Log-, Coulomb and Riesz gases: Fluctuations and microscopic behavior

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Systems

N point particles $\vec{X}_N = (x_1, \ldots, x_N)$ in $\mathbb{R}^d$

- Pairwise interaction $g(x_i - x_j)$
- External field/potential $V(x_i)$

Energy in the state $\vec{X}_N$

$$\mathcal{H}_N(\vec{X}_N) := \sum_{i \neq j} g(x_i - x_j) + N \sum_{i=1}^{N} V(x_i)$$

Typical example: $d = 1, 2$, $g(x) = -\log|x|$, $V(x) = |x|^2$.

Choice of g and V

Interaction potential g:

$$g(x) = \begin{cases} -\log|x| & (d=1) \\ -\log|x| & (d=2) \\ |x|^{-(d-2)} & (d \geq 3) \\ |x|^{-s} & (\max(d-2,0) < s < d) \end{cases}$$

Coulomb interactions, Riesz interactions

External field V: continuous and “strongly confining”.

A random point configuration

Canonical Gibbs measure at (inverse) temperature β

$$d\mathbb{P}_{N,\beta}(\vec{X}_N) := \frac{1}{Z_{N,\beta}} \exp\left(-\frac{\beta}{2} N^{-s/d} \mathcal{H}_N(\vec{X}_N)\right) d\vec{X}_N$$

with $Z_{N,\beta}$ (the partition function)

$$Z_{N,\beta} := \int_{(\mathbb{R}^d)^N} \exp\left(-\frac{\beta}{2} N^{-s/d} \mathcal{H}_N(\vec{X}_N)\right) d\vec{X}_N.$$

Questions

Asymptotic behavior of the system ($N \to \infty$)? Fluctuations?
Dependency on β? Dependency on V (universality)?

Motivations

Statistical physics

- Toy model with singular, long-range interactions in $\mathbb{R}^d$.
- "Real-life" implementations (vortex systems, electrostatics, Calogero-Sutherland model)

Random matrix theory (RMT)

$d = 1, 2$, logarithmic interactions

For some classical models (Gaussian ensembles in $d = 1$, Ginibre ensemble in $d = 2$) the law of N random eigenvalues coincide with $\mathbb{P}_{N,\beta}$.

Also approximation theory, etc.

Global behavior

Empirical measure

Encodes the global/macroscopic behavior

$$\mu_N := \frac{1}{N}\sum_{i=1}^N \delta_{x_i} \longrightarrow \mu_{\mathrm{eq},V} \text{ “equilibrium measure”}$$

where $\mu_{\mathrm{eq},V}$ is the unique minimizer on $\mathcal{P}(\mathbb{R}^d)$ of

$$I_V(\mu) := \iint g(x-y)d\mu(x)d\mu(y) + \int V(x)d\mu(x).$$

Its support Σ_V is compact.
$\mu_{\mathrm{eq},V}$ depends on V, d but not on β. Examples: semi-circle, circular law...

Splitting formula

$$\mathcal{H}_N(\vec{X}_N) = N^2 I_V(\mu_{\mathrm{eq},V}) - \frac{N \log N}{d} + F_N^{\mu_{\mathrm{eq},V}}(\vec{X}_N) + 2N\zeta_N(\vec{X}_N)$$

- $I_V(\mu_{\mathrm{eq}})$ first-order energy
- ζ_N confining term
- $F_N^{\mu_{\mathrm{eq},V}}$ interaction energy of the new system

$$F_N^{\mu_{\mathrm{eq},V}}(\vec{X}_N) = \iint_{(\mathbb{R}^d \times \mathbb{R}^d)\setminus\triangle} g(x-y)(d\nu'_N - d\mu'_{\mathrm{eq},V})^{\otimes 2}(x,y)$$

$$\nu'_N = \sum_{i=1}^N \delta_{N^{1/d}x_i} \text{ and } \mu'_{\mathrm{eq},V}(N^{1/d}x) = \mu_{\mathrm{eq},V}(x)$$

Splitting formula

$$\mathcal{H}_N(\vec{X}_N) = N^2 I_V(\mu_{\mathrm{eq},V}) + N^{s/d} F_N^{\mu_{\mathrm{eq},V}}(\vec{X}_N) + 2N\zeta_N(\vec{X}_N)$$

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$$F_N^{\mu_{\mathrm{eq},V}}(\vec{X}_N) = \iint_{(\mathbb{R}^d\times\mathbb{R}^d)\setminus\triangle} g(x-y)(d\nu_N' - d\mu_{\mathrm{eq},V}')^{\otimes 2}(x,y)$$

$$\nu_N' = \sum_{i=1}^N \delta_{N^{1/d}x_i} \text{ and } \mu_{\mathrm{eq},V}'(N^{1/d}x) = \mu_{\mathrm{eq},V}(x)$$

Questions

Fluctuations

In what sense does $\mu_N \approx \mu_{\mathrm{eq},V}$?

- At small scales ($O(1) \to O(N^{-1/d+\varepsilon})$)?
- Deviations bounds?
- Central limit theorem?

Microscopic behavior

Zoom into the system by $N^{1/d}$ $\to$ point configuration. What does it look like?

Fluctuations of linear statistics

Given $\varphi \in C_c^0(\mathbb{R}^d)$, a scale $N^{-1/d} \ll \ell_N \leq 1$, $x_0 \in \mathbb{R}^d$, we let

$$\varphi_N(x) := \varphi\left(\frac{x - x_0}{\ell_N}\right)$$

and the "fluctuation of φ_N":

$$\mathrm{Fluct}_N[\varphi_N] := N \int \varphi_N \left(d\mu_N - d\mu_{\mathrm{eq},V}\right)$$

$$= \sum_{i=1}^{N} \varphi_N(x_i) - N \int \varphi_N d\mu_{\mathrm{eq},V}.$$

a) What is the order of magnitude of $\mathrm{Fluct}_N[\varphi_N]$?
b) Is there a limit as $N \to \infty$?

- a) is very well understood in $d = 1$, log-gas case (Bourgade-Erdös-Yau). Rigidity estimates...
- Now also in 2d (Bauerschmidt-Bourgade-Nikula-Yau).
- In general for $\ell_N = 1$

$$|\mathrm{Fluct}_N[\varphi_N]| \le \varepsilon N \text{ with proba } 1 - \exp(-N^2).$$

- Can be pushed to

$$|\mathrm{Fluct}_N[\varphi_N]| = O\left(N^{1/2}\right) \text{ with proba } 1 - \exp(-N).$$

- Also for ℓ_N close to 1 (up to $N^{-\frac{1}{2d}}$)

$$|\mathrm{Fluct}_N[\varphi_N]| \ll N\ell_N^d \text{ with proba } 1 - \exp(-N).$$

Central Limit Theorem - I

$d = 2$, logarithmic interaction (2d Coulomb gas).
$V \in C^4$, $\Delta V > 0$ on Σ_V (+ some regularity on Σ_V)

Theorem (L. - Serfaty)

Assume $\varphi \in C^4(\mathbb{R}^2)$. Then $\mathrm{Fluct}_N[\varphi]$ converges in law to a Gaussian random variable with mean (if $\ell_N = 1$)

$$\mathrm{Mean}(\varphi) = \frac{1}{2\pi}\left(\frac{1}{\beta} - \frac{1}{4}\right)\int_{\mathbb{R}^2} \Delta\varphi \left(\mathbf{1}_{\Sigma_V} + (\log \Delta V)^{\Sigma_V}\right)$$

and variance

$$\mathrm{Var}(\varphi) = \frac{1}{2\pi\beta}\int_{\mathbb{R}^2} |\nabla \varphi^{\Sigma_V}|^2.$$

f^{Σ_V} denotes the harmonic extension of f outside Σ_V

Central Limit Theorem - I

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Theorem (L. - Serfaty)

Assume $\varphi \in C^4(\mathbb{R}^2)$. Then $\mathrm{Fluct}_N[\varphi]$ converges in law to a Gaussian random variable with mean (if $\ell_N \ll 1$)

$$\mathrm{Mean}(\varphi) = 0$$

and variance

$$\mathrm{Var}(\varphi) = \frac{1}{2\pi\beta} \int_{\mathbb{R}^2} |\nabla \varphi^{\Sigma_V}|^2.$$

f^{Σ_V} denotes the harmonic extension of f outside Σ_V

Central Limit Theorem - II

- Remarkable feature: no $\frac{1}{\sqrt{N}}$ normalization ("there must be very effective cancellation in the sum").
- Convergence of $N(d\mu_N - d\mu_{\mathrm{eq},V})$ to a Gaussian Free Field.
- CLT known in the $1d$ log-gas case for any value of β (Johansson, Shcherbina, Borot-Guionnet).
- Mesoscopic CLT in $1d$ Bekerman-Lodhia.
- Only for $\beta = 2$ in the $2d$ Coulomb case (Rider-Virag in the Ginibre ($V(x) = |x|^2$) case, Ameur-Hedenmalm-Makarov in the analytic case).
- "Correct" assumption should be $\varphi \in H^1$, or at most in C^2...

CLT III

- Extends to a fixed number of test functions $(\mathrm{Fluct}_N[\varphi^{(1)}], \dots, \mathrm{Fluct}_N[\varphi_N^{(m)}]) \to$ some Gaussian vector (Rider-Virag for the Ginibre case)
- Moderate deviations bounds. For any $1 \ll r_N \ll N\ell_N^2$ we have

$$\mathbb{P}_{N,\beta}\left(|\mathrm{Fluct}_N[\xi_N]| \geq c r_N\right) \leq \exp\left(-\frac{c}{2} r_N^2\right),$$

as in BBNY.

CLT IV - Overview of the method

- Computing the Laplace transform of the fluctuations

$$\mathbf{E}_{\mathbb{P}_{N,\beta}}\left[\exp(tN\mathrm{Fluct}_N[\varphi_N])\right],$$

  amounts to computing the ratio of two partition functions: the original one and that of a new gas with potential $V - \frac{2t}{\beta}\Delta\varphi_N$.
- Finding a transport map from $\mu_{\mathrm{eq},V}$ to the new equilibrium measure $\mu_{\mathrm{eq},V,t}$ is always possible (but finding a nice one can be more delicate).
- Comparing the energies before/after transport allows to estimate the ratio of partition functions.

Idea of transport already present in Bekerman-Figalli-Guionnet, Shcherbina.

Assume $\ell_N = 1$, $\varphi \in C^4(\mathbb{R}^2)$ compactly supported inside Σ_V. In particular the harmonic extension is φ itself.

$$\mathbf{E}_{\mathbb{P}_{N,\beta}}\left[\exp(tN\mathrm{Fluct}_N[\varphi_N])\right] = \frac{K_{N,\beta}(\mu_t)}{K_{N,\beta}(\mu_0)} \exp\left(\frac{N^2 t^2}{4\pi\beta}\int_{\mathbb{R}^2} |\nabla\varphi|^2\right),$$

Partition function

$$K_{N,\beta}(\mu_t) := \int_{(\mathbb{R}^2)^N} \exp\left(-\frac{\beta}{2}\left(F_N^{\mu_t}(\vec{X}_N) + 2N\sum_{i=1}^{N}\zeta(\vec{X}_N)\right)\right) d\vec{X}_N,$$

μ_t is the equilibrium measure associated to $V - \frac{2t}{\beta}\Delta\varphi$,

$$\mu_t = \frac{1}{4\pi}\left(V - \frac{2t}{\beta}\Delta\varphi\right)\mathbf{1}_{\Sigma_V}.$$

Reachability

Construct a diffeomorphism $\Phi_t : \mathbb{R}^2 \to \mathbb{R}^2$ which transports μ_0 on μ_t and

$$\Phi_t = \mathrm{Id} + t\Psi + O(t^{1+\sigma}) \text{ in } C^{1,1}(\mathbb{R}^2).$$

Comparing $K_{N,\beta}(\mu_t)$ and $K_{N,\beta}(\mu_0)$ amounts to comparing

$$F_N^{\mu_t}(\Phi_t(\vec{X}_N)) \text{ and } F_N^{\mu_0}(\vec{X}_N)$$

"Taylor expanding the energy", one finds

$$F_N^{\mu_t}(\Phi_t(\vec{X}_N)) - F_N^{\mu}(\vec{X}_N) = t\mathbf{Ani}(\vec{X}_N) + \frac{1}{2}\sum_{i=1}^{N} \log|\det D\Phi_t(x_i)|$$
$$+ \text{ error terms}$$

$$F_N^{\mu_t}(\Phi_t(\vec{X}_N)) - F_N^{\mu}(\vec{X}_N) = t\mathbf{Ani}(\vec{X}_N) + \frac{1}{2}\sum_{i=1}^{N} \log|\det D\Phi_t(x_i)|$$

$\sum_{i=1}^{N} \log|\det D\Phi_t(x_i)|$ is also the Jacobian.

$$\sum_{i=1}^{N} \log|\det D\Phi_t(x_i)| \approx N\int \log|\det D\Phi_t|(x)d\mu_0$$

$$\approx N\left(\int \mu_0 \log \mu_0 - \int \mu_t \log \mu_t\right)$$

"Trick" needed to show that $\mathbf{Ani}(\vec{X}_N)$ is negligible.

Know how to compute $\frac{K_{N,\beta}(\mu_t)}{K_{N,\beta}(\mu_0)}$ up to order $\exp(o(N))$, for t of order 1.

There is no **Ani** term !

Thus

$$\mathbf{E}_{\mathbb{P}_{N,\beta}}\left[\exp(t\,\mathbf{Ani})\right] = \exp(o(N)).$$

$+$ Hölder's inequality, implies for t of order $1/N$

$$\mathbf{E}_{\mathbb{P}_{N,\beta}}\left[\exp\left(\frac{t}{N}\mathbf{Ani}\right)\right] = \exp(o(1)).$$

We may then prove that $\mathbf{E}_{\mathbb{P}_{N,\beta}}\left[t\mathrm{Fluct}_N[\varphi_N]\right]$ converges to the Laplace transform of a Gaussian random variable.

Microscopic behavior I

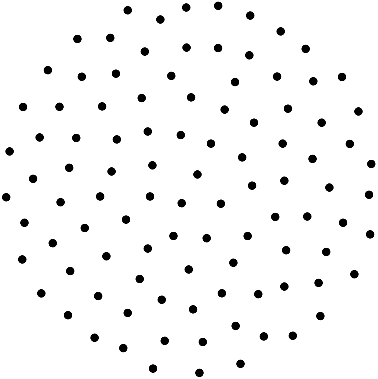

Figure: $\beta = 400$

Microscopic behavior I

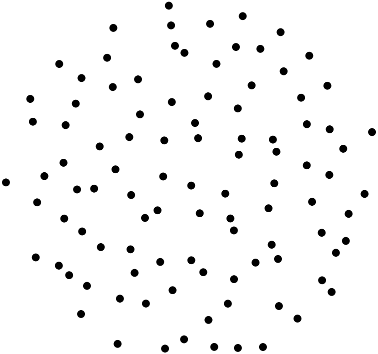

Figure: $\beta = 5$

Microscopic behavior II

Non-averaged point process

Let $z \in \mathring{\Sigma}$ be fixed.

$$\mathcal{C}_{N,z} : \vec{X}_N \mapsto \sum_{i=1}^{N} \delta_{N^{1/d}(x_i - z)}.$$

Values in $\mathcal{X}$, the space of point configurations.

Empirical field

Let $\Omega \subset \Sigma$ be fixed.

$$\overline{\mathcal{C}}_{N,\Omega} := \frac{1}{|\Omega|} \int_{\Omega} \delta_{\mathcal{C}_{N,z}} dz$$

Values in $\mathcal{P}(\mathcal{X})$.

- Ω of size independent of N: macroscopic average.
- Ω of size $N^{-\frac{1}{d}+\delta}$ mesoscopic average.

Microscopic behavior - III

Assumptions: Σ is a C^1 compact set, and $\mu_{\rm eq}$ has Hölder density. Take $\Omega = B(x, \varepsilon)$ and for simplicity, assume $\mu_{\rm eq}(x) = m$ on Ω.

Theorem (L. - Serfaty)

There exists a functional $\mathcal{F}_\beta^m$ on the space $\mathcal{P}(\mathcal{X})$ such that:
The law of the empirical field $\overline{\mathcal{C}}_{N,\Omega}$ concentrates on minimizers of $\mathcal{F}_\beta^m$ as $N \to \infty$, with proba $1 - \exp(-N|\Omega|)$.

For $d = 2, g(x) = -\log|x|$, true for mesoscopic average (i.e. $\Omega = B(x, \varepsilon)$ with $\varepsilon = N^{-1/2+\delta}$).

Rate function

For $m > 0$, define $\mathcal{F}_\beta^m$ by

$$\mathcal{F}_\beta^m(P) := \beta \mathbb{W}_m^{\mathrm{elec}}(P) + \mathbf{ent}[P|\Pi^m]$$

$\mathbb{W}_m^{\mathrm{elec}}(P)$ is an energy functional, $\mathbf{ent}[P|\Pi^m]$ is a relative entropy functional, $\Pi^m =$ Poisson point process.
Minimizers of $\mathcal{F}_\beta^m$ depend on m only through a scaling. In the logarithmic cases, the dependency on m "decouples" and the microscopic behavior is thus largely independent of V (and we may restrict to study $m = 1$).

Some known facts

$$\mathcal{F}_\beta(P) := \beta \mathbb{W}^{\mathrm{elec}}(P) + \mathbf{ent}[P|\Pi^1]$$

- The Sine_β point processes of Valko-Virag are minimizers of $\mathcal{F}_\beta$ for $\beta > 0$ in the $d = 1, g(x) = -\log|x|$ case
- The Ginibre point process minimizes $\mathcal{F}_\beta$ for $\beta = 2$ in the $d = 2, g(x) = -\log|x|$ case.
- Minimizers of $\mathcal{F}_\beta$ tend (in entropy sense) to a Poisson point process as $\beta \to 0$.
- In dimension 1 minimizers of $\mathcal{F}_\beta$ converge to $P_{\mathbb{Z}}$ as $\beta \to \infty$.

Relative specific entropy

P stationary,

$$\mathbf{ent}[P|\Pi^1] = \lim_{R\to\infty} \frac{1}{R^d} \mathrm{Ent}[P_R|\Pi^1_R].$$

P_R, Π_R = restrictions to $[-R/2, R/2]^d$.
Hard to compute explicitely.

Energy functional I

$\mathbb{W}^{\text{elec}}$ is defined using the "electric approach" of Sandier-Serfaty (& Rougerie, & Petrache). An alternative, more explicit formulation: define $\mathbb{W}^{\text{int}}(P)$ as

$$\liminf_{R\to\infty} \frac{1}{R^d} \mathbf{E}_P \left[\iint_{C_R \times C_R \setminus \triangle} g(x-y)\,(d\mathcal{C}(x) - dx)\,(d\mathcal{C}(y) - dy) \right]$$

Inspired by Borodin-Serfaty.

Energy functional II

If P stationary and has intensity 1, let $\rho_{2,P}$ be its pair correlation function.

$$\mathbb{W}^{\text{int}}(P) := \liminf_{R\to\infty} \int_{[-R,R]^d} g(v)\,(\rho_{2,P}-1)\prod_{i=1}^{d}\left(1-\frac{|v_i|}{R}\right),$$

where $v=(v_1,\dots,v_d)$.
For "decorrelating" systems ($\rho_{2,P}-1\to 0$ fast enough)

$$\mathbb{W}^{\text{int}}(P) := \int_{\mathbb{R}^d} g(v)\,(\rho_{2,P}-1)$$

Some properties and questions

- For $d = 1$ and $g(x) = -\log|x|$ or $|x|^{-s}$ (g convex...), $P_{\mathbb{Z}}$ is the unique minimizer.
- What about $d \geq 2$? Can we minimize $\mathbb{W}^{\mathrm{elec}}$ or $\mathbb{W}^{\mathrm{int}}$?
- If $\mathbb{W}^{\mathrm{elec}}(P)$ is finite then the number variance scales as R^{d+s}. In the $d = 1$, $g(x) = -\log|x|$ case, $\mathbb{W}^{\mathrm{elec}}(P) < +\infty$ implies hyperuniformity, but Poisson always has finite Riesz energy.
- What about the $d = 2, g(x) = -\log|x|$ case?
- There is a minimizing sequence of "decorrelating" P_k.
- Disordered system with minimal energy?

Other settings

- Hypersingular Riesz gases $g(x) = |x|^{-s}$, $s > d$. No equilibrium measure from potential theory (depends on β), microscopic behavior determined by a similar free energy functional (Hardin - L. - Saff - Serfaty).
- Two-component plasma: ± 1 charges, $d = 2$, logarithmic interactions. No equilibrium measure from potential theory, microscopic behavior determined by a similar free energy functional (L.-Serfaty-Zeitouni + Wu).
- Other RMT ensembles? Zeroes of random polynomials? Other physically relevant interactions?

Thank you for your attention!