Girko's Lévy-Khintchine-type Random Matrices

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History and Motivation

- Wigner matrices ('55, '58).
- Heavy tailed i.i.d. entries (up to symmetry).
  Cizeau/Bouchaud ('94), Soshnikov ('04), Ben Arous/Guionnet ('08)
  Bordenave/Caputo/Chafai ('11).
- Adjacency matrices of Erdős-Rényi graphs with $p = 1/n$.
  Rogers/Bray ('88),
  Bordenave/Lelarge ('10).
- General symmetric matrices with i.i.d. entries Girko ('75):
  Sum of a row converges weakly as $n \to \infty$.
  Limits are infinitely divisible $ID(\sigma^2, b, \nu)$.

Normalization for Wigner matrices

$$\frac{1}{n}\sum_{j=1}^{n}\delta_{\lambda_j(\omega)} = \mathrm{ESD}_n \quad \Longrightarrow \quad \frac{1}{2\pi}\sqrt{4-x^2}\,dx$$

Simple argument for the scaling:

- Tightness (in expectation) of random probability measures:

$$\mathbf{E}(\text{Second Moment}(\mathrm{ESD}_n)) = \mathbf{E}\frac{1}{n}\operatorname{Tr}(A_n^2) = n\mathbf{E}A_n(i,j)^2.$$

- So we need

$$\mathbf{E}A_n(i,j)^2 = \mathcal{O}\left(\frac{1}{n}\right).$$

- Instead of normalizing, change the distribution as n varies:

$$A_n(i,j) \;\sim\; \text{Bernoulli}\,(\Lambda/n) \quad \text{so that} \quad \mathbf{E}A_n(i,j)^2 = \Lambda/n.$$

3 Different normalizations for random matrices

- Wigner:

$$\mathbf{E}A_n(i,j)^2 \sim \frac{1}{n} \qquad \rightsquigarrow \qquad A_n(i,j) \sim \frac{1}{\sqrt{n}} N(0,\sigma^2).$$

- Sparse:

$$\mathbf{E}A_n(i,j)^2 \sim \frac{1}{n} \qquad \rightsquigarrow \qquad A_n(i,j) \sim Poisson\left(\frac{\Lambda}{n}\right).$$

- Heavy-tailed:

$$\mathbf{P}(|A_n(i,j)| > Cn^{\frac{1}{\alpha}}) \sim \frac{1}{n} \qquad \rightsquigarrow \qquad A_n(i,j) \sim \frac{1}{n^{1/\alpha}} \text{Stable}_\alpha(\sigma).$$

Infinite divisibility

X (symmetric) is infinitely divisible if for every n

$$X \overset{d}{=} A_n(1,1) + A_n(1,2) + \cdots + A_n(1,n) \quad (i.i.d.)$$

and it is determined by (σ^2, ν) satisfying, when $X \overset{d}{=} -X$,

$$\mathbf{E} e^{i\theta X} = \exp\left[-\frac{1}{2}\sigma^2\theta^2 + \int_{\mathbb{R}} (e^{i\theta x} - 1)\nu(dx)\right]$$

Existence of the LSD

Suppose each A_n has i.i.d. entries up to self-adjointness such that for each i:

$$\lim_{n\to\infty} \sum_{j=1}^{n} A_n(i,j) \stackrel{d}{=} ID(\sigma^2, \nu).$$

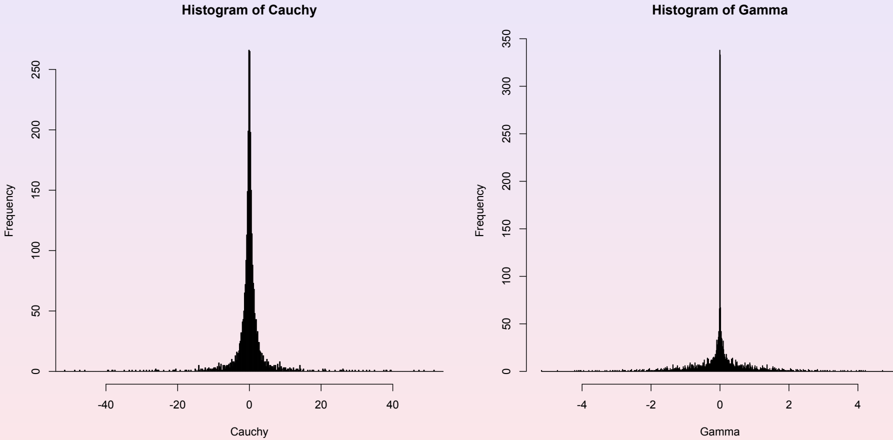
J. (2018)

- With probability 1, ESD_n weakly converges to a symm. prob. meas. μ_∞.
- μ_∞ is the expected spectral measure for vector δ_{root} of a self-adjoint operator on $L^2(G)$.

(Spectral measure for v is defined as $d\langle v, E(t)v\rangle$)

- Wigner matrices: $G = \mathbb{N}$
- Sparse matrices: G is a Poisson Galton-Watson tree

ESD of heavy-tailed and gamma random matrices



Proof sketch: Existence of the LSD

(1) As rooted graphs, Erdős-Rényi(Λ/n) *locally converge* to a branching process with Poisson(Λ) offspring distribution.

(2)

Bordenave-Lelarge (2010)

If $G_n[1] \Rightarrow G_\infty[1]$, then for all $z \in \mathbb{C}_+$,

$$(A_n - zI)^{-1}_{11} \to (A_\infty - zI)^{-1}_{11} := \langle \delta_1, (A_\infty - zI)^{-1} \delta_1 \rangle,$$

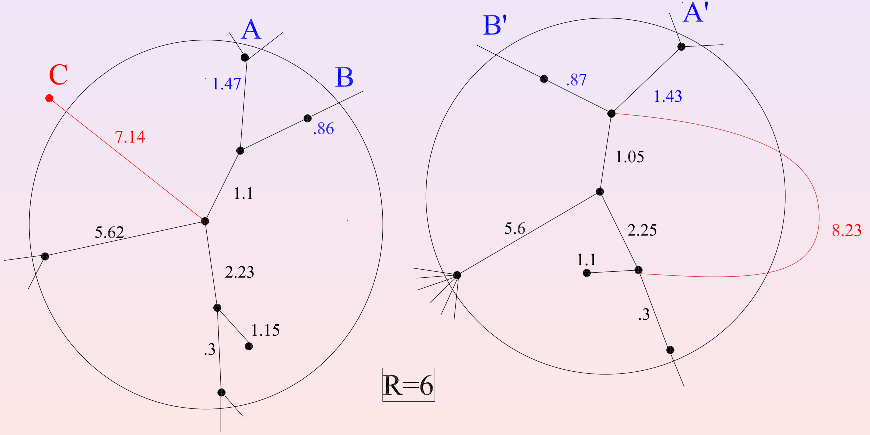
where A_∞ is an adjacency operator on $L^2(G_\infty)$.

(3)

$$\mathbf{E}(A_n - zI)^{-1}_{11} = \mathbf{E}\frac{1}{n}\operatorname{Tr}(A_n - zI)^{-1} = \int \frac{1}{x - z}\, d\mathbf{E}(\mathrm{ESD}_n).$$

Topology for rooted weighted graphs (locally-finite)

Such graphs form a Polish space so weak convergence makes sense.



Aldous' Poisson weighted infinite tree

Heavy-tailed matrices: weights $|A_n(i,j)|^{-1}$ are arrivals of a PPP of intensity $|x|^{\alpha-1}dx$.

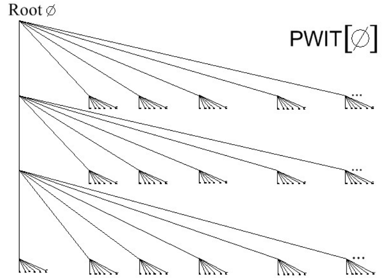


Figure 2: Each [symbol] represents a copy of the PWIT. Weights on offspring edges from any vertex are determined by a Poisson process.

Let $\{a_j\}$ be the order statistics of $\{|A_n(1,j)|\}$, then

$$R_{00}(z) \stackrel{d}{=} -\left(z + \sigma^2 R_{\infty\infty}(z) + \sum_{j\geq 1} a_j^2 R_{jj}(z)\right)^{-1}$$

How to get $\sigma^2 R_{\infty\infty}(z)$?

For the step in the proof where Benjamini-Schramm convergence $\Rightarrow$ Strong resolvent convergence, we need

$$\lim_{\varepsilon \searrow 0} \lim_{n\to\infty} \sum_{j=1}^{n} |a_{1j}|^2 1_{\{|a_{1j}|^2 \le \varepsilon\}} = 0.$$

The cords to infinity: $\sigma^2 > 0$

- Interpret the weights as the lengths of edges.
  Thus, v and ∞_v are infinitely far apart, but have infinitely many parallel edges between.

- Conductance of each parallel edge is zero, but they have a collective effective conductance to infinity.

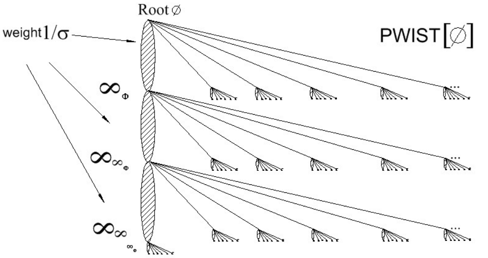
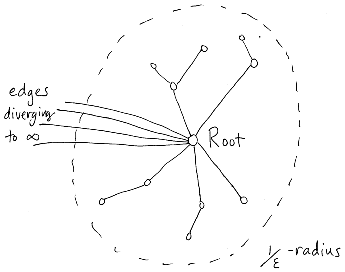


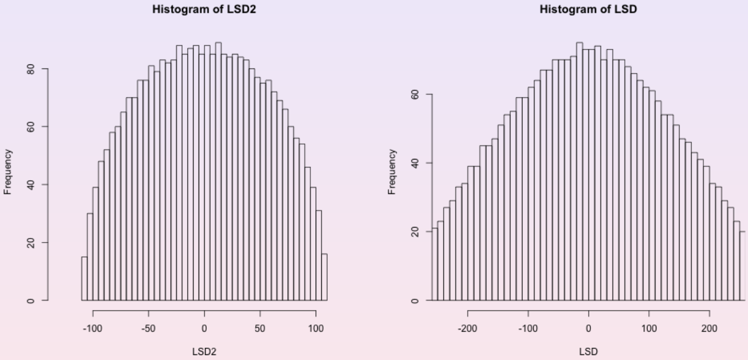
Figure 3: Each represents a copy of the PWIST. Weights on cords to infinities are deterministic. All other weights are random and determined by Poisson processes as before.



We handle infinite variance in the Gaussian domain of attraction.

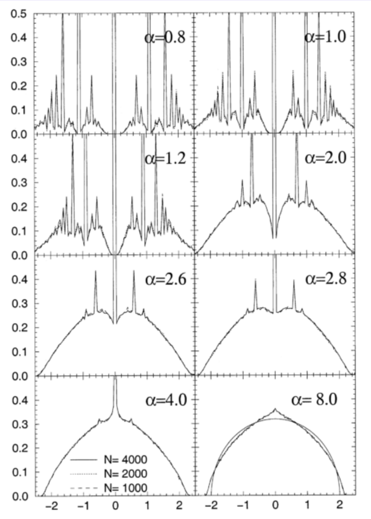
When there is no Levy measure, the PWIST is the half-line N.
It is well-known that the spectral measure at the root is semi-circle.

Slow convergence for density $1/x^3$



Erdős-Rényi graphs and the semicircle

As $\alpha = np_n$ increases:

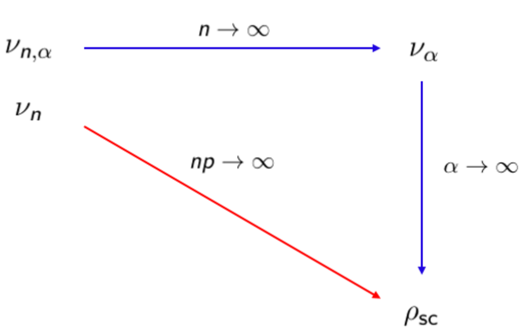


[BG01] M. Bauer and O. Golinelli, Random Incidence Matrices: Moments of the Spectral Density

LSD converges to the semicircle

J. and Lee (2018)

As $\alpha \to \infty$, the LSD ν_α converges weakly to the semicircle distribution ρ_{sc}.



Delocalization for Dilute Erdős-Rényi graphs

If the unit eigenvectors $\{u\}_{i=1}^{n}$ satisfy $\|u\|_\infty = o(1)$, a.s., we say they delocalize.

Tran, Vu, Wang (2012)

Suppose

$$\lim_{n\to\infty} \frac{np_n}{\log n} = \infty.$$

Then there exists, a.s., an orthonormal eigenvector basis $\{u_i(A_n)\}_{i=1}^{n}$ such that

$$\|u_i(A_n)\|_\infty = o(1) \quad \text{for } 1 \le i \le n.$$

Near Delocalization for Sparse Erdős-Rényi graphs

If $np = \alpha$ is fixed, we have localization since there are $O(n)$ isolated vertices.

However, as α increases, the infinity norms of most unit eigenvectors are small.

J. and Lee (2018)

Let $\varepsilon > 0$. Then, there exists an orthonormal eigenvector basis $\{u_i(A_n)\}_{i=1}^n$ satisfying

$$\liminf_{\alpha\to\infty} \liminf_{n\to\infty} \frac{|U(n,\alpha,\varepsilon)|}{n} = 1 \quad \text{almost surely,}$$

where $U(n,\alpha,\varepsilon) := \{i \in \{1,2,\cdots,n\} : \|u_i(A_n)\|_\infty < \varepsilon\}$.