Mesoscopic eigenvalue correlations of random matrices

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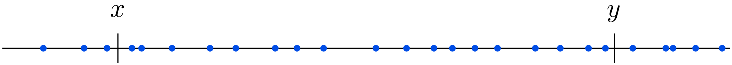
Eigenvalue correlation functions

Hermitian $N \times N$ random matrix H with eigenvalues $\lambda_1, \ldots, \lambda_N$.

Goal: eigenvalue density-density correlations at two different energies.



Eigenvalue process $\sum_i \delta_{\lambda_i}$, with correlation functions

$$\rho_1(x) := \mathbb{E} \sum_i \delta(x - \lambda_i)\,, \qquad \rho_2(x,y) := \mathbb{E} \sum_{i \neq j} \delta(x - \lambda_i)\delta(y - \lambda_j)\,, \qquad \ldots .$$

Connected (or truncated) two-point function

$$p(x,y) := \rho_2(x,y) - \rho_1(x)\rho_1(y)\,,$$

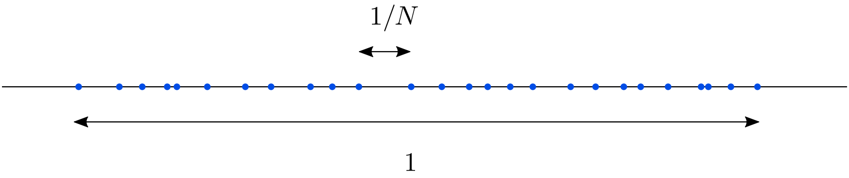
measures eigenvalue density-density correlations at the energies x and y.

Spectral scales

Let H be a Wigner matrix: $(H_{ij} : i \leqslant j)$ are independent with $\mathbb{E}H_{ij} = 0$, $\mathbb{E}|\sqrt{N}H_{ij}|^2 = 1$, and $|\sqrt{N}H_{ij}|^C = O(1)$.

Scales $\omega = \frac{y-x}{2}$:

- Macroscopic: $\omega \asymp 1$ (global extent of the spectrum).
- Microscopic: $\omega \asymp 1/N$ (eigenvalue spacing).
- Mesoscopic: $1/N \ll \omega \ll 1$.



Microscopic eigenvalue correlations

Let $\varrho_E := \frac{1}{2\pi}\sqrt{(4-E^2)_+}$ be the semicircle law.

To analyse the microscopic correlations, choose an energy E and consider rescaled eigenvalue process

$$\sum_i \delta_{N\varrho_E(\lambda_i - E)}$$

with connected two-point function

$$p_E(u,v) = \frac{1}{(N\varrho_E)^2}\, p\left(E + \frac{u}{N\varrho_E}, E + \frac{v}{N\varrho_E}\right).$$

Microscopic density-density correlations $\longleftrightarrow$ behaviour of p_E for fixed u, v.

Wigner-Gaudin-Mehta-Dyson (WGDM) statistics

Let H be GUE ($\beta = 2$) or GOE ($\beta = 1$). Then

$$\lim_{N \to \infty} p_E(u, v) = Y_\beta(u - v) \tag{1}$$

weakly, where

$$Y_2(u) := -s(u)^2\,, \qquad Y_1(u) := -s'(u) \int_u^\infty s(v)\,\mathrm{d}v - s(u)^2\,,$$

with the sine kernel

$$s(u) := \frac{\sin(\pi u)}{\pi u}\,.$$

For $u \to \infty$ we have the asymptotic expansion

$$Y_1(u) = -\frac{1}{\pi^2 u^2} + \frac{1 + \cos^2(\pi u)}{\pi^4 u^4} + O\left(\frac{1}{u^6}\right).$$

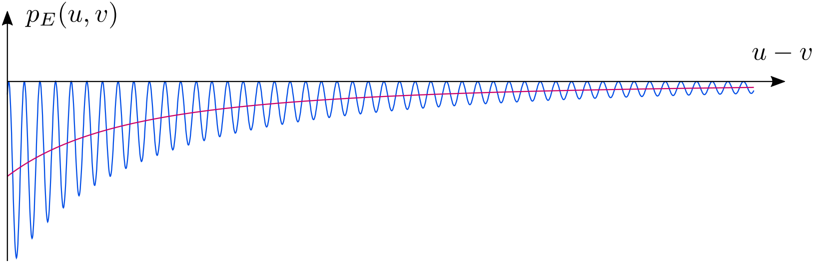
Theorem [Universality (ABEKSTVYY, 2009-2016)]. (1) holds for arbitrary Wigner matrices.

WGMD statistics for mesoscopic separations

Goal:

WGMD statistics for p_E on mesoscopic scales $1 \ll u - v \leqslant N$?

I.e., analyse the density-density correlations between energies u and v at mesoscopic separations.



Macroscopic fluctuations of linear statistics

A different type of fluctuation result, involving correlations of $O(N^2)$ eigenvalues of typical macroscopic separation $\omega \asymp 1$.

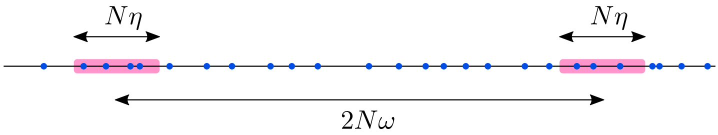
E.g. for $\beta = 1$ we have [Lytova, Pastur – 2009]

$$\operatorname{Var} \sum_i f(\lambda_i) = \frac{1}{2\pi^2} \int_{-2}^{2} \mathrm{d}x \int_{-2}^{2} \mathrm{d}y \left(\frac{f(x) - f(y)}{x - y} \right)^2 \frac{4 - xy}{\sqrt{4 - x^2}\sqrt{4 - y^2}} + \frac{N^2 \mathcal{C}_4(H_{12})}{2\pi^2} \left(\int_{-2}^{2} \mathrm{d}x\, f(x)\, \frac{2 - x^2}{\sqrt{4 - x^2}} \right)^2,$$

where $\mathcal{C}_4(\cdot)$ is the fourth cumulant.

Any relation to WGDM statistics?

Result



Fix $E \in (-2,2)$ and $f, g \in C_c^\infty(\mathbb{R})$. Let

$$1/N \ll \eta \leqslant \omega \leqslant 1\,.$$

Using the notation

$$f_\pm(u) := \frac{1}{N\eta} f\left(\frac{u \mp N\omega}{N\eta}\right).$$

we have

$$\int p_E(u,v) f_+(u) g_-(v)\,\mathrm{d}u\,\mathrm{d}v = \int \Upsilon_{E,\beta}(u,v)\, f_+(u) g_-(v)\,\mathrm{d}u\,\mathrm{d}v\,,$$

where,

$$\Upsilon_{E,1}(u,v) = -\frac{1}{\pi^2(u-v)^2} + \frac{3}{2\pi^4(u-v)^4} + \mathcal{E}(u,v) + \frac{1}{N^2\kappa_E^2}\left(F_1(u,v) + F_2(u,v)\sum_{i,j}\mathcal{C}_4(H_{ij}) + F_3(u,v)\sum_i \mathcal{C}_3(H_{ii})\right)$$

and

$$\Upsilon_{E,2}(u,v) = -\frac{1}{2\pi^2(u-v)^2} + \mathcal{E}(u,v) + \frac{1}{N^2\kappa_E^2}\left(\frac{1}{2}F_1(u,v) + F_2(u,v)\sum_{i,j}\mathcal{C}_{2,2}(H_{ij}) + F_3(u,v)\sum_i \mathcal{C}_3(H_{ii})\right).$$

Here $\mathcal{C}_{\cdot}(\cdot)$ is the cumulant, $\kappa_E := 2\pi\varrho_E = \sqrt{4-E^2}$, and F_1, F_2, F_3 are explicit bounded elementary functions.

Blue = Average of Y_β, Red = non-universal, $\mathcal{E}(u,v)$ = quantitatively controlled error term.

Remarks

- WGDM statistics valid to leading order on all mesoscopic scales $N\omega \ll N$. Fails at macroscopic scale $N\omega \asymp N$.
- For $\beta = 1$, the subleading corrections of WGDM are leading order for $N\omega \ll \sqrt{N}$. For $N\omega \gg \sqrt{N}$, dominant subleading corrections are non-universal. These become leading order on macroscopic scales $N\omega \asymp N$.
- Result completely insensitive to size of spectral window $N\eta \gg 1$. Even at macroscopic scale $N\omega \asymp 1$, our result is much more precise than [Lytova, Pastur, 2009].
- After the local rescaling by $N\varrho_E$, the WGDM terms do not depend on E but the non-universal ones are proportional to ϱ_E^{-2}.

Comparison to Gustavsson's theorem

Gustavsson [2005] analyses mesoscopic correlations of eigenvalue locations instead of densities for GUE.

Order eigenvalues $\lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_N$ and introduce quantiles γ_i defined by $i/N = \int_{-2}^{\gamma_i} \varrho_x \, \mathrm{d}x$. Define normalized eigenvalues

$$\tilde{\lambda}_i := \frac{\pi \varrho_{\gamma_i} N(\lambda_i - \gamma_i)}{\sqrt{\log N}} \,.$$

Theorem. $\tilde{\lambda}_i \sim \mathcal{N}(0,1)$ and $\mathrm{Cov}(\tilde{\lambda}_i, \tilde{\lambda}_j) \sim \beta$, where $\beta = 1 - \log_N (j-i)$.

Locations of two mesoscopically separated eigenvalues have a covariance of the same order as their individual variances.

Interpretation: eigenvalues fluctuate as a semi-rigid jelly on the scale $\sqrt{\log N}/N$.

Fluctuations of eigenvalue locations and density have little to do with one another.

Some ideas of proof

We can rewrite

$$p_E(u,v) = \frac{1}{(N\varrho_E)^2}\,\mathrm{Cov}(X^\eta(u), X^\eta(v))\,,$$

where $X^\eta(u) := \sum_i f^{\eta/\varrho_E}\big(E + \frac{u}{N\varrho_E} - \lambda_i\big)$ and $f^\varepsilon(x) := \frac{1}{\varepsilon} f\big(\frac{x}{\varepsilon}\big)$.

Key difficulty, appearing throughout the proof: we are computing the covariance (to an arbitrary precision) of two weakly correlated random variables:

$$\mathrm{Cov}(X^\eta(u), X^\eta(v)) \asymp \frac{1}{\omega^2}\,, \qquad \mathrm{Var}(X^\eta(u)) \asymp \frac{1}{\eta^2}\,,$$

with $1/N \ll \eta \leqslant \omega \leqslant 1$.

Main work: compute covariance of Green functions (with $E_1 - E_2 = 2\omega$)

$$G = (H - E_1 - \mathrm{i}\eta)^{-1}\,, \qquad F = (H - E_2 - \mathrm{i}\eta)^{-1}\,.$$

With notations

$$\underline{M} := \frac{1}{N}\operatorname{Tr} M\,, \qquad \langle X \rangle := X - \mathbb{E}X\,,$$

we have to compute

$$\mathbb{E}\langle \underline{G} \rangle \langle \underline{F^*} \rangle\,.$$

We do this by deriving a recursive family of self-consistent equations, indexed by a finite tree, in polynomials in expectations of polynomials of the variables $\underline{A^m}$, $\langle \underline{A^m} \rangle$, A^m_{xy}, where $A = G, G^*, F, F^*$.

Simple tools

- Resolvent identity $zG = H(H-z)^{-1} - I\,.$
- Cumulant expansion [Khorunzhy, Khoruzhenko, Pastur – 1996]

$$\mathbb{E}\big[h \cdot f(h)\big] = \sum_{k=0}^{l} \frac{1}{k!}\mathcal{C}_{k+1}(h)\mathbb{E}[f^{(k)}(h)] + (\mathrm{Error})\,.$$

Can be viewed as a generalization of Gaussian integration by parts to arbitrary random variables. Alternatively, a quantitative and more precise version of Stein's method.

Very powerful for deriving recursive high moment estimates in RMT [He, K – 2016]. This strategy was subsequetly used to derive local laws [Lee, Schnelli – 2017], [He, K, Rosenthal – 2017].

How to start

Apply resolvent identity and cumulant expansion to $\mathbb{E}\langle \underline{G} \rangle \langle \underline{F^*} \rangle$, and get

$$\mathbb{E}\langle \underline{G} \rangle \langle \underline{F^*} \rangle = \frac{1}{-z_1 - 2\mathbb{E}\underline{G}} \left(\frac{2}{N^2} \mathbb{E}\underline{GF^{*2}} + \frac{1}{N} \mathbb{E}\langle \underline{G^2} \rangle \langle \underline{F^*} \rangle + \mathbb{E}\langle \underline{G} \rangle^2 \langle \underline{F^*} \rangle + \mathcal{W}_1 \right).$$

Leading term of order $1/(N\omega)^2$ (will have to be computed precisely by another self-consistent equation).

Error terms have to be estimated. Naive attempt: using the local semicircle law, we obtain

$$\mathbb{E}\langle \underline{G} \rangle^2 \langle \underline{F^*} \rangle = O\left(\frac{1}{(N\eta)^3} \right).$$

Much too big!

Reason: we did not exploit that $\langle \underline{G} \rangle^2$ and $\langle \underline{F^*} \rangle$ are weakly correlated. Solution: self-consistent equations for error terms.

General scheme for a term X:

- If X is an error term and the naive bound is too large, derive a self-consistent equation that expresses X in terms of a family of other terms.
- If X is a term we wish to compute, derive a self-consistent equation that extracts its main contribution plus error terms.

At each step, every term X gives rise to a set of children $\mathcal{S}(X)$ of further terms (tree).

How do we stop?

1. Identify a large enough set $\mathcal{F}$ of terms that is closed under the map $X \mapsto \mathcal{S}(X)$.
2. Find bounds that allows to estimate all $X \in \mathcal{F}$ of a sufficiently high generation.

Algebra gets somewhat involved.

Tools for stopping

For the stopping in 2, we need much more than the local semicircle law: a priori bounds from [He, K – 2016] on $(G^m)_{ij}$, and the estimate

$$\mathbb{E}\underline{G^m} = O_m(1) \tag{2}$$

for all $m \in \mathbb{N}$.

(Local semicircle law gives $\mathbb{E}\underline{G^m} = O(1/\eta^{m-1})$.)

Interpretation: expected eigenvalue density ρ has uniformly bounded derivatives of all order down to all mesoscopic scales:

$$\operatorname{Im} \mathbb{E}\underline{G^m} = \operatorname{Im} \int \frac{\rho(x)}{(x-z)^m}\, \mathrm{d}x = (-1)^{m-1} \operatorname{Im} \int \frac{\rho_1^{(m-1)}(x)}{x-z}\, \mathrm{d}x$$

$$= (-1)^{m-1} \int \rho^{(m-1)}(x) \cdot \frac{\eta}{(x-E)^2+\eta^2}\, \mathrm{d}x = (-1)^{m-1} (\rho^{(m-1)} * \delta^\eta)(E)\,.$$

Proof: another recursive family of self-consistent equations, except that now it is possible to stop just using the local semicircle law and [He, K – 2016].

Define

$$F_i(u,v) = g_i\Big(\frac{u-E}{N\varrho_E}, \frac{v-E}{N\varrho_E}\Big).$$

where

$$g_1(x_1,x_2) = -\frac{4\big(4 + x_1x_2 + \sqrt{(4-x_1^2)(4-x_2^2)}\,\big)}{\sqrt{(4-x_1^2)(4-x_2^2)}\big(\sqrt{4-x_1^2} + \sqrt{4-x_2^2}\,\big)^2}\,,$$

$$g_2(x_1,x_2) = \frac{2(x_1^2-2)(x_2^2-2)}{\sqrt{(4-x_1^2)(4-x_2^2)}}\,,$$

$$g_3(x_1,x_2) = \frac{x_1^2x_2 + x_1x_2^2 - 2x_1 - 2x_2}{\sqrt{(4-x_1^2)(4-x_2^2)}}\,.$$

Error term

$$\mathcal{E}(u,v) = O\bigg(\frac{1}{(u-v)^5} + \frac{1}{N(u-v)^3} + \frac{1}{N^{3/2}(u-v)^2} + \frac{1}{N^2(u-v)}\bigg)\,.$$