# Extreme events in Voronoi tessellations

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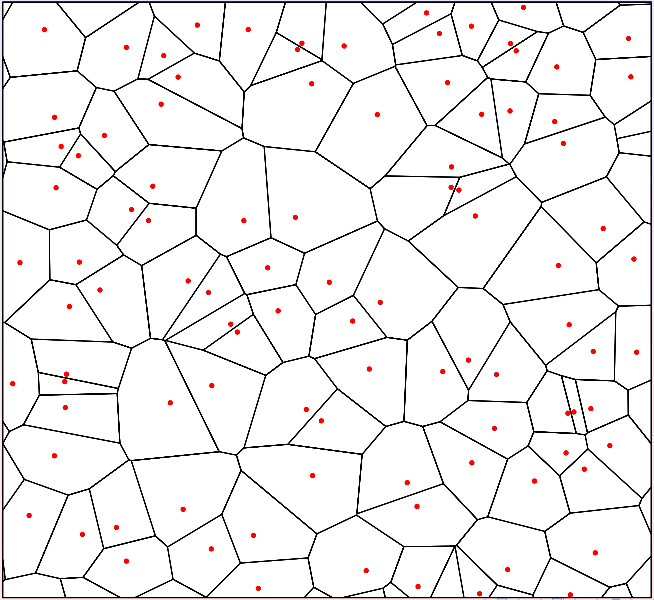
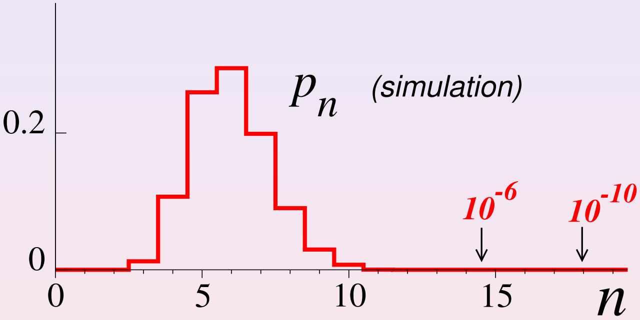
# The Voronoi construction



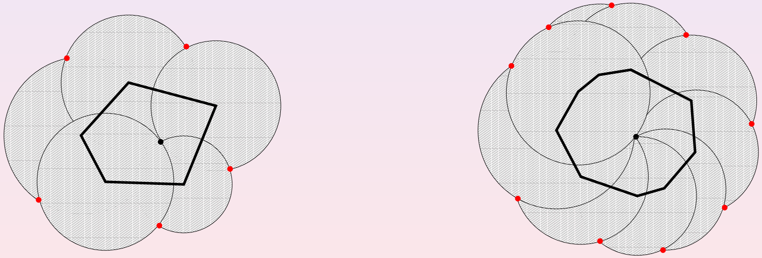
# The sidedness probability $p_n$ in 2D



# Basic expression for $p_n$

$$p_n = \int \mathrm{d}R_1 \ldots \mathrm{d}R_n \underbrace{\chi_n(R_1,\ldots,R_n)}_{\boxed{\text{indicator}}} \quad \underbrace{e^{-\rho A_n(R_1,\ldots,R_n)}}_{\boxed{\text{excluded domain}}}$$

The *flower* :



$n = 5$ $\qquad$ $n = 9$

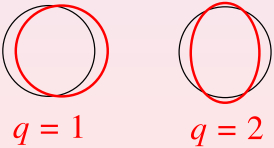
# Exact 2D results (HJH, 2005)

Final result for 2D cell when $n \to \infty$:

$$p_n \simeq C_2 \frac{(8\pi^2)^n}{(2n)!}$$

Byproduct : cell becomes circular with radius $\quad R_n \simeq \left(\frac{n}{4\pi}\right)^{1/2}$

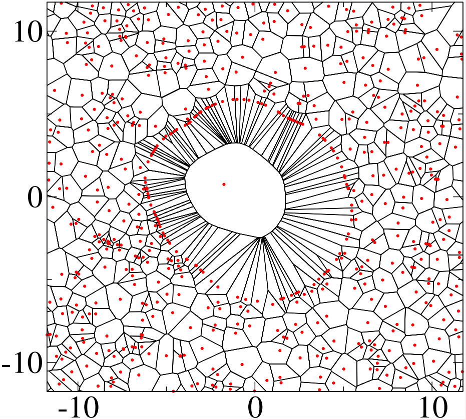
$$C_2 = \frac{1}{4\pi^2} \prod_{q=1}^{\infty} \left(1 - \frac{1}{q^2} + \frac{4}{q^4}\right)^{-1} \quad = 0.344\,347$$



# A very-many-sided cell

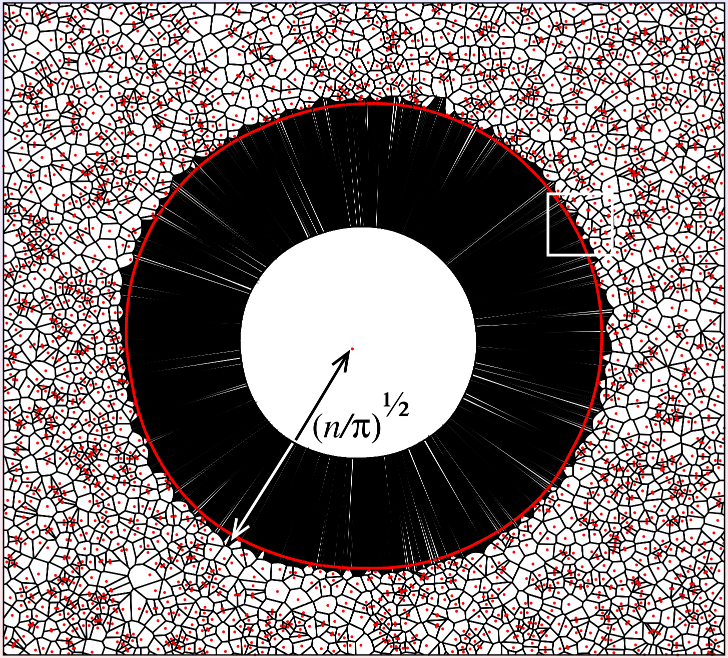
$n = 96$

$p_n \approx 10^{-177}$



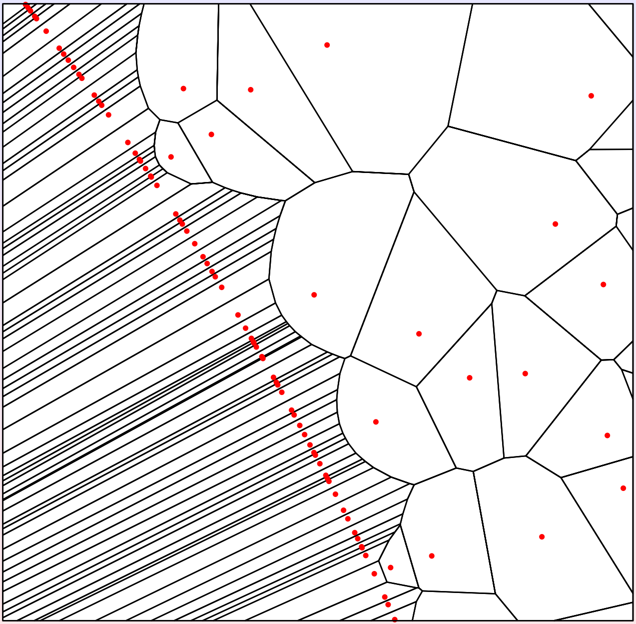
$n = 1536$

$p_n \approx 10^{-6472}$

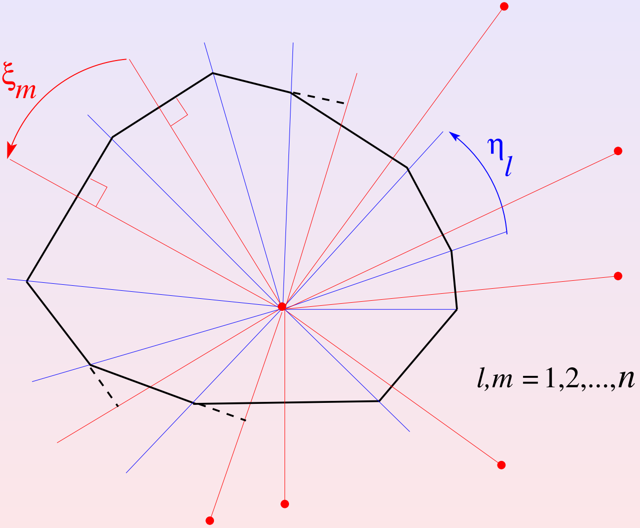


$n = 1536$

zoom



# *Method*: the $2n$ angles



# *Method*: the $C_2$ integral

Then

$$C_2 = \lim_{n\to\infty} \int^{*} \underbrace{\prod_{m=1}^{n} \mathrm{d}\xi_m \, u(\xi_m) \prod_{\ell=1}^{n} \mathrm{d}\eta_\ell \, v(\eta_\ell)}_{\text{asymptotically independent}} \exp\Big[ - \mathcal{W}_n(\{\xi_m, \eta_\ell\}) \Big]$$

in which

$$u(\xi) = \frac{n^2 \xi}{\pi^2} \exp\Big( - \frac{n\,\xi}{\pi} \Big), \qquad v(\eta) = \frac{n}{2\pi} \exp\Big( - \frac{n\,\eta}{2\pi} \Big).$$

Define

$$\delta\xi_m = \xi_m - \frac{2\pi}{n}, \qquad \delta\eta_m = \eta_m - \frac{2\pi}{n}$$

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# *Method*: formal perturbation expansion

- Expand $\mathcal{W}_n$ in powers of $n^{-1/2}$:

non-Gaussian

$$\mathcal{W}_n = -\frac{1}{n}\sum_{j=1}^{n}\sum_{k=1}^{n}(\delta\xi_j,\delta\eta_j)\cdot\mathbf{W}_{jk}\cdot(\delta\xi_k,\delta\eta_k)^{\mathrm{T}}$$

long-range, O(1)

$+$ negligible as $n\to\infty$

- Diagonalize **W** and integrate $\Rightarrow$ $C_2$ (*"Debye-Hückel"*)

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non-Gaussian: $(\delta\xi_j,\delta\eta_j)$, $(\delta\xi_k,\delta\eta_k)$

$\mathbf{W}_{jk}$: long-range, O(1)

- Diagonalize **W** and integrate $\Rightarrow$ $C_2$ (*"Debye-Hückel"*)

# How about three dimensions?

Questions you may ask:

$p_n^{\mathrm{F}}$ = probability that a *cell* have *n* *faces* ??

Problem with *spherical* symmetry

$p_n^{\mathrm{E}}$ = probability that a *face* have *n* *edges*

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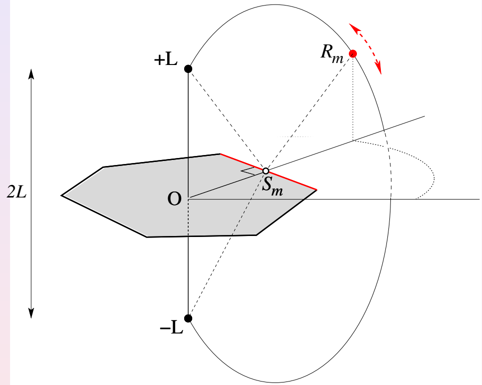
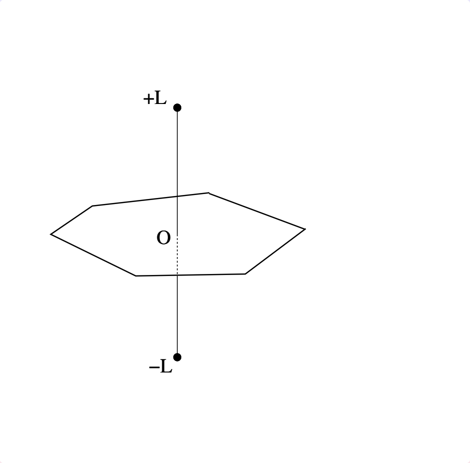
# A face shared by two 3D cells

*E. Lazar et al. (simulations) 2013; E. Lazar and HJH 2014; HJH 2016.*



# The “pumpkin” of a cell face in 3D

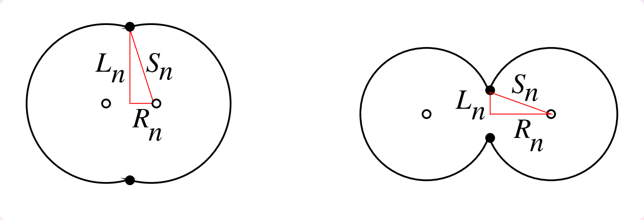



*Excluded domain:*

*Excluded domain* =
union of $n$ balls centered on the face’s vertices $T_\ell$
and having $|OT_\ell|$ as their radius.

# The pumpkin in the limit $n \to \infty$

*a “spindle” torus*



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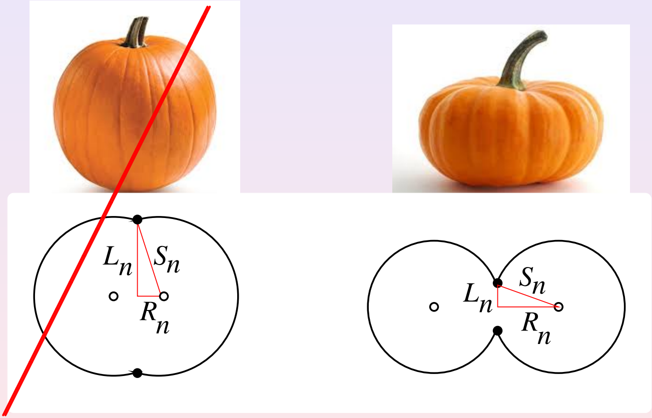
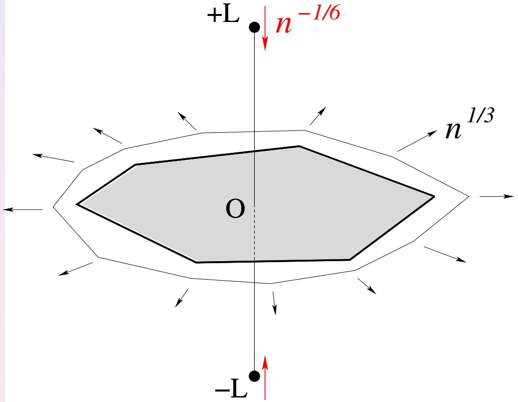



$$R_n \simeq S_n \simeq c\, n^{1/3}, \qquad L_n \sim n^{-1/6}$$

# Imposing $n \implies$ entropic attraction



# Exact results for the $n$-edged face

(1) For $n \to \infty$ the $n$-edged face becomes circular

(2) Probability of an $n$-edged face

$$p_n \simeq \frac{(12\pi^2)^n}{(2n)!} C_3, \qquad C_3 = \prod_{q=1}^{n} \left(1 + \frac{9}{q^4}\right)^{-1}$$

(3) The excluded domain tends towards a torus of major and minor radii both equal to

$$R_n \simeq \left(\frac{n}{2\pi^2}\right)^{1/3}$$

(4) Conditional distribution $Q_n(n^{1/6}L)$ of $L$

$$Q_n(y) \;\to\; Q_\infty(y) = c_0 y^2 \mathrm{e}^{-c_1 y^2}, \qquad y > 0,$$

(5) Finite-size corrections

$$Q_n(y) = Q_\infty(y)\left[\,1 \,+\, \frac{1}{n} F_1(y) + \ldots\,\right]$$

# Final remarks

- Line tessellations:

a coefficient $C_\alpha$ appears *(with Pierre Calka, 2008).*

- Random acceleration process:

$R(\phi) \equiv$ radius of the $n$-sided cell, or interface;
then under suitable scaling for $n \to \infty$

$$\frac{\mathrm{d}^2 R}{\mathrm{d}\phi^2} = \xi(\phi), \qquad \xi = \text{Gaussian white noise s.t. } \int_0^{2\pi} \xi(\phi)\mathrm{d}\phi = 0.$$

*(with Pierre Calka and Grégory Schehr, 2008).*

- Exact result *versus* conjecture ...

Thank you.