Stolarsky-type identities, energy optimization, uniform tessellations, and one-bit sensing

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Optimal and random point configurations:
From Statistical Physics to Approximation Theory

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Good point distributions

- Lattices
- Energy minimization, polarization
- Monte-Carlo
- Other random point processes (jittered sampling, determinantal)
- Covering/packing problems
- Low-discrepancy sets
- Cubature formulas
- Uniform tessellation, almost isometric embeddings

Discrepancy

- U: a set with a natural probability measure μ (e.g., $[0,1]^d$, $\mathbb{S}^d$, $\mathbb{R}^d$, etc.)

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- sup $\to$ L^2-average: L^2 discrepancy.

Spherical cap discrepancy

For $x \in \mathbb{S}^d$, $t \in [-1, 1]$ define spherical caps:

$$C(x,t) = \{y \in \mathbb{S}^d : \langle x, y\rangle \geq t\}.$$

For a finite set $Z = \{z_1, z_2, ..., z_N\} \subset \mathbb{S}^d$ define

$$D_{cap}(Z) = \sup_{x\in\mathbb{S}^d, t\in[-1,1]} \left| \frac{\#\big(Z \cap C(x,t)\big)}{N} - \sigma\big(C(x,t)\big) \right|.$$

Theorem (Beck, '84)

There exists constants c_d, $C_d > 0$ *such that*

$$c_d N^{-\frac{1}{2}-\frac{1}{2d}} \leq \inf_{\#Z=N} D_{cap}(Z) \leq C_d N^{-\frac{1}{2}-\frac{1}{2d}} \sqrt{\log N}.$$

Spherical caps: L^2 Stolarsky Principle

Define the spherical cap L^2 discrepancy

$$D_{cap,L^2}(Z) = \left(\int_{\mathbb{S}^d} \int_{-1}^{1} \left| \frac{\#\big(Z \cap C(x,t)\big)}{N} - \sigma\big(C(x,t)\big) \right|^2 dt\, d\sigma(x) \right)^{\frac{1}{2}}.$$

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For any finite set $Z = \{z_1, \ldots, z_N\} \subset \mathbb{S}^d$

$$\frac{1}{N^2} \sum_{i,j=1}^{N} \|z_i - z_j\| \; + \; c_d \Big[D_{L^2,cap} \Big]^2 = \text{const}$$

$$= \int_{\mathbb{S}^d} \int_{\mathbb{S}^d} \|x - y\| \, d\sigma(x) d\sigma(y).$$

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- Proofs: Stolarsky ('73), Brauchart, Dick ('12), DB ('16).

Spherical caps: L^2 Stolarsky Principle

- Define the spherical cap discrepancy of fixed height t:

$$D^{(t)}_{L^2,\text{cap}}(Z) := \left(\int_{\mathbb{S}^d} \left| \frac{1}{N} \sum_{j=1}^{N} \mathbf{1}_{C(x,t)}(z_j) - \sigma\big(C(x,t)\big) \right|^2 d\sigma(x) \right)^{1/2}$$

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- $$\left[D^{(t)}_{L^2,\mathrm{cap}}(Z) \right]^2 = \frac{1}{N^2} \sum_{i,j=1}^{N} \sigma\big(C(z_i,t) \cap C(z_j,t)\big) - \Big(\sigma\big(C(p,t)\big)\Big)^2 .$$

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- Averaging over $t \in [-1,1]$

$$\int_{-1}^{1} \sigma\big(C(x,t) \cap C(y,t)\big)\, dt \quad = \quad 1 - C_d \|x-y\|$$

$$\int_{-1}^{1} \big(\sigma\big(C(p,t)\big)\big)^2 \, dt \quad = \quad 1 - C_d \int_{\mathbb{S}^d} \int_{\mathbb{S}^d} \|x-y\| \, d\sigma(x) d\sigma(y).$$

Hemisphere discrepancy

- L^2 discrepancy for spherical cap discrepancy of fixed height t satisfies:
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- Taking $t = 0$ (i.e. hemispheres)

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Theorem (Stolarsky for hemispheres, DB '16, Skriganov '16)

$$[D_{L^2,\text{hem}}(Z)]^2 = [D^{(0)}_{L^2,\text{cap}}(Z)]^2$$
$$= \frac{1}{2}\left(\int_{\mathbb{S}^d}\int_{\mathbb{S}^d} d(x,y)\, d\sigma(x)\, d\sigma(y) - \frac{1}{N^2}\sum_{i,j=1}^{N} d(z_i,z_j)\right).$$

Hemisphere Stolarsky: simple corollaries

$$[D_{L^2,\mathrm{hem}}(Z)]^2 = \frac{1}{2}\left(\frac{1}{2} - \frac{1}{N^2}\sum_{i,j=1}^{N} d(z_i, z_j)\right).$$

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- For any $Z = \{z_1, ..., z_N\} \subset \mathbb{S}^d$

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- For even N:

$$\frac{1}{N^2}\sum_{i,j=1}^{N} d(z_i, z_j) = \frac{1}{2} \Longleftrightarrow Z \text{ - symmetric.}$$

- For odd N the maximal value is

$$\frac{1}{N^2}\sum_{i,j=1}^{N} d(z_i, z_j) = \frac{1}{2} - \frac{1}{2N^2}.$$

Hemisphere Stolarsky: simple corollaries

- Fejes-Toth '59: $d = 1$ and conjectured for $d \geq 2$.
- Sperling, '60 (even N)
- Larcher, '61 (odd N)

Hemisphere Stolarsky for general measures

Let μ be a probability measure on $\mathbb{S}^d$. Define the geodesic distance energy integral

$$I_g(\mu) = \int\limits_{\mathbb{S}^d} \int\limits_{\mathbb{S}^d} d(x,y)\, d\mu(x) d\mu(y).$$

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Let $H(x) = C(x,0)$ denote the hemisphere with center at x. Then the following version of the Stolarsky principle holds:

$$\int_{\mathbb{S}^d} \left(\mu\big(H(x)\big) - \frac{1}{2} \right)^2 d\sigma(x) = \frac{1}{2} \cdot \left(\frac{1}{2} - I_g(\mu) \right).$$

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- For any probability measure μ: $I_g(\mu) \leq \frac{1}{2}$.
- $I_g(\mu) = \frac{1}{2}$ (i.e. μ is a maximizer) iff $\mu(H(x)) = \frac{1}{2}$ for σ-a.e. $x \in \mathbb{S}^d$

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Distance energy integrals

Let μ be a Borel probability measure on $\mathbb{S}^d$.
Then

$$I_E(\mu) = \int\limits_{\mathbb{S}^d} \int\limits_{\mathbb{S}^d} \|x - y\| \, d\mu(x) d\mu(y)$$

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However,

$$I_g(\mu) = \int_{\mathbb{S}^d}\int_{\mathbb{S}^d} d(x, y) \, d\mu(x) d\mu(y)$$

is maximized by any symmetric measure μ.

Euclidean distance energy integrals

Let μ be a Borel probability measure on the sphere $\mathbb{S}^d$. For $\lambda > 0$ define the energy integral

$$I_\lambda = \int_{\mathbb{S}^d}\int_{\mathbb{S}^d} |x-y|^\lambda d\mu(x) d\mu(y)$$

Maximizers (Bjorck '56):

- $0 < \lambda < 2$: unique maximizer is surface measure,
- $\lambda = 2$: any measure with center of mass at 0,
- $\lambda > 2$: mass $\frac{1}{2}$ at two opposite poles.

Geodesic distance energy integrals

Let μ be a Borel probability measure on the sphere $\mathbb{S}^d$. For $\lambda > 0$ define the energy integral

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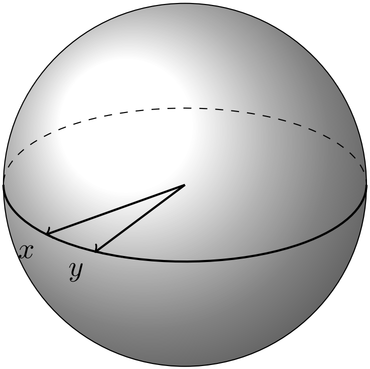
$d = 1$: Brauchart, Hardin, Saff, '12



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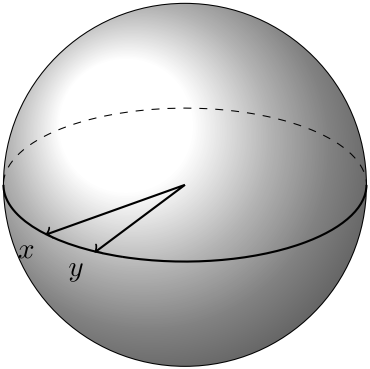
choose a random hyperplane $z^{\perp}$, $z \in \mathbb{S}^d$.



Let x, $y \in \mathbb{S}^d$
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Then

$$\begin{aligned}&\mathbb{P}(z^\perp \text{ separates } x \text{ and } y)\\ &= \mathbb{P}(\operatorname{sign}\langle z, x\rangle \neq \operatorname{sign}\langle z, y\rangle)\\ &= d(x, y),\end{aligned}$$

where d is the normalized geodesic distance on the sphere, i.e. $d(x, y) = \frac{\cos^{-1}\langle x,y\rangle}{\pi}$.

Hamming distance

Consider a set of vectors $Z = \{z_1, z_2, ..., z_N\}$ on the sphere $\mathbb{S}^d$. Define the Hamming distance as

$$d_H(x,y) := \frac{\#\{z_k \in Z : \operatorname{sign}(x \cdot z_k) \neq \operatorname{sign}(y \cdot z_k)\}}{N},$$

i.e. the proportion of hyperplanes $z_k^\perp$ that *separate* x and y. In other words,

$$d_H(x,y) = \frac{1}{2N} \cdot \|\phi_Z(x) - \phi_Z(y)\|_1,$$

where $\phi_Z : \mathbb{S}^d \to \mathcal{H}^N = \{-1,+1\}^N \subset \mathbb{R}^N$ is given by

$$\phi_Z(x) = \{\operatorname{sign}(x \cdot z_k)\}_{k=1}^N = \operatorname{sign}(Zx).$$

Define

$$\Delta_Z(x, y) := d_H(x, y) - d(x, y).$$

The main question

Define

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Let $K \subset \mathbb{S}^d$. We say that Z *induces a δ-uniform tessellation of* K if

$$\sup_{x,y\in K} \left|\Delta_Z(x,y)\right| \leq \delta.$$

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Examples of K:

- $K = \mathbb{S}^d$
- K finite
- sparse vectors

Motivation: almost isometric embeddings

Definition

Let X, Y be metric spaces. A δ-isometric embedding of X into Y (a δ-RIP map) is a map $f : X \to Y$ such that for each $x, y \in X$

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$\phi_Z(x) = \{\text{sign}(x \cdot z_k)\}_{k=1}^{N}\text{sign}(Zx)$ is a δ-RIP map from K into the Hamming cube $\mathcal{H}^N = \{-1, 1\}^N$.

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Question: Given $K \subset \mathbb{S}^d$ and $\delta > 0$, what is the smallest value of N so that K can be δ-isometrically embedded into $\mathcal{H}^N$?

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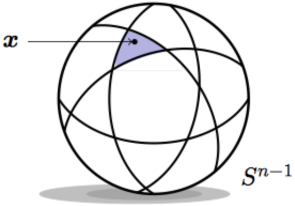
Prior results:

Plan, Vershynin, '13: $N = C\delta^{-6}\omega(K)^2$ random points yield a δ-uniform tessellation of K with high probability.

Motivation: cells with small diameter

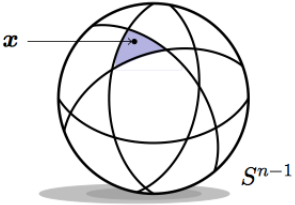


Picture from Baraniuk, Foucart, Needell, Plan, Wooters

Lemma

Every cell of a δ-uniform tessellation of K by hyperplanes has diameter at most δ.

Motivation: cells with small diameter



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Lemma

Every cell of a δ-uniform tessellation of K by hyperplanes has diameter at most δ.

Proof:
if x and y are in the same cell then

$$d(x,y) = |d(x,y) - \underbrace{d_H(x,y)}_{=0}| \leq \delta.$$

Motivation: one-bit compressed sensing

- Let $x \in K \subset \mathbb{S}^{n-1} \subset \mathbb{R}^n$ represent a signal.
- $\langle x, z_k \rangle$ are linear measurements, $k = 1, ..., m$, $m \ll n$.

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- $\text{sign}\langle x, z_k \rangle$ are quantized linear measurements.
- Can one reconstruct/approximate x from these measurements?

Motivation: one-bit compressed sensing

- Let $x \in K \subset \mathbb{S}^{n-1} \subset \mathbb{R}^n$ represent a signal.
- $\langle x, z_k \rangle$ are linear measurements, $k = 1, ..., m$, $m \ll n$.
- $\mathrm{sign}\langle x, z_k \rangle$ are quantized linear measurements.
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- Jaques, Laska, Boufounos, Baraniuk:
  embeddings to Hamming cube through $\phi_Z(x) = \mathrm{sign}(Zx)$.

Mean Gaussian width and "hemisphere" width

- Let γ be the standard Gaussian vector in $\mathbb{R}^{d+1}$. The Gaussian mean width of K is defined as

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- "Hemisphere" process: mean zero Gaussian process with $\mathbb{E} G_x^2 = \frac{1}{4}$ with increments

$$\big(\mathbb{E} |G_x - G_y|^2 \big)^{1/2} = \| \mathbf{1}_{H(x)} - \mathbf{1}_{H(y)} \|_2 = \sqrt{d(x,y)},$$

where $H(x)$ is the hemisphere $H(x) = \{ z \in \mathbb{S}^d : z \cdot x > 0 \}$. Hemisphere mean width

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- Sudakov's inequality:

$$\sqrt{\log N(K, \delta)} \lesssim \begin{cases} \delta^{-1} \omega(K) \\ \delta^{-1/2} H(K) \end{cases}$$

Main results (DB, Lacey, '15-16)

- **Small cells:** If $m \gtrsim \delta^{-1} \log N(K, c\delta)$, then w.h.p. m random vectors induce a tessellation with δ-small cells.

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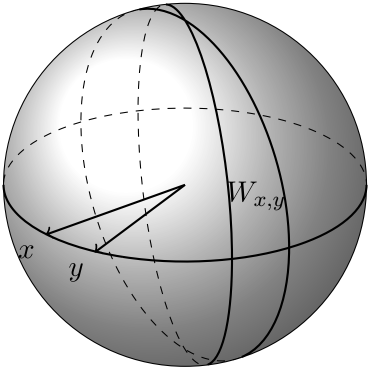
- **One-bit Johnson-Lindenstrauss lemma:** If K is finite and $m \gtrsim \delta^{-2} \log(\# K)$, then there exists a δ-isometry between $K \subset \mathbb{S}^d$ and the Hamming cube $\mathcal{H}^m$.



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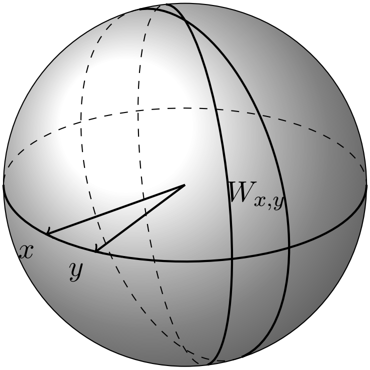

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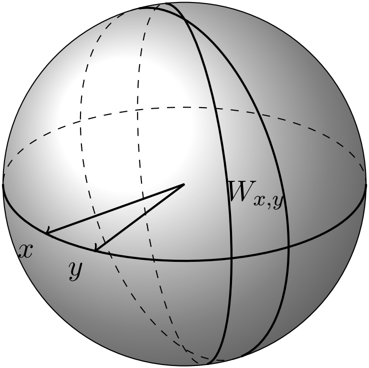
Tessellations and discrepancy



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$$\Delta_Z(x, y) = d_H(x, y) - d(x, y) = \frac{\#(Z \cap W_{xy})}{N} - \sigma(W_{xy})$$

$$D_{\text{wedge}}(Z) = \big\|\Delta_Z(x, y)\big\|_\infty = \sup_{x,y\in\mathbb{S}^d} \left| \frac{\#(Z \cap W_{xy})}{N} - \sigma(W_{xy}) \right|.$$

Lemma

There exists an N-point set $Z \subset \mathbb{S}^d$ with

$$D_{\text{wedge}}(Z) \leq C_d N^{-\frac{1}{2}-\frac{1}{2d}} \sqrt{\log N}.$$

Tessellation/"Wedge" discrepancy

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Corollary

This implies that for $\delta > 0$ there exists a δ-uniform tessellation of $\mathbb{S}^d$ by N hyperplanes with

$$N \le C'_d \delta^{-2+\frac{2}{d+1}} \cdot \left(\log \frac{1}{\delta}\right)^{\frac{d}{d+1}}.$$

Tessellation/"Wedge" discrepancy

Lemma (Blümlinger, 1991)

For any N-point set $Z \subset \mathbb{S}^d$

$$D_{slice}(Z) \gtrsim N^{-\frac{1}{2}-\frac{1}{2d}},$$

where D_{slice} is the spherical discrepancy with respect to "slices" $S_{xy} = \{z \in \mathbb{S}^d : \langle z, x\rangle > 0 \ \& \ \langle z, y\rangle > 0\}$.

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This implies that for any $\delta > 0$, if there exists a δ-uniform tessellation of $\mathbb{S}^d$ by N hyperplanes, then

$$N \geq c_d \delta^{-2+\frac{2}{d+1}}.$$

Summary

- There exist constants c_d, C_d, such that the following discrepancy bounds hold:

$$c_d N^{-\frac{1}{2}-\frac{1}{2d}} \le \inf_{Z\subset \mathbb{S}^d:\, \#Z=N} \Delta(Z) \le C_d N^{-\frac{1}{2}-\frac{1}{2d}} \sqrt{\log N}.$$

Inverting this we find that the optimal value of N satisfies

$$\delta^{-2-\frac{2}{d+1}} \lesssim N \lesssim \delta^{-2-\frac{2}{d+1}} \left(\log \frac{1}{\delta} \right)^{\frac{d}{d+1}}.$$

Stolarsky principle for wedge discrepancy

Define the L^2 discrepancy for wedges

$$[D_{L^2,\text{wedge}}(Z)]^2 = \int_{\mathbb{S}^d}\int_{\mathbb{S}^d} \left(\frac{1}{N} \sum_{k=1}^{N} \mathbf{1}_{W_{xy}}(z_k) - \sigma(W_{xy}) \right)^2 d\sigma(x)\, d\sigma(y)$$

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Theorem (Stolarsky for wedges, DB, Lacey, '15)

For any finite set $Z = \{z_1, \ldots, z_N\} \subset \mathbb{S}^d$

$$[D_{L^2,\text{wedge}}(Z)]^2 = \frac{1}{N^2} \sum_{i,j=1}^{N} \left(\frac{1}{2} - d(z_i, z_j) \right)^2 - \int_{\mathbb{S}^d}\int_{\mathbb{S}^d} \left(\frac{1}{2} - d(x,y) \right)^2 d\sigma(x)\, d\sigma(y).$$

Frame potential

- $Z = \{z_1, \dots, z_N\} \subset \mathbb{S}^d$ is a frame in $\mathbb{R}^d$ iff there exist $c, C > 0$ such that for any $x \in \mathbb{R}^{d+1}$
$$c\|x\|^2 \leq \sum_k |\langle x, z_k \rangle|^2 \leq C\|x\|^2.$$

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Theorem (Benedetto, Fickus)

A set $Z = \{z_1, \dots, z_N\} \subset \mathbb{S}^d$ is a tight frame in $\mathbb{R}^{d+1}$ if and only if Z is a local minimizer of the frame potential:

$$F(Z) = \sum_{i,j=1}^{N} |\langle z_i, z_j\rangle|^2.$$

Stolarsky principle for slices

Define the L^2 discrepancy for slices
$S_{xy} = \{z \in \mathbb{S}^d : \langle z, x\rangle > 0 \ \& \ \langle z, y\rangle > 0\}$

$$[D_{L^2,\text{slice}}(Z)]^2 = \int\limits_{\mathbb{S}^d}\int\limits_{\mathbb{S}^d} \left(\frac{1}{N}\sum_{k=1}^{N} \mathbf{1}_{S_{xy}}(z_k) - \sigma(S_{xy}) \right)^2 d\sigma(x)\, d\sigma(y)$$

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Theorem (Stolarsky for slices, DB, '16)

For any finite set $Z = \{z_1, \dots, z_N\} \subset \mathbb{S}^d$

$$4[D_{L^2,\text{slice}}(Z)]^2 = \frac{1}{N^2}\sum_{i,j=1}^{N} \big(1 - d(z_i, z_j)\big)^2 - \int_{\mathbb{S}^d}\int_{\mathbb{S}^d} \big(1 - d(x,y)\big)^2 d\sigma(x)\, d\sigma(y).$$