Determinantal Point Processes and Products of Random Matrices

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Optimal and Random Point Configurations
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Biorthogonal Ensembles

Determinantal point processes on $\mathbb{R}$

Orthogonal Polynomial Ensemble

$\subseteq$

Polynomial Ensemble

$\subseteq$

Biorthogonal Ensemble

$\subseteq$

Determinantal Point Process on $\mathbb{R}$

Determinantal point process

DPP: random point configuration with correlation kernel $K(x, y)$

- **Correlation functions are determinantanl**

$$\rho_k(x_1, \ldots, x_k) = \det \left[K(x_i, x_j)\right]_{i,j=1}^{k}$$

- K **is not unique:**

$$\frac{h(x)}{h(y)} K(x, y)$$

is correlation kernel for same DPP

Orthogonal ensembles

Orthonormal functions φ_k, $k = 0, 1, \ldots,$

$$\int \varphi_k(x)\varphi_j(x)dx = \delta_{j,k}$$

- n **th reproducing kernel**

$$K_n(x, y) = \sum_{k=0}^{n-1} \varphi_k(x)\varphi_k(y)$$

- **Probability density**

$$\frac{1}{n!} \det \left[K_n(x_i, x_j)\right]_{i,j=1}^{n} = \frac{1}{n!} \left(\det \left[\varphi_{k-1}(x_j)\right]_{j,k=1}^{n}\right)^2$$

OP ensembles

Orthogonal polynomials **w.r.t. weight w on $\mathbb{R}$,**

$$\int_{-\infty}^{\infty} p_k(x)p_j(x)w(x)dx = \delta_{j,k}$$

- $\varphi_k(x) = \sqrt{w(x)}p_k(x)$, $k = 0, 1, \ldots$, **are orthonormal functions and kernel is**

$$K_n(x,y) = \sqrt{w(x)}\sqrt{w(y)}\sum_{k=0}^{n-1} p_k(x)p_k(y)$$

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Proposition

$$\frac{1}{n!}\det\left[K_n(x_i,x_j)\right]_{i,j=1}^{n} = \frac{1}{Z_n}\prod_{1\le i<j\le n}(x_j-x_i)^2\prod_{j=1}^{n} w(x_j)$$

Biorthogonal ensemble

Two sequences of functions φ_k, ψ_k, $k = 0, 1, \ldots$ are biorthogonal if

$$\int_{-\infty}^{\infty} \varphi_k(x)\psi_j(x)dx = \delta_{j,k}$$

- **nth reproducing kernel**

$$K_n(x, y) = \sum_{k=0}^{n-1} \varphi_k(x)\psi_k(y)$$

- **K_n is correlation kernel for DPP iff**

$$\det\left[K_n(x_i, x_j)\right]_{i,j=1}^{n} = \det\left[\varphi_{k-1}(x_j)\right]_{j,k=1}^{n} \cdot \det\left[\psi_{k-1}(x_j)\right]_{j,k=1}^{n} \geq 0$$

for all $x_1, \ldots, x_n$.

Biorthogonal ensemble, 2

Assume probability density function on $\mathbb{R}^n$ of the form

$$\frac{1}{Z_n} \det \left[f_{k-1}(x_j)\right]_{j,k=1}^n \cdot \det \left[g_{k-1}(x_j)\right]_{j,k=1}^n$$

Then we biorthogonalize the functions

$$\begin{aligned} f_0, \ldots, f_{n-1} &\mapsto \varphi_0, \ldots, \varphi_{n-1}, \\ g_0, \ldots, g_{n-1} &\mapsto \psi_0, \ldots, \psi_{n-1}, \end{aligned} \qquad \int_{-\infty}^{\infty} \varphi_k(x)\psi_j(x)dx = \delta_{j,k}$$

Corollary

This is biorthogonal ensemble with correlation kernel

$$K_n(x, y) = \sum_{k=0}^{n-1} \varphi_k(x)\psi_k(y).$$

Example: Nonintersecting path ensembles

Theorem (Karlin McGregor (1959))

Let $X_1, \dots, X_n$ be independent copies of a one-dimensional strong Markov process with continuous sample paths, conditioned such that

$$X_j(0) = a_j, \qquad \text{for given } a_1 < \cdots < a_n$$
$$X_j(T) = b_j, \qquad \text{for given } b_1 < \cdots < b_n$$

and conditioned not to intersect for any $0 < t < T$. Then the random positions $X_1(t), \dots, X_n(t)$ have joint density

$$\frac{1}{Z_n} \det\left[p_t(a_k, x_j)\right]_{j,k=1}^{n} \cdot \det\left[p_{T-t}(x_j, b_k)\right]_{j,k=1}^{n}$$

where $p_t(x, y)$ is the transition probability density

Non-intersecting path ensembles

$$\frac{1}{Z_n} \det\left[p_t(a_k, x_j)\right]_{j,k=1}^{n} \cdot \det\left[p_{T-t}(x_j, b_k)\right]_{j,k=1}^{n}$$

- **Biorthogonal ensemble with functions**

$$f_k(x) = p_t(a_k, x), \qquad g_k(x) = p_{T-t}(x, b_k)$$

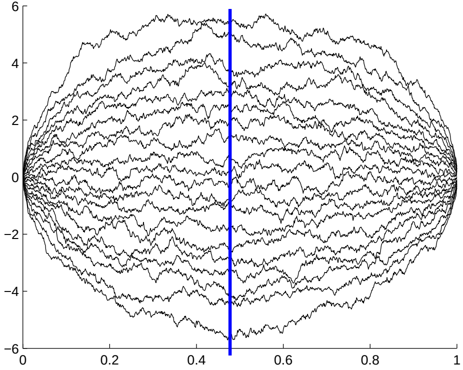
Main example

- **Brownian motion has transition densities**

$$p_t(x, y) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{(x-y)^2}{2t}}$$

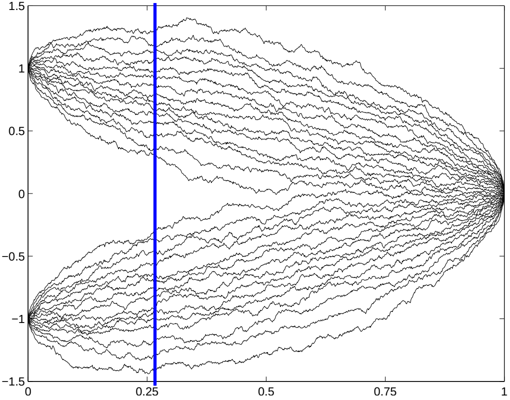
Confluent case

- **Brownian motion in the limit** $a_j \to 0, \quad b_j \to 0$**.**
- **This leads to same p.d.f. (after scaling) as for the eigenvalues of GUE.**



Two starting points

- **Brownian motion in the limit** $a_j \to \pm a, \quad b_j \to 0.$



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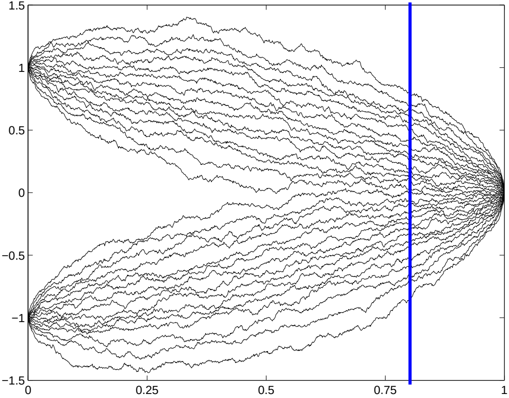

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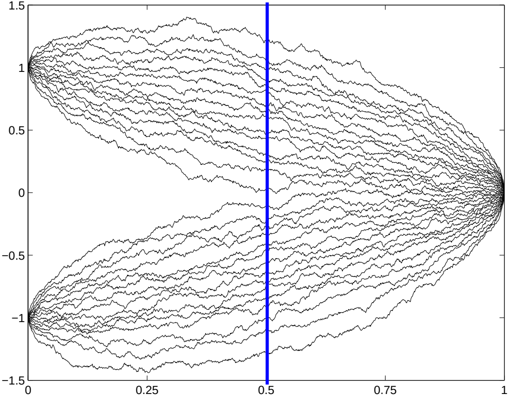

Polynomial Ensembles

Polynomial ensemble

Polynomial ensemble is

$$\frac{1}{Z_n}\Delta_n(x) \cdot \det\left[w_{k-1}(x_j)\right]_{j,k=1}^{n}$$

with Vandermonde determinant

$$\Delta_n(x) = \det\left[x_j^{k-1}\right]_{j,k=1}^{n} = \prod_{i<j}(x_j - x_i)$$

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$$\Delta_n(x) = \det \left[x_j^{k-1} \right]_{j,k=1}^{n} = \prod_{i<j} (x_j - x_i)$$

- **After biorthogonalization**

$$\begin{aligned} 1, x, \ldots, x^{n-1} &\mapsto P_0, \ldots, P_{n-1} \quad \textbf{polynomials} \\ w_0, \ldots, w_{n-1} &\mapsto Q_0, \ldots, Q_{n-1} \end{aligned}$$

- P_j **is a monic polynomial of degree** j **such that**

$$\int_{-\infty}^{\infty} P_j(x) w_k(x) dx = 0, \qquad \textbf{for } k = 0, \ldots, j-1$$

Average characteristic polynomial

P_n **is such that**

$$\int_{-\infty}^{\infty} P_n(x) w_k(x) dx = 0 \qquad \textbf{for } k = 0, 1, \ldots, n-1$$

Lemma

P_n **is the average characteristic polynomial**

$$P_n(x) = \mathbb{E}\left[\prod_{j=1}^{n}(x - x_j)\right]$$
$$= \frac{1}{Z_n} \int_{\mathbb{R}^n} \prod_{j=1}^{n}(x - x_j) \cdot \Delta_n(x) \ \det\left[w_{k-1}(x_j)\right] \ \prod_j dx_j$$

Example 1

Random matrices

- **Hermitian $n \times n$ matrices with probability measure**

$$\frac{1}{Z_n} e^{-\operatorname{Tr} V(M)} dM$$

- **Eigenvalue density**

$$\frac{1}{Z_n} \Delta_n(x)^2 \prod_{j=1}^{n} e^{-V(x_j)}$$

  is OP ensemble

- **Also polynomial ensemble with** $w_k(x) = x^k e^{-V(x)}$

Example 2

Random matrix with external source

- **Hermitian $n \times n$ matrices with probability measure**

$$\frac{1}{Z_n} e^{-\operatorname{Tr}(V(M)-AM)} dM$$

  A **is a fixed Hermitian matrix with eigenvalues** $a_1, \ldots, a_n$.

- **Polynomial ensemble with**

$$w_k(x) = e^{-V(x)+a_k x}, \qquad k = 1, \ldots, n$$

  in case all a_k **are distinct**

Example 3

Biorthogonal ensemble

$$\frac{1}{Z_n}\prod_{j<k}(x_k - x_j)\cdot\prod_{j<k}(x_k^\theta - x_j^\theta)\cdot\prod_{j=1}^{n} e^{-V(x_j)}$$

defined on $[0,\infty)^n$ **with some** $\theta > 0$

- **Polynomial ensemble with**

$$w_k(x) = x^{\theta k} e^{-V(x)}, \qquad k = 0, \ldots, n-1.$$

Muttalib (1995), Borodin (1998)

Multiplication with complex Ginibre

Complex Ginibre matrix

G **is size** $(n+\nu) \times n$ **with Gaussian distribution**

$$\frac{1}{Z_n} e^{-\operatorname{Tr} G^* G} dG$$

- **Entries are independent standard complex Gaussians**
- **Eigenvalues of** $G^* G$ **have joint density**

$$\frac{1}{Z_n} \Delta_n(x)^2 \prod_{j=1}^{n} x_j^{\nu} e^{-x_j} \qquad \textbf{all } x_j > 0.$$

- **This is Laguerre ensemble.**

Products of complex Ginibre

Product of Ginibre matrices

$$Y = G_r \cdots G_2 G_1$$

- Both eigenvalues and singular values of Y have determinantal structure

Eigenvalues Akemann, Burda (2013)
Adhikari, Reddy, Reddy, Saha (2016)

Singular values Akemann, Kieburg, Wei (2013)
Akemann, Ipsen, Kieburg (2013)

Eigenvalues

Product of complex Ginibre matrices

$$Y = G_r \cdots G_2 G_1$$

G_j **has size** $(n+\nu_j) \times (n+\nu_{j-1})$ **with all** $\nu_j \geq 0$, $\nu_0 = \nu_r = 0$

- **Joint density for eigenvalues of** Y

$$\frac{1}{Z_n} \prod_{j<k} |z_k - z_j|^2 \prod_{j=1}^{n} w(|z_j|^2)$$

with a Meijer G-function

$$w(x) = G^{r,0}_{0,r}\left(\begin{matrix} - \\ \nu_1, \cdots, \nu_r \end{matrix} \middle| x \right)$$

$$= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \prod_{j=1}^{r} \Gamma(\nu_j + s) x^{-s} ds$$

Squared singular values

Product of complex Ginibre matrices

$$Y = G_r \cdots G_2 G_1$$

G_j **has size** $(n+\nu_j) \times (n+\nu_{j-1})$ **with all** $\nu_j \geq 0$, $\nu_0 = 0$

- **Joint density for eigenvalues of** $Y^* Y$

$$\frac{1}{Z_n} \Delta_n(x) \det \left[w_{k-1}(x_j) \right]_{j,k=1}^{n}$$

  is a polynomial ensemble
- **Weight** w_0 **is same Meijer G-function**

$$w_0(x) = G^{r,0}_{0,r}\left(\begin{matrix} - \\ \nu_1, \dots, \nu_r \end{matrix} \middle| x \right)$$

$$= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \prod_{j=1}^{r} \Gamma(\nu_j + s) \, x^{-s} ds$$

Multiplication with Ginibre

Theorem (Kuijlaars-Stivigny (2014))

Suppose G **is** $(n+\nu) \times n$ **complex Ginibre matrix and** X **is independent random matrix with squared singular value density**

$$\frac{1}{Z_n} \Delta_n(x) \det \left[f_k(x_j) \right]_{j,k=1}^{n}$$

Then squared singular values of $Y = GX$ **have density**

$$\frac{1}{\tilde{Z}_n} \Delta_n(y) \det \left[g_k(y_j) \right]_{j,k=1}^{n}$$

with g_k **the Mellin convolution of** f_k **with** $x^\nu e^{-x}$

$$g_k(y) = \int_0^\infty x^\nu e^{-x} f_k\left(\frac{y}{x}\right) \frac{dx}{x}$$

Ingredients in the proof

Harish-Chandra/Itzykson-Zuber formula

$$\int_{U\in U(n)} e^{-\operatorname{Tr}(U^* AUB)}\, dU = c_n \frac{\det\left[e^{-a_j b_k}\right]_{j,k=1}^{n}}{\Delta_n(a)\Delta_n(b)}$$

where A **and** B **are Hermitian matrices with respective eigenvalues** $a_1, \ldots, a_n$ **and** $b_1, \ldots, b_n$

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Andreief identity (a.k.a. generalized Cauchy-Binet)

$$\int_{[0,\infty)^n} \det\left[\varphi_k(x_j)\right]_{j,k=1}^n \det\left[\psi_k(x_j)\right]_{j,k=1}^n dx_1\cdots dx_n$$

$$= n!\det\left[\int_0^\infty \varphi_j(x)\psi_k(x)dx\right]_{j,k=1}^n$$

Ingredients 2

Singular value decomposition of Y of size $(n+\nu) \times n$:

$$Y = V\Sigma U, \qquad \Sigma = \mathrm{diag}(\sigma_1, \ldots, \sigma_n)$$

U is unitary and $V^* V = I$

- **Mapping**

$$Y \mapsto (U, V, y_1, \ldots, y_n), \qquad y_j = \sigma_j^2$$

is a change of variable with

$$dY = \left(\prod_{j=1}^{n} y_j^{\nu}\right) \Delta_n(y)^2 \, dy_1 \cdots dy_n \, dU \, dV$$

Proof of Theorem, step 1

First consider fixed X

Step 1: Complex Ginibre matrix G has distribution

$$\propto \; e^{-\operatorname{Tr} G^* G} dG$$

Change of variables $G \mapsto Y = GX$

- **has Jacobian** $\det(X^*X)^{-n-\nu} = \prod_j x_j^{-n-\nu}$
- **Then $G = YX^{-1}$ and**

$$e^{-\operatorname{Tr}(G^*G)} dG = \left(\prod_{j=1}^{n} x_j^{-n-\nu} \right) e^{-\operatorname{Tr}(Y^*Y(X^*X)^{-1})} dY$$

Proof of Theorem, steps 2 and 3

Step 2: **Singular value decomposition** $Y = V\Sigma U$

- **Then**

$$e^{-\operatorname{Tr}(G^*G)} dG \propto$$

$$\left(\prod_{k=1}^{n} x_k^{-n-\nu}\right) \underbrace{\left(\prod_{k=1}^{n} y_k^{\nu}\right) \Delta_n(y)^2}_{\text{Jacobian of SVD}} dy_1 \cdots dy_n$$

$$e^{-\operatorname{Tr}(U^*\Sigma^2 U (X^*X)^{-1})} \, dU \, dV$$

Proof of Theorem, steps 2 and 3

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$$e^{-\operatorname{Tr}(U^*\Sigma^2 U(X^*X)^{-1})} \, dU \, dV$$

Step 3: **Recall** **Harish-Chandra/Itzykson-Zuber formula**

$$\int_{U \in U(n)} e^{-\operatorname{Tr}(U^*AUB)} \, dU = c_n \frac{\det \left[e^{-a_j b_k}\right]_{j,k=1}^{n}}{\Delta_n(a)\Delta_n(b)}$$

Proof of Theorem, steps 3 and 4

Step 3: Intregate out U and V

- **Density for $y_1, \ldots, y_n$, after averaging over U and V**

$$\propto \left(\prod_{j=1}^{n} x_j^{-n-\nu}\right) \left(\prod_{k=1}^{n} y_k^{\nu}\right) \Delta_n(y)^2$$

$$\times \underbrace{\frac{1}{\Delta_n(y)\Delta_n(x^{-1})} \det\left[e^{-\frac{y_k}{x_j}}\right]_{j,k=1}^{n}}_{\text{result of HCIZ}}$$

Step 4: Clean up the formula

- **Use $\Delta_n(x^{-1}) = \pm \left(\prod_j x_j^{-n+1}\right) \Delta_n(x)$ and bring factors into the determinant:**

$$\propto \frac{\Delta_n(y)}{\Delta_n(x)} \det\left[\frac{y_k^{\nu}}{x_j^{\nu+1}} e^{-\frac{y_k}{x_j}}\right]_{j,k=1}^{n}$$

Proof of Theorem, step 5

Step 5: Density for fixed matrix X is

$$\propto \frac{\Delta_n(y)}{\Delta_n(x)} \det \left[\frac{y_k^\nu}{x_j^{\nu+1}} e^{-\frac{y_k}{x_j}} \right]_{j,k=1}^n$$

- **Average over** $\frac{1}{Z_n} \Delta_n(x) \det \left[f_k(x_j) \right]_{j,k=1}^n$
- **By Andreief identity**

$$\propto \Delta_n(y) \det \left[g_k(y_j) \right]_{j,k=1}^n$$

with

$$g_k(y) = \int_0^\infty \frac{y^\nu}{x^{\nu+1}} e^{-\frac{y}{x}} f_k(x) dx = \int_0^\infty x^\nu e^{-x} f_j\left(\frac{y}{x}\right) \frac{dx}{x}$$

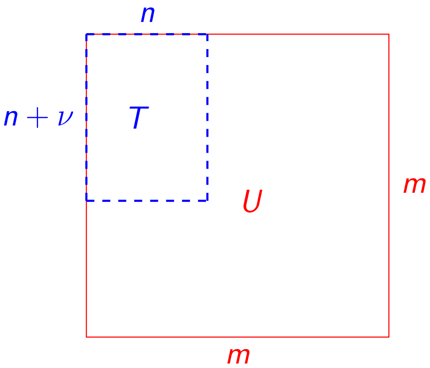
Multiplication with truncated unitary matrix

Other products

- **Products with inverses of complex Ginibre matrices**
  Forrester (2014)
- **Products with truncations of unitary matrices**
  Kieburg-Kuijlaars-Stivigny (2016)

Truncated unitary matrix

- **Unitary matrix** U **has size** $m \times m$
- **Truncation** T **has size** $(n+\nu) \times n$ **with** $n \leq n+\nu \leq m$



Transformation of polynomial ensemble

T **is** $(n+\nu) \times n$ **truncation of Haar distributed** $m \times m$ **unitary matrix**

Theorem (Kieburg-Kuijlaars-Stivigny (2016))

If squared singular values of X **have joint density**

$$\frac{1}{Z_n} \Delta_n(x) \det \left[f_k(x_j)\right]_{j,k=1}^{n} \qquad \textbf{all } x_j > 0$$

then squared singular values of $Y = TX$ **have density**

$$\frac{1}{\tilde{Z}_n} \Delta_n(y) \det \left[g_k(y_j)\right]_{j,k=1}^{n} \qquad \textbf{all } y_j > 0$$

with $$g_k(y) = \int_0^1 x^{\nu}(1-x)^{m-n-\nu-1} f_k\left(\frac{y}{x}\right) \frac{dx}{x}$$

Mellin convolution with Beta density.

Proofs

We have 4 proofs of this theorem.

Proofs

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- First proof mimics the proof for products of complex Ginibre matrices.
  It works only if $m \geq 2n + \nu$ since then there is a density for T

$$\propto \det(I - T^* T)^{m-2n-\nu} \, \mathbb{1}_{\{T^* T \leq I\}} \, dT$$

- Second proof uses more involved matrix integrals
- Third proof uses interlacing of eigenvalues of restricted matrices Kuijlaars (2016)
- Fourth proof uses spherical functions Kieburg-Kösters (arXiv 2016)

Ingredient in first proof

- **Analogue of HCIZ integral**

$$\int_{U \in U(n)} \det_+ (A - UBU^*)^p \, dU = c_{n,p} \frac{\det \left[(a_j - b_k)_+^{p+n-1} \right]_{j,k=1}^n}{\Delta_n(a)\Delta_n(b)}$$

where A and B are Hermitian matrices with eigenvalues $a_1, \ldots, a_n$, and $b_1, \ldots, b_n$, and

$$\det_+(X) = \begin{cases} \det(X) & \textbf{if } X \geq 0 \\ 0 & \textbf{otherwise} \end{cases}$$

$$x_+ = \begin{cases} x & \textbf{if } x \geq 0 \\ 0 & \textbf{otherwise} \end{cases} \qquad c_{n,p} = \prod_{j=0}^{n-1} \binom{p+n-1}{j}^{-1}$$

Ingredient in first proof

- **Analogue of HCIZ integral**

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- **Known formula without $+$** **Gross-Richards (1989)**

Third proof

Observation: suppose $U = \begin{pmatrix} T & * \\ * & * \end{pmatrix}$

then

$$U \begin{pmatrix} XX^* & 0 \\ 0 & 0 \end{pmatrix} U^* = \begin{pmatrix} TXX^*T^* & * \\ * & * \end{pmatrix}$$

- **Squared singular values of** TX **are non-zero eigenvalues of leading principal submatrix of** UAU^* **where**

$$A = \begin{pmatrix} XX^* & 0 \\ 0 & 0 \end{pmatrix}$$

- A **is a semi positive-definite matrix whose non-zero eigenvalues are the squared singular values of** X**.**

Known result

Baryshnikov (2001)
Suppose A is $(n+1)\times(n+1)$ Hermitian matrix with eigenvalues

$$a_0 < a_1 < \cdots < a_n$$

B is the $n \times n$ principal submatrix of UAU^* where U is Haar distributed unitary matrix

- **Eigenvalues of B almost surely interlace**

$$a_0 < b_1 < a_1 < b_2 < \cdots < a_{n-1} < b_n < a_n$$

- **Joint density of eigenvalues**

$$n!\,\frac{\Delta_n(b)}{\Delta_{n+1}(a)}$$

subject to the interlacing condition.

Variation on this theme

Forrester-Rains (2005)

Suppose A **is** $(n+p)\times(n+p)$ **positive semidefinite with eigenvalues**

$$0 < a_1 < \cdots < a_n, \qquad 0 \text{ has multiplicity } p \geq 1.$$

B **is** $(n+p-1)\times(n+p-1)$ **principal submatrix of** UAU^*

- B **has** $p-1$ **eigenvalues at** 0 **and remaining** n **eigenvalues almost surely interlace**

$$0 < b_1 < a_1 < b_2 < \cdots < a_{n-1} < b_n < a_n$$

- **Joint density**

$$\propto \frac{\prod_{k=1}^{n} b_k^{p-1}}{\prod_{k=1}^{n} a_k^{p}} \frac{\Delta_n(b)}{\Delta_n(a)}$$

subject to the interlacing condition.

Transformation of polynomial ensemble

Suppose A **is** $(n+p) \times (n+p)$ **positive semidefinite with eigenvalues**

$$0 < a_1 < \cdots < a_n, \qquad 0 \text{ has multiplicity } p \geq 1,$$

whose non-zero eigenvalues are polynomial ensemble

$$\frac{1}{Z_n} \Delta_n(a) \det \left[f_k(a_j) \right]_{j,k=1}^{n}$$

B **is** $(n+p-1) \times (n+p-1)$ **principal submatrix of** UAU^*

Then non-zero eigenvalues of B **have density**

$$\frac{1}{\tilde{Z}_n} \Delta_n(b) \det \left[g_k(b_j) \right]_{j,k=1}^{n}$$

where

$$g_k(b) = \int_0^1 x^{p-1} f_k\left(\frac{b}{x}\right) \frac{dx}{x}$$

Repeat the transformation

Suppose A **is** $(n+p) \times (n+p)$ **is positive semidefinite**

$$0 < a_1 < \cdots < a_n, \qquad 0 \text{ has multiplicity } p \geq 1,$$

whose non-zero eigenvalues are polynomial ensemble

$$\frac{1}{Z_n} \Delta_n(a) \det\left[f_k(a_j)\right]_{j,k=1}^{n}$$

B **is** $(n+q) \times (n+q)$ **principal submatrix of** UAU^* **with**

$$0 \leq q < p$$

Then non-zero eigenvalues of B **have density**

$$\frac{1}{\tilde{Z}_n} \Delta_n(b) \det\left[g_k(b_j)\right]_{j,k=1}^{n}$$

where

$$g_k(b) = \int_0^1 x^q (1-x)^{p-q-1} f_k\left(\frac{b}{x}\right) \frac{dx}{x}$$

Fourth proof (sketch)

Kieburg-Kösters (arXiv 2016)
Group theoretic point of view

$G = GL(n, \mathbb{C})$ **complex Lie group**
$K = U(n)$ **maximal compact subgroup**

- **A function** $f : G \to \mathbb{C}$ **is** **K-biinvariant** **if**

$$f(UAV) \qquad \textbf{for all } A \in G, \; U, V \in K.$$

- **Then** f **depends only on the squared singular values of** A

$$f(A) = f_{ssv}(x_1, \ldots, x_n)$$

Convolution

Suppose f and g are probability densities of independent random matrices X and Y that are both K-biinvariant.

- **Then**

$$(g * f)(A) = \int_G g(B) f(AB^{-1}) \frac{dB}{\det(B^* B)}$$

  is density for YX.
- $\dfrac{dB}{\det(B^* B)}$ **is Haar measure on $G = GL(n, \mathbb{C})$**
- **Convolution is commutative for biinvariant functions (Gelfand pair)**

Spherical function

$\varphi : G \to \mathbb{C}$ **is a** **spherical function** **if** $\varphi(A) = 1$ **and**

$$\int_K \varphi(AUB)dU = \varphi(A)\varphi(B)$$

- φ **is** K**-biinvariant.**

Spherical transform $f \mapsto \widehat{f}$

$$\widehat{f}(\varphi) = \int_G f(A)\varphi(A)\frac{dA}{\det A^*A}$$

has property

$$\widehat{f * g} = \widehat{f} \cdot \widehat{g}$$

Spherical functions

Spherical functions for $(G, K) = (GL(n,\mathbb{C}), U(n))$ **are labeled by** $s = (s_1, \ldots, s_n) \in \mathbb{C}^n$**:**

$$\varphi_s(A) = \left(\prod_{j=1}^{n-1} j!\right) \frac{\det\left[x_j^{s_k}\right]_{j,k=1}^{n}}{\Delta_n(x)\Delta_n(s)}$$

where $x_1, \ldots, x_n$ **are eigenvalues of** A^*A**.**

Spherical transform

Spherical transform $\widehat{f}(s) = \int_G f(A)\varphi_s(A)\frac{dA}{\det(A^*A)}$

can be calculated for

- **Gaussian density**

$$\widehat{g}_1(s) = \prod_{j=1}^{n} \Gamma(s_j + \nu)$$

- **Truncated unitary matrix density**

$$\widehat{g}_2(s) = \prod_{j=1}^{n} B(s_j + \nu, m - n - \nu)$$

- **Polynomial ensemble** $f(A) \propto \Delta_n(x)\det\left[f_k(x_j)\right]$

$$\widehat{f}(s) \propto \frac{1}{\Delta_n(s)}\det\left[\int_0^\infty x^{s_j} f_k(x)\frac{dx}{x}\right]$$

Spherical transform

- **Thus**

$$\widehat{g_2 * f}(s) \propto \frac{\prod_{j=1}^{n} B(s_j + \nu, m - n - \nu)}{\Delta_n(s)} \det\left[\int_0^\infty x^{s_j} f_k(x) \frac{dx}{x}\right]$$

$$= \frac{1}{\Delta_n(s)} \det\left[B(s_j + \nu, m - n - \nu) \int_0^\infty x^{s_j} f_k(x) \frac{dx}{x}\right]$$

Spherical transform

- **Thus**

$$\widehat{g_2 * f}(s) \propto \frac{\prod_{j=1}^{n} B(s_j+\nu, m-n-\nu)}{\Delta_n(s)} \det\left[\int_0^\infty x^{s_j} f_k(x) \frac{dx}{x}\right]$$
$$= \frac{1}{\Delta_n(s)} \det\left[B(s_j+\nu, m-n-\nu)\int_0^\infty x^{s_j} f_k(x) \frac{dx}{x}\right]$$

- **Note**

$$B(s_j+\nu, m-n-\nu) = \int_0^\infty x^{s_j} x^{\nu}(1-x)^{m-n-\nu-1}\frac{dx}{x}$$

is Mellin transform of $x^{\nu}(1-x)^{m-n-\nu-1}$

- **Product of Mellin transforms is transform of the Mellin convolution**

$$B(s_j+\nu, m-n-\nu)\int_0^\infty x^{s_j} f_k(x)\frac{dx}{x} = \int_0^\infty x^{s_j} g_k(x)\frac{dx}{x}$$

Proucts of Ginibre matrices

Multiplication with Ginibre

Theorem (Kuijlaars-Stivigny (2014))

Suppose G **is** $(n+\nu)\times n$ **complex Ginibre matrix and** X **is independent random matrix with squared singular value density**

$$\frac{1}{Z_n}\Delta_n(x)\det\left[f_k(x_j)\right]_{j,k=1}^{n}$$

Then squared singular values of $Y = GX$ **have density**

$$\frac{1}{\tilde{Z}_n}\Delta_n(y)\det\left[g_k(y_j)\right]_{j,k=1}^{n}$$

with

$$g_k(y) = \int_0^\infty x^\nu e^{-x} f_k\left(\frac{y}{x}\right)\frac{dx}{x}$$

Transformation of polynomial ensemble

Multiplication by complex Ginibre matrix transforms polynomial ensembles

- **From**

$$\frac{1}{Z_n} \Delta_n(x) \det \left[f_k(x_j) \right]_{j,k=1}^{n}$$

to

$$\frac{1}{\tilde{Z}_n} \Delta_n(y) \det \left[g_k(y_j) \right]_{j,k=1}^{n}$$

where $$g_k(y) = \int_0^\infty x^\nu e^{-x} f_k\left(\frac{y}{x}\right) \frac{dx}{x}$$

- g_k **is Mellin convolution of** f_k **with** $x^\nu e^{-x}$

Mellin transform

Mellin transform

$$\widehat{f}(s) = \int_0^\infty x^{s-1} f(x)\, dx$$

- **If g is the Mellin convolution of f_1 and f_2, then**

$$\widehat{g}(s) = \widehat{f_1}(s) \cdot \widehat{f_2}(s)$$

- **Since g_k is the Mellin convolution of $x^\nu e^{-x}$ with f_k**

$$\widehat{g}_k(s) = \Gamma(s+\nu) \cdot \widehat{f}_k(s)$$

Mellin transform

Mellin transform

$$\widehat{f}(s) = \int_0^\infty x^{s-1} f(x)\, dx$$

- **If g is the Mellin convolution of f_1 and f_2, then**

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- **Since g_k is the Mellin convolution of $x^\nu e^{-x}$ with f_k**

$$\widehat{g_k}(s) = \Gamma(s+\nu) \cdot \widehat{f_k}(s)$$

Inverse Mellin transform

$$f(x) = \frac{1}{2\pi i} \int_L \widehat{f}(s) x^{-s} ds$$

with suitable contour L in the complex plane

Products of Ginibre matrices

Suppose G_j **has size** $(n+\nu_j) \times (n+\nu_{j-1})$ **with all** $\nu_j \geq 0$ **and** $\nu_0 = 0$**. Make product**

$$Y = G_r \cdots G_2 G_1 \qquad \text{of size } (n+\nu_r) \times n$$

- **Eigenvalues of** $Y^* Y$ **are polynomial ensemble**

$$\frac{1}{Z_n} \Delta_n(x) \ \det \left[w_{k-1}(x_j) \right]_{j,k=1}^{n}$$

where

$$\int_0^\infty x^{s-1} w_k(x) dx = s^k \prod_{j=1}^{r} \Gamma(s+\nu_j)$$

Akemann-Kieburg-Wei (2013)
Akemann-Ipsen-Kieburg (2013)

Meijer G-functions

By inverse Mellin transform

$$w_k(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} s^k \prod_{j=1}^{r} \Gamma(s+\nu_j)\, x^{-s} ds, \qquad x > 0$$

- **Such functions are known as Meijer G-functions**

Beals-Szmigielski, Notices AMS, 2013

Correlation kernel

Correlation kernel is

$$K_n(x,y) = \sum_{k=0}^{n-1} P_k(x) Q_k(y)$$

- P_k **is a polynomial of degree** k
- Q_k **is a linear combination of** $w_0, \ldots, w_{n-1}$

with biorthogonality

$$\int_0^\infty P_k(x) Q_j(x) dx = \delta_{j,k}.$$

Integral representation

Theorem

$$P_k(x) = \frac{\gamma_k}{2\pi i} \oint_\Sigma \frac{\Gamma(t-k)}{\prod_{j=0}^r \Gamma(t+\nu_j+1)} x^t \, dt$$

were Σ is a closed contour around the interval $[0, k]$.

$$Q_k(x) = \frac{1}{\gamma_k} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} (s-k)_k \prod_{j=1}^{r} \Gamma(s+\nu_j) \, x^{-s} \, ds$$

Proof.

Residue calculus + properties of Mellin transform □

Integral representation

Theorem

$$P_k(x) = \frac{\gamma_k}{2\pi i} \oint_\Sigma \frac{\Gamma(t-k)}{\prod_{j=0}^{r} \Gamma(t+\nu_j+1)} x^t \, dt$$

were Σ is a closed contour around the interval $[0, n]$.

Corollary: P_k **is a hypergeometric polynomial**

$$P_k(x) = \gamma_k \; {}_1F_r \left(\begin{array}{c} -k \\ 1+\nu_1, \ldots, 1+\nu_r \end{array} \middle| x \right)$$

Double integral representation

Theorem

The polynomial ensemble has correlation kernel

$$K_n(x,y) = \frac{1}{(2\pi i)^2} \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} ds \oint_{\Sigma} dt \left(\prod_{j=0}^{r} \frac{\Gamma(s+\nu_j+1)}{\Gamma(t+\nu_j+1)} \right) \frac{\Gamma(t-n+1)}{\Gamma(s-n+1)} \frac{x^t y^{-s-1}}{s-t}$$

Proof.

Combine integral representations of P_k and Q_k

Telescoping sum □

Large n limit: global regime

Global limit

- **Let $n \to \infty$ with $\nu_1, \nu_2, \ldots, \nu_r$ fixed.**
- **Largest eigenvalue of Y^*Y grows like n^r.**

Global limit

- **Let $n \to \infty$ with $\nu_1, \nu_2, \ldots, \nu_r$ fixed.**
- **Largest eigenvalue of Y^*Y grows like n^r.**
- **The limit**

$$\rho_r(x) = \lim_{n\to\infty} n^{r-1} K_n(n^r x, n^r x)$$

  is the density of Fuss-Catalan distribution and does not depend on $\nu_1, \ldots, \nu_r$.
- **It is the r-fold multiplicative free convolution of the Marchenko-Pastur distribution.**

Equilibrium problem

Forrester, Liu, Zinn-Justin (2015)

The DPP looks like a biorthogonal ensemble

$$\frac{1}{Z_n}\prod_{j<k}(x_k - x_j)\prod_{j<k}(x_k^{1/r} - x_j^{1/r})\ \prod_{j=1}^{n} x_j^{\frac{r-1}{2}} e^{-r x_j^{1/r}}$$

in the global regime as $n \to \infty$

- **After rescaling $x_j \mapsto n^r x_j$**

$$\frac{1}{\widehat{Z}_n}\prod_{j<k}(x_k - x_j)\prod_{j<k}(x_k^{1/r} - x_j^{1/r})\ \prod_{j=1}^{n} x_j^{\frac{r-1}{2}} e^{-rn x_j^{1/r}}$$

- **It leads to an equilibrium problem**

Equilibrium problem

- **Equilibrium problem** **Minimize over μ on $[0, \infty)$,**

$$\frac{1}{2}\iint \log\frac{1}{|x-y|}d\mu(x)d\mu(y)$$

$$+\frac{1}{2}\iint \log\frac{1}{|x^{1/r}-y^{1/r}|}d\mu(x)d\mu(y) + r\int x^{1/r}d\mu(x)$$

Equilibrium problem

- **Equilibrium problem** **Minimize over** μ **on** $[0,\infty)$,

$$\frac{1}{2}\iint \log\frac{1}{|x-y|}d\mu(x)d\mu(y)$$
$$+\frac{1}{2}\iint \log\frac{1}{|x^{1/r}-y^{1/r}|}d\mu(x)d\mu(y)+r\int x^{1/r}d\mu(x)$$

- **Fuss-Catalan distribution** $\rho_r(x)dx$ **has compact support** $[0,x^*]$ **with** $x^*=\frac{(r+1)^{r+1}}{r^r}$
- **Density blows up at hard edge**

$$\rho_r(x)\sim x^{-r/(r+1)} \qquad \text{as } x\to 0+$$

- **Square root decay at soft edge** x^*

Burda-Jarosz-Livan-Nowak-Swiech (2011)
Penson-Zyczkowski (2011), Neuschel (2014)
Forrester-Liu (2015)

Vector equilibrium problem

Notation

$$I(\mu,\nu) = \iint \log \frac{1}{|x-y|} d\mu(x) d\nu(y), \qquad I(\mu) = I(\mu,\mu)$$

Minimize

$$\sum_{j=0}^{r-1} I(\mu_j) - \sum_{j=0}^{r-2} I(\mu_j, \mu_{j+1}) + \int V(x) d\mu_0(x)$$

- μ_j **is on** $(-1)^j[0,\infty)$ **with total mass** $1 - \frac{j}{r}$

Unique minimizing vector of measures $(\mu_0, \ldots, \mu_{r-1})$

- μ_0 **is compactly supported (if** V **grows faster than logarithm)**
- **Other measures have full support** $(-\infty, 0]$ **or** $[0,\infty)$**.**

Vector equilibrium problem

Proposition (arXiv 2016)

μ_0 **is the minimizer for**

$$\frac{1}{2} I(\mu) + \frac{1}{2} I_{1/r}(\mu) + \int V(x) d\mu(x)$$

among probability measures on $[0, \infty)$**.**

$$I_{1/r}(\mu) = \iint \log \frac{1}{|x^{1/r} - y^{1/r}|} d\mu(x) d\mu(y)$$

Large n limit: local regime

Local scaling limits

Sequence of correlation kernels K_n

- **Scale points around a**

$$x_j \mapsto cn^{\gamma}(x_j - a)$$

- **New kernel**

$$\frac{1}{cn^{\gamma}} K_n\left(a + \frac{x}{cn^{\gamma}}, a + \frac{y}{cn^{\gamma}}\right)$$

- **Is there a limit**

$$\lim_{n\to\infty} \frac{1}{cn^{\gamma}} K_n\left(a + \frac{x}{cn^{\gamma}}, a + \frac{y}{cn^{\gamma}}\right) = ??$$

Known limits

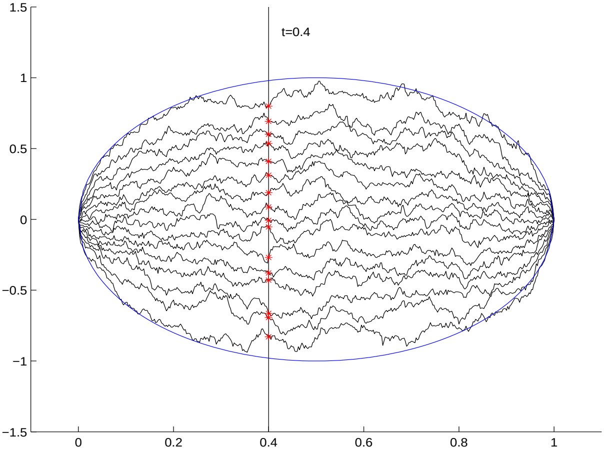
- **Sine kernel in the bulk** $$\frac{\sin \pi(x-y)}{\pi(x-y)}$$
- **Airy kernel at soft edge**

$$\frac{\mathrm{Ai}(x)\,\mathrm{Ai}'(y) - \mathrm{Ai}'(x)\,\mathrm{Ai}(y)}{x-y}$$

- **Bessel kernels at hard edge**

$$\frac{J_\nu(\sqrt{x})\sqrt{y}J_\nu'(\sqrt{y}) - \sqrt{x}J_\nu'(\sqrt{x})J_\nu(\sqrt{y})}{2(x-y)}$$

GUE scaling limits



- **Sine kernel in the bulk** $\dfrac{\sin \pi(x-y)}{\pi(x-y)}$
- **Airy kernel at the edge**

Airy function

- **The** **Airy equation**

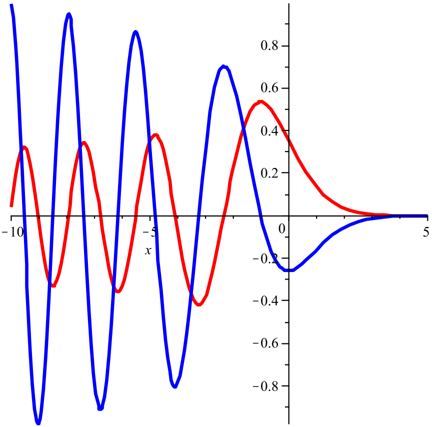
$$y''(z) = zy(z)$$

has special solution $\mathrm{Ai}(z)$ **that decays as** $z \to +\infty$

$$\mathrm{Ai}(z) = \frac{1}{2\sqrt{\pi} z^{1/4}} e^{-\frac{2}{3} z^{3/2}} \left(1 + \mathcal{O}(z^{-3/2})\right)$$

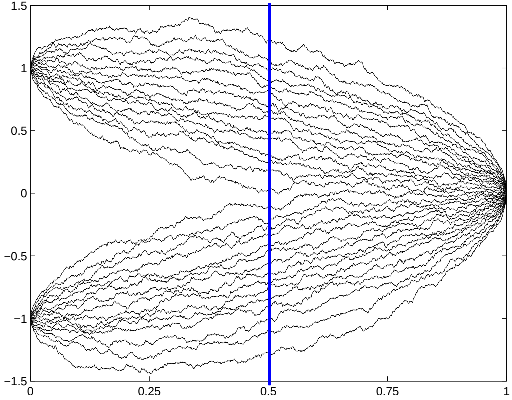
and oscillates on negative real axis

Plot



- **Plot of** Ai (**red**) **and its derivative** Ai′ (**blue**)

Cusp point



- **Again sine kernel in the bulk, and Airy kernel at the edge.**
- **New limiting kernels at cusp point**

Pearcey kernels

- **New family of limiting kernels built out of solutions of third order Pearcey ODEs**

$$p'''(x) = xp(x) - sp'(x) \quad \textbf{and} \quad q'''(y) = yq(y) + sq'(y)$$

- **Double scaling limit are the Pearcey kernels**

$$\frac{p(x)q''(y) - p'(x)q'(y) + p''(x)q(y) - sp(x)q(y)}{x - y}$$

Brézin-Hikami (1998), Tracy-Widom (2006)
Bleher-Kuijlaars (2007)

Local scaling limits

What are scaling limits of

$$K_n(x,y) = \frac{1}{(2\pi i)^2} \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} ds \oint_{\Sigma} dt \left(\prod_{j=0}^{r} \frac{\Gamma(s+\nu_j+1)}{\Gamma(t+\nu_j+1)} \right) \frac{\Gamma(t-n+1)}{\Gamma(s-n+1)} \frac{x^t y^{-s-1}}{s-t}$$

as $n \to \infty$ **with** $\nu_1, \ldots, \nu_r$ **fixed?**

- **Sine kernel in the bulk and Airy kernel at the soft right edge** **Liu-Wang-Zhang (2016)**
- **Bessel kernel at the hard edge if** $r = 1$**.**
- **something new at the hard edge if** $r \geq 2$**.**

Hard edge scaling limit

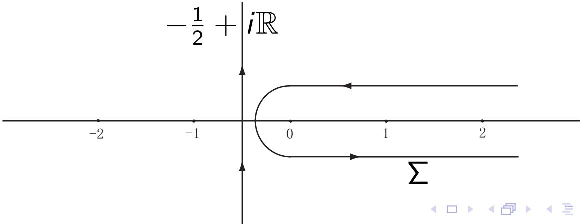
Theorem (Kuijlaars-Zhang (2014))

$$\lim_{n\to\infty} \frac{1}{n} K_n\left(\frac{x}{n}, \frac{y}{n}\right) = K_{\nu_1,\ldots,\nu_r}(x,y), \qquad x, y > 0,$$

exists with limiting kernel

$$K_{\nu_1,\ldots,\nu_r}(x,y) =$$

$$\frac{1}{(2\pi i)^2} \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} ds \int_{\Sigma} dt \left(\prod_{j=0}^{r} \frac{\Gamma(s+\nu_j+1)}{\Gamma(t+\nu_j+1)}\right) \frac{\sin \pi s}{\sin \pi t} \frac{x^t y^{-s-1}}{s-t}$$



Meijer G-kernel

Alternative expression in terms of Meijer G-functions

$$K_{\nu_1,\ldots,\nu_r}(x,y) =$$

$$\int_0^1 G^{1,0}_{0,r+1}\left(\begin{matrix} - \\ 0, -\nu_1, \ldots, -\nu_r \end{matrix} \middle| ux \right) G^{r,0}_{0,r+1}\left(\begin{matrix} - \\ \nu_1, \ldots, \nu_r, 0 \end{matrix} \middle| uy \right) du$$

Universality?

Meijer G-kernels appear as scaling limits in

- **Cauchy multi matrix model:**

  Bertola-Gekhtman-Szmigielski (2013)

  Bertola-Bothner (2015)

- **Biorthogonal ensemble**

$$\frac{1}{Z_n} \prod_{j<k} (x_k - x_j) \prod_{j<k} (x_k^\theta - x_j^\theta) \prod_{j=1}^{n} x_j^\nu e^{-x_j}$$

  if $\theta = 1/r$ **Kuijlaars-Stivigny (2014)**

- **Products of Ginibre + inverse Ginibre**

  Forrester (2014)

- **Products with truncated unitary matrices**

  Kieburg-Kuijlaars-Stivigny (2016)

That's all

Thank you