# Free Lévy processes in large and small time limits

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# Classical limit theorem for random walks

Let $\{Z_i\}$ be iid random variables ($\mathbb{R}$-valued) and

$$S_n = X_1 + \cdots + X_n.$$

**Question**

*When does $a_n S_n + b_n$ converge in law as $n \to \infty$ for some deterministic sequences $a_n > 0$ and $b_n \in \mathbb{R}$?*

The answer is well known (Lévy, Khintchine,...):

(1) the possible limit distributions of $a_n S_n + b_n$ are stable distributions and delta measures;

(2) Given a stable distribution $\mu$, a necessary and sufficient condition for the convergence $a_n S_n + b_n \Rightarrow \mu$ for some $a_n, b_n$ can be given in terms of $X_1$.

Reference: Gnedenko & Kolmogorov's book

# Limit theorem for Lévy processes

A continuous-time version is, for a (additive) Lévy process $\{X_t\}$ on $\mathbb{R}$,

**Question**

*When does $a(t)X_t + b(t)$ converge in law as $t \to \infty$ for some deterministic functions $a(t) > 0$ and $b(t) \in \mathbb{R}$?*

[Bertoin 96, Doney & Maller 02, de Weert 03]

(1) the possible limit distributions of $a(t)X_t + b(t)$ are stable distributions and delta measures;

(2) given a stable distribution, a necessary and sufficient condition for the convergence is known.

We can also discuss the convergence as $t \to 0$. Then similar results hold (Maller & Mason 09)

# Free Lévy processes

- In free probability, we have free (additive) Lévy processes. They can be realized as large dimensional limits of some Hermitian matrix-valued, unitarily invariant Lévy processes [Perez & Perez-Abreu & Rocha-Arteaga]

- There is a homeomorphism (Bercovici-Pata bijection) between classical ID distributions and free ID distributions, so the complete analogy holds for limits of free Lévy processes.

# Multiplicative free LP in large times

Classical multiplicative Lévy processes $\{M_t\}$ on the multiplicative group $(0, \infty)$ can also be defined, but it eventually means $X_t := \log M_t$ is an additive LP. Note that

$$\log e^{b(t)}(M_t)^{a(t)} = a(t)X_t + b(t).$$

However, in free probability, a very different phenomenon is known:

**Theorem ((Special case of) Tucci 10, Haagerup & Moeller 13)**

*Let $\{N_t\}$ be a multiplicative free LP (then $N_t \sim \mu^{\boxtimes t}$ where $N_1 \sim \mu$). Then*

$$\text{Law of } (N_t)^{1/t} \Rightarrow \nu \qquad (t \to \infty),$$

*where $\nu([0, x]) = S_{N_1}^{-1}(1/x) + 1$. ($S_X$ is the $S$-transform of $X$)*

In particular, the map "Law of $N_1 \mapsto \nu$ " is injective. The limit distributions are not universal

# Multiplicative FLP in small times

**Selected examples among our results**

## Theorem (1)

*Let* $\{N_t\}$ *be a multiplicative free LP such that* $S_{N_1}(z) = e^{(-z)^{\alpha-1}}$, $1 < \alpha \le 2$. *Then*

$$(N_t)^{t^{-1/\alpha}} \overset{\mathrm{d}}{\Rightarrow} e^{S_\alpha}, \qquad t \to 0,$$

*where* $S_\alpha$ *has a one-sided free* $\alpha$*-stable law. In particular,* $S_2$ *follows the standard semicircle law.*

## Theorem (2)

*Let* $\{N_t\}$ *be a multiplicative free LP such that* $S_{N_1}(z) = \frac{1}{\lambda+z}$, $\lambda \ge 1$, *namely* $N_1$ *follows the Marchenko-Pastur law. Then*

$$\text{Law of } \ t(N_t)^{1/t} \Rightarrow \mathrm{DH}, \qquad t \to 0.$$

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[Dykema & Haagerup 04]

- DH has moments $\frac{n^n}{(n+1)!}$ & support $[0, e]$ & an implicit density
- Let $\{t_{ij}\}_{1 \leq i < j \leq N}$ be indep. complex Gaussian, mean 0 and var. $1/n$;

$$T_N := \begin{pmatrix} 0 & t_{12} & t_{13} & \cdots & t_{1,N-1} & t_{1N} \\ 0 & 0 & t_{23} & \cdots & t_{2,N-1} & t_{2N} \\ 0 & 0 & 0 & \cdots & t_{3,N-1} & t_{3N} \\ \vdots & \vdots & & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 & t_{N-1,N} \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

Then the mean empirical eigenvalue distr. of $T_N^* T_N \Rightarrow \text{DH}$ $(N \to \infty)$.

By computation of the densify functions we found that:

**Proposition**

*If $X$ follows the free $1$-stable law supported on $(-\infty, 1]$ then*

$$e^X \sim \mathrm{DH}.$$

- This means that the empirical eigenvalue distribution of $\log(T_N^* T_N)$ converges to the free $1$-stable law.
- Recall that the semicircle law $\frac{1}{2\pi}\sqrt{4 - x^2}$ on $[-2, 2]$ (free $2$-stable) has a RM model (e.g. Wigner matrix)

**Question**

*Do other free stable distributions have RM models?*

# Summary

- For classical additive LPs $(X_t)$, the limit distr. of $a(t)X_t + b(t)$ ($t \to \infty$ or $0$), if exists, is stable.
- For classical multiplicative LPs $(M_t)$, the limit distr. of $e^{b(t)}(M_t)^{a(t)}$ ($t \to \infty$ or $0$), if exists, is of the form $e^S$, where $S \sim$ stable.
- For free additive LPs $(Y_t)$, the limit distr. of $a(t)Y_t + b(t)$ ($t \to \infty$ or $0$), if exists, is free stable.
- For free multiplicative LPs $(N_t)$, the limit distr. of $(N_t)^{1/t}$ ($t \to \infty$) always exists and is not universal.

**Conjecture (after our examples)**

*For free multiplicative LPs $(N_t)$, the limit distr. of $e^{b(t)}(N_t)^{a(t)}$ ($t \to 0$), if exists, must be $e^S$, where $S \sim$ free stable.*

## References


### – Free probability

1. O. Arizmendi and T. Hasebe, Limit theorems for free Lévy processes, arXiv:1711.10220.
2. H. Bercovici and V. Pata, Stable laws and domains of attraction in free probability theory (with an appendix by Philippe Biane), Ann. of Math. (2) **149**, No. 3 (1999), 1023–1060.
3. K. Dykema and U. Haagerup, DT-operators and decomposability of Voiculescu's circular operator, Amer. J. Math. **126**(1) (2004), 121–189.

### – Classical probability

1. R.A. Doney and R.A. Maller, Stability and attraction to normality for Levy processes at zero and infinity, J. Theor. Prob. 15 (2002), 751–792.
2. B.V. Gnedenko and A.N. Kolmogorov, *Limit Distributions for Sums of Independent Random Variables*, Addison-Wesley Publ. Co., Inc., 1954.
3. F.J. de Weert, Attraction to stable distributions for Levy processes at zero, M. Phil thesis, Univ. Manchester, 2003.