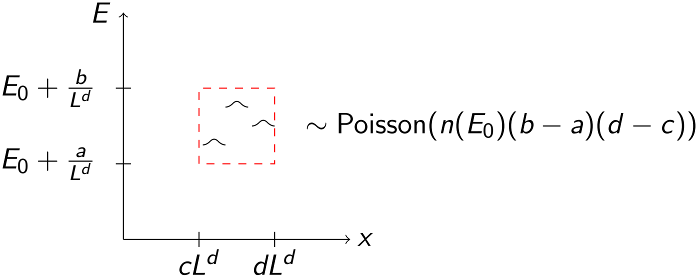
(2) (Microscopic Limit)

$$\xi_L := \sum_k \delta_{(L^d(E_k(L)-E_0),\, x_k/L^d)} \xrightarrow{d} \text{Poisson}(n(E_0)dE \otimes dx)$$

An Extension (Killip-N, N, 2007)

(2) (Microscopic Limit)

$$\xi_L := \sum_k \delta_{(L^d(E_k(L)-E_0),\, x_k/L^d)} \xrightarrow{d} \text{Poisson}(n(E_0)dE \otimes dx)$$



Decaying Potential Model

We consider

$$H := -\frac{d^2}{dt^2} + a(t)F(X_t) \quad \text{on} \quad L^2(\mathbf{R})$$

where a : decaying factor, and F : random potential.

Decaying Potential Model

We consider

$$H := -\frac{d^2}{dt^2} + a(t)F(X_t) \quad \text{on} \quad L^2(\mathbf{R})$$

where a : decaying factor, and F : random potential.

$$a(t) \in C^\infty(\mathbf{R}), \quad a(-t) = a(t), \quad \searrow \text{ for } t > 0$$
$$a(t) = t^{-\alpha}(1 + o(1)), \quad t \to \infty, \quad \alpha > 0$$

Decaying Potential Model

We consider

$$H := -\frac{d^2}{dt^2} + a(t)F(X_t) \quad \text{on} \quad L^2(\mathbf{R})$$

where a : decaying factor, and F : random potential.

$$a(t) \in C^\infty(\mathbf{R}), \quad a(-t) = a(t), \quad \searrow \text{ for } t > 0$$

$$a(t) = t^{-\alpha}(1 + o(1)), \quad t \to \infty, \quad \alpha > 0$$

$$F \in C^\infty(M), \quad M : torus, \quad \langle F \rangle := \int_M F(x)dx = 0,$$

$$\{X_t\}_{t\in\mathbf{R}} : \text{ BM. on } M.$$

Spectrum of H

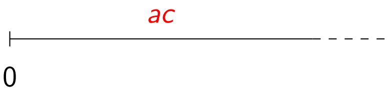
Kotani-Ushiroya(1988) : $\sigma(H) \cap [0, \infty)$ is

Spectrum of H

Kotani-Ushiroya(1988) : $\sigma(H) \cap [0, \infty)$ is

(1)(Rapid decay)

$\alpha > \frac{1}{2} \quad \Longrightarrow$

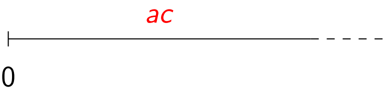


Spectrum of H

Kotani-Ushiroya(1988) : $\sigma(H) \cap [0, \infty)$ is

(1)(Rapid decay)

$\alpha > \frac{1}{2} \quad \Longrightarrow$



(2)(Slow decay)

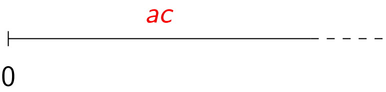
$0 \le \alpha < \frac{1}{2} \quad \Longrightarrow$

Spectrum of H

Kotani-Ushiroya(1988) : $\sigma(H) \cap [0, \infty)$ is

(1)(Rapid decay)

$$\alpha > \frac{1}{2} \quad \Longrightarrow$$

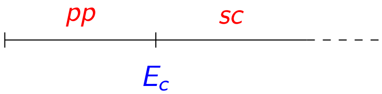


(2)(Slow decay)

$$0 \leq \alpha < \frac{1}{2} \quad \Longrightarrow$$

(3)(Critical decay)

$$\alpha = \frac{1}{2} \quad \Longrightarrow$$



IDS

We have

$$IDS(E) = IDS_{free}(E) = \frac{1}{\pi}\sqrt{E}.$$

as far as $\underline{\alpha > 0}$.

IDS

We have

$$IDS(E) = IDS_{free}(E) = \frac{1}{\pi}\sqrt{E}.$$

as far as $\underline{\alpha > 0}$. Let

$$H_L := H|_{[0,L]}, \qquad \text{Dirichlet bc}$$

$$\text{“}\kappa := \sqrt{E}\text{”}$$

$$N_L(\kappa_1, \kappa_2) := \sharp \left\{ \text{ e.v.'s of } H_L \text{ in } \left(\kappa_1^2, \kappa_2^2\right) \right\}, \quad 0 < \kappa_1 < \kappa_2.$$

IDS

We have

$$IDS(E) = IDS_{free}(E) = \frac{1}{\pi}\sqrt{E}.$$

as far as $\underline{\alpha > 0}$. Let

$$H_L := H|_{[0,L]}, \qquad \text{Dirichlet bc}$$

$$\text{“}\kappa := \sqrt{E}\text{”}$$

$$N_L(\kappa_1, \kappa_2) := \sharp \left\{ \text{e.v.'s of } H_L \text{ in } (\kappa_1^2, \kappa_2^2) \right\}, \quad 0 < \kappa_1 < \kappa_2.$$

Then

$$N_L(\kappa_1, \kappa_2) = \frac{L}{\pi}(\kappa_2 - \kappa_1)(1 + o(1)), \quad L \to \infty.$$

IDS

We have

$$IDS(E) = IDS_{free}(E) = \frac{1}{\pi}\sqrt{E}.$$

as far as $\underline{\alpha > 0}$. Let

$$H_L := H|_{[0,L]}, \quad \text{Dirichlet bc}$$

$$\text{“}\kappa := \sqrt{E}\text{”}$$

$$N_L(\kappa_1, \kappa_2) := \sharp \left\{ \text{e.v.'s of } H_L \text{ in } (\kappa_1^2, \kappa_2^2) \right\}, \quad 0 < \kappa_1 < \kappa_2.$$

Then

$$N_L(\kappa_1, \kappa_2) = \frac{L}{\pi}(\kappa_2 - \kappa_1)(1 + o(1)), \quad L \to \infty.$$

Q : 2nd order (“CLT”) ?

Fluctuation of IDS (Notation)

Let

(1) $\{G(x)\}_{x>0}$: the Gaussian field with

$$\langle G(x), G(y) \rangle = \delta_{xy} C(x),$$

(2) G_0 : a Gaussian independent of $\{G(\cdot)\}$.

Fluctuation of IDS (Notation)

Let

(1) $\{G(x)\}_{x>0}$: the Gaussian field with

$$\langle G(x), G(y) \rangle = \delta_{xy} C(x),$$

(2) G_0 : a Gaussian independent of $\{G(\cdot)\}$.

Further, let

$$G(\kappa_1, \kappa_2) := G(\kappa_2) - G(\kappa_1) + \left(\frac{1}{\kappa_1} - \frac{1}{\kappa_2} \right) G_0$$

Fluctuation of IDS (Results 1)

Theorem 0 [N2017]

(1) (AC case) $\alpha > \frac{1}{2}$:

$$N_L(\kappa_1, \kappa_2) = \frac{L}{\pi}(\kappa_2 - \kappa_1) + \text{bounded}$$

Fluctuation of IDS (Results 1)

Theorem 0 [N2017]

(1) (AC case) $\alpha > \frac{1}{2}$:

$$N_L(\kappa_1, \kappa_2) = \frac{L}{\pi}(\kappa_2 - \kappa_1) + \text{bounded}$$

(2) (Critical Case) $\alpha = \frac{1}{2}$:

$$N_L(\kappa_1, \kappa_2) = \frac{L}{\pi}(\kappa_2 - \kappa_1) + C(\kappa_1, \kappa_2) \log L + G(\kappa_1, \kappa_2)\sqrt{\log L} + \cdots$$

Fluctuation of IDS (Results 1)

Theorem 0 [N2017]

(1) (AC case) $\alpha > \frac{1}{2}$:

$$N_L(\kappa_1, \kappa_2) = \frac{L}{\pi}(\kappa_2 - \kappa_1) + \text{bounded}$$

(2) (Critical Case) $\alpha = \frac{1}{2}$:

$$N_L(\kappa_1, \kappa_2) = \frac{L}{\pi}(\kappa_2 - \kappa_1) + C(\kappa_1, \kappa_2) \log L + G(\kappa_1, \kappa_2)\sqrt{\log L} + \cdots$$

(3) (PP Case) $\alpha < \frac{1}{2}$:

$$N_L(\kappa_1, \kappa_2) = \frac{L}{\pi}(\kappa_2 - \kappa_1) + C_2(\kappa_1, \kappa_2) L^{1-2\alpha} + C_3(\kappa_1, \kappa_2) L^{1-3\alpha} + \cdots$$

$$+ \cdots C_m(\kappa_1, \kappa_2) L^{1-m\alpha} + L^{\frac{1}{2}-\alpha} G(\kappa_1, \kappa_2) + \cdots$$

Fluctuation of IDS (Results 2)

Theorem 0 (continued)

(2) (Critical Case) $\alpha = \frac{1}{2}$:

$$N_L(\kappa_1, \kappa_2) = \frac{L}{\pi}(\kappa_2 - \kappa_1) + C(\kappa_1, \kappa_2)\log L + G(\kappa_1, \kappa_2)\sqrt{\log L} + \cdots$$

Fluctuation of IDS (Results 2)

Theorem 0 (continued)

(2) (Critical Case) $\alpha = \frac{1}{2}$:

$$N_L(\kappa_1, \kappa_2) = \frac{L}{\pi}(\kappa_2 - \kappa_1) + C(\kappa_1, \kappa_2) \log L + G(\kappa_1, \kappa_2)\sqrt{\log L} + \cdots$$

(3) (PP Case) $\frac{1}{2m} \le \alpha < \frac{1}{2(m-1)}$, $m = 2, 3, \cdots,$:

$$N_L(\kappa_1, \kappa_2) = \frac{L}{\pi}(\kappa_2 - \kappa_1) + C_2(\kappa_1, \kappa_2)L^{1-2\alpha} + C_3(\kappa_1, \kappa_2)L^{1-3\alpha} + \cdots + C_m(\kappa_1, \kappa_2)L^{1-m\alpha} + L^{\frac{1}{2}-\alpha}G(\kappa_1, \kappa_2) + \cdots$$

$C_j(\kappa_1, \kappa_2)$: non-random const.'s

Level statistics

$$H_L := H|_{[0,L]}, \quad \text{Dirichlet b.c.} \ ,$$

Level statistics

$H_L := H|_{[0,L]}$, Dirichlet b.c. ,

$0 < \kappa_{n_0}^2(L) < \kappa_{n_0+1}^2(L) < \cdots$, positive e.v.'s of H_L

Level statistics

$H_L := H|_{[0,L]}$, Dirichlet b.c. ,

$0 < \kappa_{n_0}^2(L) < \kappa_{n_0+1}^2(L) < \cdots$, positive e.v.'s of H_L

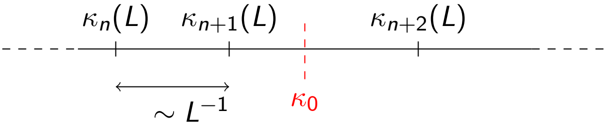
$E_0 = \kappa_0^2 > 0$: reference energy (fixed)

Level statistics

$H_L := H|_{[0,L]}$, Dirichlet b.c. ,

$0 < \kappa_{n_0}^2(L) < \kappa_{n_0+1}^2(L) < \cdots$, positive e.v.'s of H_L

$E_0 = \kappa_0^2 > 0$: reference energy (fixed)

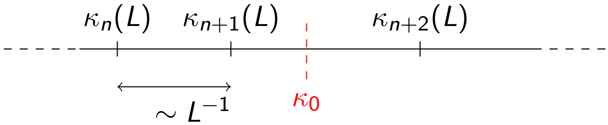


Level statistics

$H_L := H|_{[0,L]}$, Dirichlet b.c. ,

$0 < \kappa_{n_0}^2(L) < \kappa_{n_0+1}^2(L) < \cdots$, positive e.v.'s of H_L

$E_0 = \kappa_0^2 > 0$: reference energy (fixed)



To study the local statistics of e.v.'s near E_0, we consider

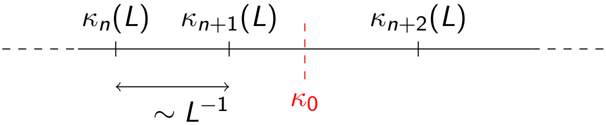
$$\xi_L := \sum_{n \geq n_0} \delta_{L(\kappa_n(L) - \kappa_0)}.$$

Level statistics

$H_L := H|_{[0,L]}$, Dirichlet b.c. ,

$0 < \kappa_{n_0}^2(L) < \kappa_{n_0+1}^2(L) < \cdots$, positive e.v.'s of H_L

$E_0 = \kappa_0^2 > 0$: reference energy (fixed)



To study the local statistics of e.v.'s near E_0, we consider

$$\xi_L := \sum_{n \geq n_0} \delta_{L(\kappa_n(L) - \kappa_0)}.$$

Problem : $\xi_L \to ?$ as $L \to \infty$.

Known Results

(1) Killip-Stoiciu (2009) : For CMV matrices,

Known Results

(1) Killip-Stoiciu (2009) : For CMV matrices,

$$\xi_L \to \begin{cases} \text{(i) } \alpha > \frac{1}{2} \ : \text{ clock process} \\ \text{(ii) } \alpha < \frac{1}{2} \ : \text{ Poisson process} \\ \text{(iii) } \alpha = \frac{1}{2} \ : \text{ limit of circular } \beta\text{-ensembles} \end{cases}$$

Known Results

(1) Killip-Stoiciu (2009) : For CMV matrices,

$$\xi_L \to \begin{cases} \text{(i) } \alpha > \frac{1}{2} \ : \ \text{clock process} \\ \text{(ii) } \alpha < \frac{1}{2} \ : \ \text{Poisson process} \\ \text{(iii) } \alpha = \frac{1}{2} \ : \ \text{limit of circular } \beta\text{-ensembles} \end{cases}$$

(2) Krichevski-Valko-Virag (2012) :
For 1-dim discrete Sch. op., $\alpha = \frac{1}{2}$,

Known Results

(1) Killip-Stoiciu (2009) : For CMV matrices,

$$\xi_L \to \begin{cases} \text{(i) } \alpha > \frac{1}{2} : \text{ clock process} \\ \text{(ii) } \alpha < \frac{1}{2} : \text{ Poisson process} \\ \text{(iii) } \alpha = \frac{1}{2} : \text{ limit of circular } \beta\text{-ensembles} \end{cases}$$

(2) Krichevski-Valko-Virag (2012) :
For 1-dim discrete Sch. op., $\alpha = \frac{1}{2}$,

$$\xi_L \to \alpha = \frac{1}{2} : \text{ Sine}_\beta\text{-process (limit of Gaussian } \beta\text{-ensembles)}$$

Fast Decay ($\alpha > \frac{1}{2}$) : Assumption

For free Hamiltonian ($V \equiv 0$), $\kappa_n = n\pi/L$, so that the atoms of ξ_L are

$$L(\kappa_n - \kappa_0) = n\pi - \kappa_0 L.$$

Fast Decay ($\alpha > \frac{1}{2}$) : Assumption

For free Hamiltonian ($V \equiv 0$), $\kappa_n = n\pi/L$, so that the atoms of ξ_L are

$$L(\kappa_n - \kappa_0) = n\pi - \kappa_0 L.$$

$\kappa_0 L$: must converge modulo π.

Fast Decay ($\alpha > \frac{1}{2}$) : Assumption

For free Hamiltonian ($V \equiv 0$), $\kappa_n = n\pi/L$, so that the atoms of ξ_L are

$$L(\kappa_n - \kappa_0) = n\pi - \kappa_0 L.$$

$\kappa_0 L$: must converge modulo π.

Assumption (A)

Subsequence $\{L_j\}$ satisfies $L_j \stackrel{j\to\infty}{\to} \infty$ and

$$\kappa_0 L_j = m_j \pi + \beta + o(1), \quad j \to \infty, \quad m_j \in \mathbf{N}, \quad \beta \in [0, \pi).$$

Fast Decay ($\alpha > \frac{1}{2}$) : Results

Theorem 1 ([KN2014] AC-case $\Longrightarrow$ clock process)

Assume (A). Then we have

$$\lim_{j\to\infty} \mathbf{E}\left[e^{-\xi_{L_j}(f)}\right] = \int_0^\pi d\mu_\beta(\phi) \exp\left(-\sum_{n\in\mathbf{Z}} f(n\pi - \phi)\right)$$

for some probability measure μ_β on $[0, \pi]$.

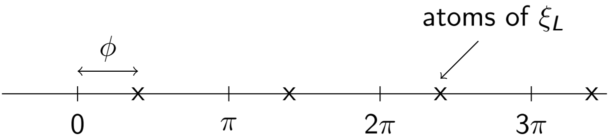
Fast Decay ($\alpha > \frac{1}{2}$) : Results

Theorem 1 ([KN2014] AC-case $\Longrightarrow$ clock process)

Assume (A). Then we have

$$\lim_{j\to\infty} \mathbf{E}\left[e^{-\xi_{L_j}(f)}\right] = \int_0^\pi d\mu_\beta(\phi) \exp\left(-\sum_{n\in\mathbf{Z}} f(n\pi - \phi)\right)$$

for some probability measure μ_β on $[0, \pi]$.

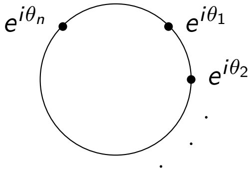


Circular β-ensemble

Definition

(1) The circular β-ensemble with n-points is given by

$$\mathbf{P}\left(\quad \right) \propto |\triangle(e^{i\theta_1}, \cdots, e^{i\theta_n})|^{\beta}$$



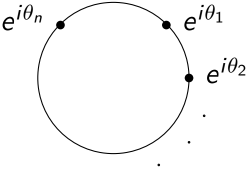
$\triangle$: Vandermonde determinant, $\beta > 0$.

Circular β-ensemble

Definition

(1) The circular β-ensemble with n-points is given by

$$\mathbf{P}\left(\quad \right) \propto \quad |\triangle(e^{i\theta_1}, \cdots, e^{i\theta_n})|^{\beta}$$



$\triangle$: Vandermonde determinant, $\beta > 0$.

(2) The scaling limit ζ_β^C of the circular β-ensemble is defined by

$$\zeta_\beta^C := \lim_{n\to\infty} \sum_{j=1}^{n} \delta_{n\theta_j}.$$

Characterization of ζ_β^C

Theorem (Killip-Stoiciu (2009))

$$\mathbf{E}[e^{-\zeta_\beta^C(f)}] = \mathbf{E}\left[\int_0^{2\pi} \frac{d\theta}{2\pi} \exp\left(-\sum_{n\in\mathbf{Z}} f\left(\Psi_1^{-1}(2n\pi+\theta)\right)\right)\right]$$

Characterization of ζ_β^C

Theorem (Killip-Stoiciu (2009))

$$\mathbf{E}[e^{-\zeta_\beta^C(f)}] = \mathbf{E}\left[\int_0^{2\pi} \frac{d\theta}{2\pi} \exp\left(-\sum_{n\in\mathbf{Z}} f\left(\Psi_1^{-1}(2n\pi+\theta)\right)\right)\right]$$

where $\{\Psi_t(\cdot)\}_{t\geq 0}$ is the strictly-increasing function valued process s.t. $\{\Psi_t(\lambda)\}_{t>0}$ is the solution to :

Characterization of ζ_β^C

Theorem (Killip-Stoiciu (2009))

$$\mathbf{E}[e^{-\zeta_\beta^C(f)}] = \mathbf{E}\left[\int_0^{2\pi} \frac{d\theta}{2\pi} \exp\left(-\sum_{n\in\mathbf{Z}} f\left(\Psi_1^{-1}(2n\pi + \theta)\right)\right)\right]$$

where $\{\Psi_t(\cdot)\}_{t\geq 0}$ is the strictly-increasing function valued process s.t. $\{\Psi_t(\lambda)\}_{t>0}$ is the solution to :

$$d\Psi_t(\lambda) = \lambda\, dt + \frac{2}{\sqrt{\beta t}} Re\left\{(e^{i\Psi_t(\lambda)} - 1)dZ_t\right\},$$

$$\Psi_0(\lambda) = 0$$

Z_t : complex B.M.

Gaussian β-ensemble

Definition

(1) The Gaussian β-ensemble with n-points is given by

$$\mathbf{P}\left(\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n\right) \propto \exp\left(-\frac{\beta}{4}\sum_{k=1}^{n}\lambda_k^2\right)|\triangle(\{\lambda_j\})|^{\beta}$$

Gaussian β-ensemble

Definition

(1) The Gaussian β-ensemble with n-points is given by

$$\mathbf{P}\left(\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n\right) \propto \exp\left(-\frac{\beta}{4}\sum_{k=1}^{n}\lambda_k^2\right)|\triangle(\{\lambda_j\})|^{\beta}$$

(2) The scaling limit ζ_β^G of the Gaussian β-ensemble is defined by

$$\zeta_\beta^G := \lim_{n\to\infty}\sum_{j=1}^{n}\delta_{\sqrt{4n}\lambda_j}$$

which is called the Sine$_\beta$-process.

Characterization of ζ_β^G

Theorem (Valko-Virag 2009)

Let $\alpha_t(\lambda)$ be the solution to the following SDE.

$$d\alpha_t(\lambda) = \lambda \cdot \frac{\beta}{4} e^{-\frac{\beta}{4}t} dt + Re\left[\left(e^{i\alpha_t} - 1\right) dZ_t\right],$$
$$\alpha_0(\lambda) = 0.$$

Then for $\lambda > 0$, $t \mapsto \lfloor \alpha_t(\lambda)/2\pi \rfloor$ is non-decreasing and $\alpha_\infty(\lambda) := \exists \lim_{t\to\infty} \alpha_t(\lambda) \in 2\pi\mathbf{Z}$, a.s. Then Sine$_\beta$-process on each interval is given by

$$\zeta_\beta^G[\lambda_1, \lambda_2] \stackrel{d}{=} \frac{\alpha_\infty(\lambda_2) - \alpha_\infty(\lambda_1)}{2\pi}.$$

Remark

(1)(Valko-Virag (2009) Universality in the bulk)
Let μ_n (reference energy) is away from the Tracy-Widom region : $n^{\frac{1}{6}}(2\sqrt{n} - |\mu_n|) \to \infty$.

Remark

(1)(Valko-Virag (2009) Universality in the bulk)
Let μ_n (reference energy) is away from the Tracy-Widom region : $n^{\frac{1}{6}}(2\sqrt{n} - |\mu_n|) \to \infty$.
Then

$$\sum_{j=1}^{n} \delta_{\Lambda_j} \to \zeta_\beta^G, \quad \text{where } \Lambda_j := \sqrt{4n - \mu_n^2}(\lambda_j - \mu_n).$$

Remark

(1)(Valko-Virag (2009) Universality in the bulk)
Let μ_n (reference energy) is away from the Tracy-Widom region : $n^{\frac{1}{6}}(2\sqrt{n} - |\mu_n|) \to \infty$.
Then

$$\sum_{j=1}^{n} \delta_{\Lambda_j} \to \zeta_\beta^G, \quad \text{where } \Lambda_j := \sqrt{4n - \mu_n^2}(\lambda_j - \mu_n).$$

(2) We have two SDE's which are similar each other, however,

Remark

(1)(Valko-Virag (2009) Universality in the bulk)
Let μ_n (reference energy) is away from the Tracy-Widom region : $n^{\frac{1}{6}}(2\sqrt{n} - |\mu_n|) \to \infty$.
Then

$$\sum_{j=1}^{n} \delta_{\Lambda_j} \to \zeta_{\beta}^{G}, \qquad \text{where } \Lambda_j := \sqrt{4n - \mu_n^2}(\lambda_j - \mu_n).$$

(2) We have two SDE's which are similar each other, however,

(i) Killip-Stoiciu : SDE has singularity at $t = 0$, but Ψ_t^{KS} is continuous for any $t > 0$

Remark

(1)(Valko-Virag (2009) Universality in the bulk)
Let μ_n (reference energy) is away from the Tracy-Widom region : $n^{\frac{1}{6}}(2\sqrt{n} - |\mu_n|) \to \infty$.
Then

$$\sum_{j=1}^{n} \delta_{\Lambda_j} \to \zeta_\beta^G, \qquad \text{where } \Lambda_j := \sqrt{4n - \mu_n^2}(\lambda_j - \mu_n).$$

(2) We have two SDE's which are similar each other, however,

(i) Killip-Stoiciu : SDE has singularity at $t = 0$, but Ψ_t^{KS} is continuous for any $t > 0$

(ii) Valko-Virag : SDE has no singularity, but $\Psi_{t-}^{VV} \in 2\pi\mathbf{Z}$

Critical Case

Go back to our model and let $\alpha = \frac{1}{2}$: $a(t) = t^{-\frac{1}{2}}(1 + o(1))$.

Theorem 3

(1) [KN2014] $\xi_L \overset{L\to\infty}{\to} \zeta_\beta^C$,

Critical Case

Go back to our model and let $\alpha = \frac{1}{2}$: $a(t) = t^{-\frac{1}{2}}(1 + o(1))$.

Theorem 3

(1) [KN2014] $\xi_L \overset{L\to\infty}{\to} \zeta_\beta^C$, (2) [N2014] $\xi_L \overset{L\to\infty}{\to} \zeta_\beta^G$

Critical Case

Go back to our model and let $\alpha = \frac{1}{2}$: $a(t) = t^{-\frac{1}{2}}(1 + o(1))$.

Theorem 3

(1) [KN2014] $\xi_L \overset{L\to\infty}{\to} \zeta_\beta^C$, (2) [N2014] $\xi_L \overset{L\to\infty}{\to} \zeta_\beta^G$

with $\beta = \beta(E_0) = 8\kappa_0^2 / C(\kappa_0) = \gamma(E_0)^{-1}$.

Critical Case

Go back to our model and let $\alpha = \frac{1}{2}$: $a(t) = t^{-\frac{1}{2}}(1 + o(1))$.

Theorem 3

(1) [KN2014] $\xi_L \overset{L\to\infty}{\to} \zeta_\beta^C$, (2) [N2014] $\xi_L \overset{L\to\infty}{\to} \zeta_\beta^G$

with $\beta = \beta(E_0) = 8\kappa_0^2 / C(\kappa_0) = \gamma(E_0)^{-1}$.

$\gamma(E)$: "Lyapunov exponent" in the sense that the solution ψ to $H\psi = E\psi$ satisfies $\psi(x) \sim |x|^{-\gamma(E)}$.

Critical Case

Go back to our model and let $\alpha = \frac{1}{2}$: $a(t) = t^{-\frac{1}{2}}(1 + o(1))$.

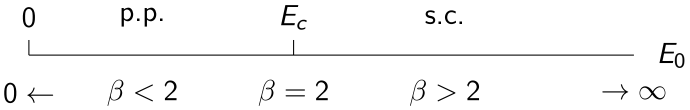
Theorem 3

(1) [KN2014] $\xi_L \overset{L\to\infty}{\to} \zeta_\beta^C$, (2) [N2014] $\xi_L \overset{L\to\infty}{\to} \zeta_\beta^G$

with $\beta = \beta(E_0) = 8\kappa_0^2 / C(\kappa_0) = \gamma(E_0)^{-1}$.

$\gamma(E)$: "Lyapunov exponent" in the sense that the solution ψ to $H\psi = E\psi$ satisfies $\psi(x) \sim |x|^{-\gamma(E)}$.

"Non-Universality"



All β's are realized.

Coincidence of two β-ensembles

Corollary 4

The limits of C_β-ensemble and G_β-ensemble are equal :

$$\zeta_\beta^C \stackrel{d}{=} \zeta_\beta^G .$$

for all $\beta > 0$.

Coincidence of two β-ensembles

Corollary 4

The limits of C_β-ensemble and G_β-ensemble are equal :

$$\zeta_\beta^C \stackrel{d}{=} \zeta_\beta^G.$$

for all $\beta > 0$.

Remark

(1) This fact had previously been known for specific β's, e.g., $\beta = 1, 2, 4$.

Coincidence of two β-ensembles

Corollary 4

The limits of C_β-ensemble and G_β-ensemble are equal :

$$\zeta_\beta^C \stackrel{d}{=} \zeta_\beta^G .$$

for all $\beta > 0$.

Remark

(1) This fact had previously been known for specific β's, e.g., $\beta = 1, 2, 4$.

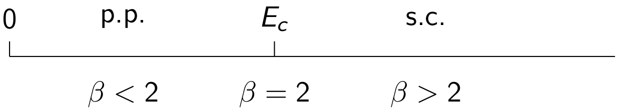
(2) Valko-Virag (2016) have "direct" proof of this fact.

Remarks

0 p.p. E_c s.c.

$\beta < 2$ $\beta = 2$ $\beta > 2$

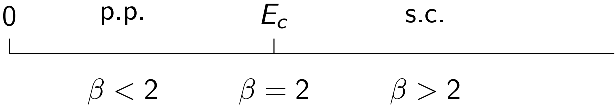
Remarks



Remarks

Sine$_\beta$-process has a "phase transition" between at $\beta = 2$.

Remarks



Remarks

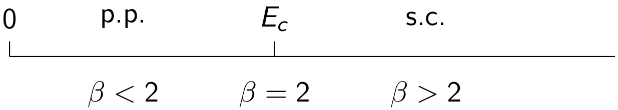
Sine$_\beta$-process has a "phase transition" between at $\beta = 2$.

(1)(Valko-Virag (2009))

(i) $\beta < 2$: $\Psi_t(\lambda)$ approaches to $2\pi\mathbf{Z}$ from above a.s.

(ii) $\beta > 2$: $\Psi_t(\lambda)$ approaches to $2\pi\mathbf{Z}$ from below with pos. prob.

Remarks



Remarks

Sine$_\beta$-process has a "phase transition" between at $\beta = 2$.

(1)(Valko-Virag (2009))

(i) $\beta < 2$: $\Psi_t(\lambda)$ approaches to $2\pi\mathbf{Z}$ from above a.s.

(ii) $\beta > 2$: $\Psi_t(\lambda)$ approaches to $2\pi\mathbf{Z}$ from below with pos. prob.

(2)(Valko, private communication)

$\exists H_{Dirac}$ on s.t. $\sigma(H_{Dirac}) \stackrel{d}{=} Sine_\beta$.

$\beta \le 2 \Longrightarrow H_{Dirac}$: limit point

$\beta > 2 \Longrightarrow H_{Dirac}$: limit circle

Remarks(Continued)

0 p.p. E_c s.c.

$0 \leftarrow$ $\beta < 2$ $\beta = 2$ $\beta > 2$ $\rightarrow \infty$

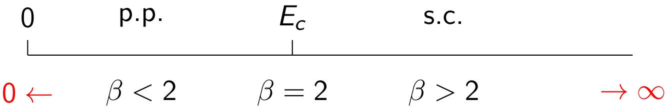
Remarks(Continued)

0 p.p. E_c s.c.

$0 \leftarrow$ $\beta < 2$ $\beta = 2$ $\beta > 2$ $\rightarrow \infty$

(1) [N2015] As $\beta \uparrow \infty$, $\mathrm{Sine}_\beta \xrightarrow{d}$ Clock process

Remarks(Continued)

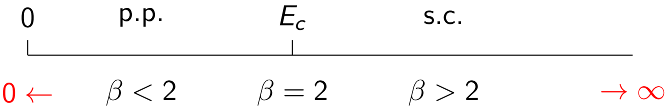


(1) [N2015] As $\beta \uparrow \infty$, $\text{Sine}_\beta \xrightarrow{d}$ Clock process

(2) (Allez - Dumaz (2014))

As $\beta \downarrow 0$, $\text{Sine}_\beta \xrightarrow{d}$ Poisson process with intensity $(2\pi)^{-1} d\lambda$.

Remarks(Continued)



(1) [N2015] As $\beta \uparrow \infty$, $\mathrm{Sine}_\beta \xrightarrow{d}$ Clock process

(2) (Allez - Dumaz (2014))
As $\beta \downarrow 0$, $\mathrm{Sine}_\beta \xrightarrow{d}$ Poisson process with intensity $(2\pi)^{-1} d\lambda$.

(3) (Benaych-Georges - Péché (2015), Duy-N (2016))
For GβE, $\xi_n \to$ Poisson$(\rho(E_0))$, if $n\beta =$ const.

PP case ($\alpha < \frac{1}{2}$)

Theorem 5 ([KN2017] PP case $\Longrightarrow$ Poisson process)

$$\xi_L(dx) \stackrel{d}{\to} \text{Poisson}\left(\frac{1}{\pi} dx\right)$$

PP case ($\alpha < \frac{1}{2}$)

Theorem 5 ([KN2017] PP case $\Longrightarrow$ Poisson process)

$$\xi_L(dx) \stackrel{d}{\to} \text{Poisson}\left(\frac{1}{\pi} dx\right)$$

Summary

(1) $\alpha > \frac{1}{2}$: $\xi_L(dx) \stackrel{d}{\to}$ Clock process

(2) $\alpha = \frac{1}{2}$: $\xi_L(dx) \stackrel{d}{\to} \text{Sine}_\beta$

(3) $\alpha < \frac{1}{2}$: $\xi_L(dx) \stackrel{d}{\to} \text{Poisson}\left(\frac{1}{\pi} dx\right)$

Outline of proof 1

Let x_t be the solution to $H_L x_t = \kappa^2 x_t$ which we write in the Prüfer coordinate.

$$\begin{pmatrix} x_t \\ x_t' / \kappa \end{pmatrix} = r_t \begin{pmatrix} \sin \theta_t \\ \cos \theta_t \end{pmatrix}, \quad \theta_0 = 0.$$

Outline of proof 1

Let x_t be the solution to $H_L x_t = \kappa^2 x_t$ which we write in the Prüfer coordinate.

$$\begin{pmatrix} x_t \\ x_t'/\kappa \end{pmatrix} = r_t \begin{pmatrix} \sin\theta_t \\ \cos\theta_t \end{pmatrix}, \quad \theta_0 = 0.$$

Let

$$\Psi_L(\lambda) := \theta_L(\kappa_0 + \frac{\lambda}{L}) - \theta_L(\kappa_0), \quad \kappa_0 := \sqrt{E_0}$$

be the relative Prüfer phase.

Outline of proof 1

Let x_t be the solution to $H_L x_t = \kappa^2 x_t$ which we write in the Prüfer coordinate.

$$\begin{pmatrix} x_t \\ x_t'/\kappa \end{pmatrix} = r_t \begin{pmatrix} \sin\theta_t \\ \cos\theta_t \end{pmatrix}, \quad \theta_0 = 0.$$

Let

$$\Psi_L(\lambda) := \theta_L(\kappa_0 + \frac{\lambda}{L}) - \theta_L(\kappa_0), \quad \kappa_0 := \sqrt{E_0}$$

be the relative Prüfer phase. Then we have

$$\mathbf{E}[e^{-\xi_L(f)}] = \mathbf{E}\left[\exp\left(-\sum_{n \geq n(L) - m(\kappa_0, L)} f\left(\Psi_L^{-1}(n\pi - \phi(\kappa_0, L))\right)\right)\right]$$

where $\theta_L(\kappa_0, L) = m(\kappa_0, L)\pi + \phi(\kappa_0, L)$, $m(\kappa_0, L) \in \mathbf{Z}$, $\phi(\kappa_0, L) \in [0, \pi)$.

Outline of Proof 2

We replace L by n, and consider

$$\Psi_t^{(n)}(\lambda) := \theta_{nt}(\kappa_\lambda) - \theta_{nt}(\kappa_0),$$

$$\sim \lambda t + \frac{1}{2\kappa_0} Re \int_0^{nt} a(s) \left(e^{2i\theta_s(\kappa_\lambda)} - e^{2i\theta_s(\kappa_0)} \right) F(X_s) ds$$

$$\kappa_\lambda := \kappa_0 + \frac{\lambda}{n} \quad n > 0, \quad t \in [0, 1].$$

Outline of Proof 2

We replace L by n, and consider

$$\Psi_t^{(n)}(\lambda) := \theta_{nt}(\kappa_\lambda) - \theta_{nt}(\kappa_0),$$
$$\sim \lambda t + \frac{1}{2\kappa_0} Re \int_0^{nt} a(s)\left(e^{2i\theta_s(\kappa_\lambda)} - e^{2i\theta_s(\kappa_0)}\right) F(X_s) ds$$
$$\kappa_\lambda := \kappa_0 + \frac{\lambda}{n} \quad n > 0, \quad t \in [0,1].$$

By using "Ito's formula",

$$e^{2i\kappa s} F(X_s) ds = d(e^{2i\kappa s} g_\kappa(X_s)) - e^{2i\kappa s} \nabla g_\kappa(X_s) dX_s$$
$$g_\kappa := (L + 2i\kappa)^{-1} F, \quad L : \text{generator of } X_s,$$

Outline of Proof 2

We replace L by n, and consider

$$\Psi_t^{(n)}(\lambda) := \theta_{nt}(\kappa_\lambda) - \theta_{nt}(\kappa_0),$$

$$\sim \lambda t + \frac{1}{2\kappa_0} Re \int_0^{nt} a(s) \left(e^{2i\theta_s(\kappa_\lambda)} - e^{2i\theta_s(\kappa_0)} \right) F(X_s) ds$$

$$\kappa_\lambda := \kappa_0 + \frac{\lambda}{n} \quad n > 0, \quad t \in [0,1].$$

By using "Ito's formula",

$$e^{2i\kappa s} F(X_s) ds = d(e^{2i\kappa s} g_\kappa(X_s)) - e^{2i\kappa s} \nabla g_\kappa(X_s) dX_s$$

$$g_\kappa := (L + 2i\kappa)^{-1} F, \quad L : \text{generator of } X_s,$$

we have

$$\Psi_t^{(n)}(\lambda) \sim \lambda t + n^{\frac{1}{2}-\alpha} \frac{1}{2\kappa_0} Re \int_0^t s^{-\alpha} (e^{2i\Psi_s^{(n)}(\lambda)} - 1) \nabla g_\kappa dX_s$$

Outline of Proof 3

$$\Psi_t^{(n)}(\lambda) \sim \lambda t + n^{\frac{1}{2}-\alpha} \frac{1}{2\kappa_0} Re \int_0^t s^{-\alpha} (e^{2i\Psi_s^{(n)}(\lambda)} - 1) \nabla g_\kappa dX_s$$

Outline of Proof 3

$$\Psi_t^{(n)}(\lambda) \sim \lambda t + n^{\frac{1}{2}-\alpha} \frac{1}{2\kappa_0} Re \int_0^t s^{-\alpha} (e^{2i\Psi_s^{(n)}(\lambda)} - 1) \nabla g_\kappa dX_s$$

(1) AC case $(\alpha > \frac{1}{2})$: $\Psi_t^{(n)}(\lambda) \to \lambda t$, a.s.

Outline of Proof 3

$$\Psi_t^{(n)}(\lambda) \sim \lambda t + n^{\frac{1}{2}-\alpha} \frac{1}{2\kappa_0} Re \int_0^t s^{-\alpha}(e^{2i\Psi_s^{(n)}(\lambda)} - 1)\nabla g_\kappa dX_s$$

(1) AC case ($\alpha > \frac{1}{2}$) : $\Psi_t^{(n)}(\lambda) \to \lambda t$, a.s.

(2) Critical Case ($\alpha = \frac{1}{2}$) : $\Psi_t^{(n)}(\lambda) \stackrel{d}{\to} \Psi_t(\lambda)$: sol. to SDE,

Outline of Proof 3

$$\Psi_t^{(n)}(\lambda) \sim \lambda t + n^{\frac{1}{2}-\alpha} \frac{1}{2\kappa_0} Re \int_0^t s^{-\alpha} (e^{2i\Psi_s^{(n)}(\lambda)} - 1) \nabla g_\kappa dX_s$$

(1) AC case ($\alpha > \frac{1}{2}$) : $\Psi_t^{(n)}(\lambda) \to \lambda t$, a.s.

(2) Critical Case ($\alpha = \frac{1}{2}$) : $\Psi_t^{(n)}(\lambda) \stackrel{d}{\to} \Psi_t(\lambda)$: sol. to SDE,

(3) PP case ($\alpha < \frac{1}{2}$) : $\Psi_t^{(n)}(\lambda) \stackrel{d}{\to}$ Poisson jump process.
(Using the idea of Allez - Dumaz(2014))

Outline of Proof 3

$$\Psi_t^{(n)}(\lambda) \sim \lambda t + n^{\frac{1}{2}-\alpha} \frac{1}{2\kappa_0} Re \int_0^t s^{-\alpha}(e^{2i\Psi_s^{(n)}(\lambda)} - 1)\nabla g_\kappa dX_s$$

(1) AC case $(\alpha > \frac{1}{2})$: $\Psi_t^{(n)}(\lambda) \to \lambda t$, a.s.

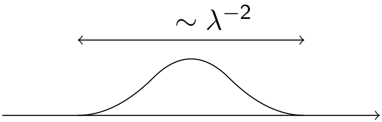
(2) Critical Case $(\alpha = \frac{1}{2})$: $\Psi_t^{(n)}(\lambda) \stackrel{d}{\to} \Psi_t(\lambda)$: sol. to SDE,

(3) PP case $(\alpha < \frac{1}{2})$: $\Psi_t^{(n)}(\lambda) \stackrel{d}{\to}$ Poisson jump process.
(Using the idea of Allez - Dumaz(2014))

Moreover in (3), $\Psi_t^{(n)}(\lambda) \stackrel{d}{\to} \pi \int_{[0,t]\times[0,\lambda]} \hat{P}(dsd\lambda')$, where
$\hat{P} := Poisson(\pi^{-1} 1_{[0,1]}(s) dsd\lambda')$.

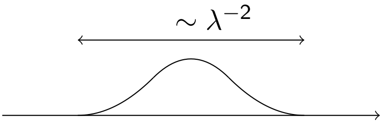
Decaying Coupling Model

In 1-dim, $H = -\triangle + \lambda V$ generically has localization length $\sim \lambda^{-2}$.



Decaying Coupling Model

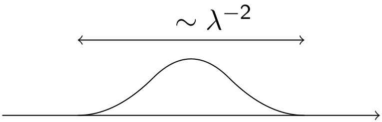
In 1-dim, $H = -\triangle + \lambda V$ generically has localization length $\sim \lambda^{-2}$.



So, for $H_L := H|_{[0,L]}$,

Decaying Coupling Model

In 1-dim, $H = -\triangle + \lambda V$ generically has localization length $\sim \lambda^{-2}$.

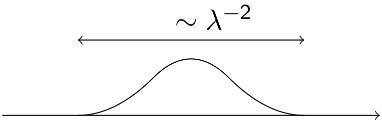


So, for $H_L := H|_{[0,L]}$, we expect

(1) $L \ll \frac{1}{\lambda^2} (\Leftrightarrow \lambda \ll \frac{1}{\sqrt{L}}) \Longrightarrow$ "extended" $\Longrightarrow \xi_L \to$ clock

Decaying Coupling Model

In 1-dim, $H = -\triangle + \lambda V$ generically has localization length $\sim \lambda^{-2}$.



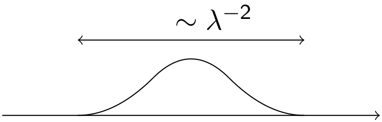
So, for $H_L := H|_{[0,L]}$, we expect

(1) $L \ll \frac{1}{\lambda^2} (\Leftrightarrow \lambda \ll \frac{1}{\sqrt{L}}) \Longrightarrow$ "extended" $\Longrightarrow \xi_L \to$ clock

(2) $L \gg \frac{1}{\lambda^2} (\Leftrightarrow \lambda \gg \frac{1}{\sqrt{L}}) \Longrightarrow$ "localized" $\Longrightarrow \xi_L \to$ Poisson

Decaying Coupling Model

In 1-dim, $H = -\triangle + \lambda V$ generically has localization length $\sim \lambda^{-2}$.



So, for $H_L := H|_{[0,L]}$, we expect

(1) $L \ll \frac{1}{\lambda^2} (\Leftrightarrow \lambda \ll \frac{1}{\sqrt{L}}) \Longrightarrow$ "extended" $\Longrightarrow \xi_L \to$ clock

(2) $L \gg \frac{1}{\lambda^2} (\Leftrightarrow \lambda \gg \frac{1}{\sqrt{L}}) \Longrightarrow$ "localized" $\Longrightarrow \xi_L \to$ Poisson

(3) $L \sim \frac{1}{\lambda^2} (\Leftrightarrow \lambda \sim \frac{1}{\sqrt{L}}) \Longrightarrow$ "critical" $\Longrightarrow \xi_L \to \beta$-ensemble ?

Hamiltonian

$$H_\lambda := -\frac{d^2}{dt^2} + \lambda F(X_t)$$

Hamiltonian

$$H_\lambda := -\frac{d^2}{dt^2} + \lambda F(X_t)$$

$$H_L := H_{\lambda_L}|_{[0,L]}, \quad \lambda_L = L^{-\alpha}$$

Hamiltonian

$$H_\lambda := -\frac{d^2}{dt^2} + \lambda F(X_t)$$
$$H_L := H_{\lambda_L}|_{[0,L]}, \quad \lambda_L = L^{-\alpha}$$

In this section, we always assume :

Assumption Subseq. $\{L_j\}$ satisfies $L_j \stackrel{j\to\infty}{\to} \infty$ and

$$\kappa_0 L_j = m_j \pi + \beta + o(1), \quad j \to \infty.$$

for some $m_j \in \mathbf{N}$, $\beta \in [0, \pi)$.

Results

Theorem 3.1 [N2017]

(1) (Extended : $\alpha > \frac{1}{2}$) $N_n(\kappa_1, \kappa_2) = \left\lfloor \frac{\kappa_2 n}{\pi} \right\rfloor - \left\lfloor \frac{\kappa_1 n}{\pi} \right\rfloor$

Results

Theorem 3.1 [N2017]

(1) (Extended : $\alpha > \frac{1}{2}$) $N_n(\kappa_1, \kappa_2) = \left\lfloor \frac{\kappa_2 n}{\pi} \right\rfloor - \left\lfloor \frac{\kappa_1 n}{\pi} \right\rfloor$

(2) (Critical : $\alpha = \frac{1}{2}$) $N_n(\kappa_1, \kappa_2) = \frac{(\kappa_2 - \kappa_1) n}{\pi} + \text{(bounded)}$

Results

Theorem 3.1 [N2017]

(1) (Extended : $\alpha > \frac{1}{2}$) $\quad N_n(\kappa_1, \kappa_2) = \left\lfloor \frac{\kappa_2 n}{\pi} \right\rfloor - \left\lfloor \frac{\kappa_1 n}{\pi} \right\rfloor$

(2) (Critical : $\alpha = \frac{1}{2}$) $\quad N_n(\kappa_1, \kappa_2) = \frac{(\kappa_2 - \kappa_1)n}{\pi} + \text{(bounded)}$

(3) (Localized : $\alpha < \frac{1}{2}$)

$$N_n(\kappa_1, \kappa_2) = \frac{(\kappa_2 - \kappa_1)n}{\pi} + \sum_j C_j n^{1-j\alpha} + \text{Gaussian}$$

Results

Theorem 3.1 [N2017]

(1) (Extended : $\alpha > \frac{1}{2}$) $N_n(\kappa_1, \kappa_2) = \left\lfloor \frac{\kappa_2 n}{\pi} \right\rfloor - \left\lfloor \frac{\kappa_1 n}{\pi} \right\rfloor$

(2) (Critical : $\alpha = \frac{1}{2}$) $N_n(\kappa_1, \kappa_2) = \frac{(\kappa_2 - \kappa_1) n}{\pi} + (\text{bounded})$

(3) (Localized : $\alpha < \frac{1}{2}$)

$$N_n(\kappa_1, \kappa_2) = \frac{(\kappa_2 - \kappa_1) n}{\pi} + \sum_j C_j n^{1 - j\alpha} + \text{Gaussian}$$

Theorem 3.2 [N2014], [KN2017]

(1) (Extended : $\alpha > \frac{1}{2}$) $\xi_L \to$ clock process

Results

Theorem 3.1 [N2017]

(1) (Extended : $\alpha > \frac{1}{2}$) $N_n(\kappa_1, \kappa_2) = \left\lfloor \frac{\kappa_2 n}{\pi} \right\rfloor - \left\lfloor \frac{\kappa_1 n}{\pi} \right\rfloor$

(2) (Critical : $\alpha = \frac{1}{2}$) $N_n(\kappa_1, \kappa_2) = \frac{(\kappa_2 - \kappa_1)n}{\pi} + (\text{bounded})$

(3) (Localized : $\alpha < \frac{1}{2}$)

$$N_n(\kappa_1, \kappa_2) = \frac{(\kappa_2 - \kappa_1)n}{\pi} + \sum_j C_j n^{1 - j\alpha} + \text{Gaussian}$$

Theorem 3.2 [N2014], [KN2017]

(1) (Extended : $\alpha > \frac{1}{2}$) $\xi_L \to$ clock process

(2) (Critical : $\alpha = \frac{1}{2}$) $\xi_L \to \mathrm{Sch}_\tau$-process

Results

Theorem 3.1 [N2017]

(1) (Extended : $\alpha > \frac{1}{2}$) $\quad N_n(\kappa_1, \kappa_2) = \left\lfloor \frac{\kappa_2 n}{\pi} \right\rfloor - \left\lfloor \frac{\kappa_1 n}{\pi} \right\rfloor$

(2) (Critical : $\alpha = \frac{1}{2}$) $\quad N_n(\kappa_1, \kappa_2) = \frac{(\kappa_2 - \kappa_1) n}{\pi} + \text{(bounded)}$

(3) (Localized : $\alpha < \frac{1}{2}$)

$$N_n(\kappa_1, \kappa_2) = \frac{(\kappa_2 - \kappa_1) n}{\pi} + \sum_j C_j n^{1 - j\alpha} + \text{Gaussian}$$

Theorem 3.2 [N2014], [KN2017]

(1) (Extended : $\alpha > \frac{1}{2}$) $\quad \xi_L \to$ clock process

(2) (Critical : $\alpha = \frac{1}{2}$) $\quad \xi_L \to \mathrm{Sch}_\tau$-process

(3) (Localized : $\alpha < \frac{1}{2}$) $\quad \xi_L \to$ Poisson process

References

[1] S. Kotani and F. Nakano, Level statistics for the one-dimensional Schrödinger operators with random decaying potential, Interdiscipl. Math. Sci., **17**(2014), 343-373.

[2] F. Nakano, Level statistics for one-dimensional Schrödinger operators and Gaussian beta ensemble, J. Stat. Phys. **156**(2014), 66-93.

[3] S. Kotani and F. Nakano, Poisson statistics for 1d Schrödinger operators with random decaying potentials, Electronic Journal of Probability (2017), Vol. 22, no. 69, 1-31.

[4] F. Nakano, Fluctuation of density of states for 1d Schrödinger operators, J. Stat. Phys. (2017) 166 : 1393-1404.

[5] T.K. Duy and F. Nakano, Gaussian beta ensembles at high temperature : eigenvalue fluctuations and bulk statistics, arXiv : 1611.09476