Normal Matrix Model and Laplacian Growth

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**Recent Advances on Log-Gases
Paris, IHP, 21 March, 2014**

Log gases

Log-gases

- **Interacting particle system $x_1, \ldots, x_n$ with energy**

$$E(x_1, \ldots, x_n) = -\frac{1}{n^2} \sum_{i \neq j} \log |x_i - x_j| + \frac{1}{n} \sum_{j=1}^{n} V(x_j)$$

Log-gases have applications in

- **Random matrix theory**
- **Orthogonal polynomials and approximation theory**
- **Equidistribution of points**

Log-gases

Gibbs measure

$$\frac{1}{Z_n} e^{-\frac{\beta}{2} n^2 E(x_1, x_2, \ldots, x_n)} \frac{1}{Z_n} \prod_{i<j} |x_i - x_j|^\beta \prod_{j=1}^{n} e^{-\frac{\beta}{2} n V(x_j)}$$

- β **ensembles in random matrix theory**
- **For** $\beta = 1, 2, 4$ **these are eigenvalue distributions of random matrices from invariant ensembles**

$$e^{-\frac{\beta}{2} n \operatorname{Tr} V(M)} \, dM$$

$$\begin{cases} \beta = 1: & \textbf{real symmetric matrices} \\ \beta = 2: & \textbf{Hermitian matrices} \\ \beta = 4: & \textbf{quaternionic self-dual matrices} \end{cases}$$

Large n limit

- **Log-energy**

$$E(x_1, \ldots, x_n) = -\frac{1}{n^2} \sum_{i \neq j} \log |x_i - x_j| + \frac{1}{n} \sum_{j=1}^{n} V(x_j)$$

- **Continuum limit = log. energy in external field**

$$E(\mu) = \iint \log \frac{1}{|x-y|} d\mu(x) d\mu(y) + \int V(x) d\mu(x)$$

Equilibrium measure in external field

Assume V is continuous and $\dfrac{V(x)}{\log|x|} \to +\infty$ **as** $|x| \to \infty$

Theorem (Frostman)

There is a unique probability measure μ_V with

$$E(\mu_V) = \min_{\mu} E(\mu)$$

The measure is compactly supported and for some ℓ_V

$$2\int \log\frac{1}{|x-y|}d\mu_V(y) + V(x) \begin{cases} = \ell_V & \text{on support of } \mu_V \\ \geq \ell_V & \text{elsewhere} \end{cases}$$

These conditions characterize μ_V.

μ_V **is equilibrium measure in external field**

Large deviation principle and a.s. convergence

Emperical measure for points $x_1, \ldots, x_n$

$$\frac{1}{n}\sum_{j=1}^{n}\delta_{x_j}$$

Theorem (Ben Arous–Guionnet)

Empirical measures satisfy a large deviation principle with speed n^2 and good rate function

$$E(\mu) - E(\mu_V)$$

The empirical measures converge weakly to μ_V almost surely

$$\frac{1}{n}\sum_{j=1}^{n}\delta_{x_j} \xrightarrow{*} \mu_V \qquad \text{a.s.}$$

Log gases in 2D

Ginibre random matrix

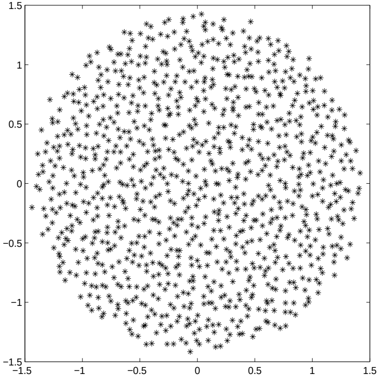
- $n \times n$ **matrix with independent complex Gaussian entries**
- **Joint p.d.f. for eigenvalues**

$$\frac{1}{Z_n} \prod_{i<j} |z_i - z_j|^2 \prod_{j=1}^{n} e^{-|z_j|^2}, \qquad z_j \in \mathbb{C}$$

Circular law

- **Eigenvalues in the Ginibre ensemble (after scaling by $\sqrt{n}$) have a limiting distribution as $n \to \infty$ that is uniform in a disk.**

Ginibre (1965)



Products of random matrices

Products of Ginibre matrices

$$M = G_k \cdots G_1$$

Theorem (Akemann-Burda (2012))

Eigenvalues of M have joint p.d.f.

$$\frac{1}{Z_n} \prod_{i<j} |z_i - z_j|^2 \prod_{j=1}^{n} w(|z_j|), \qquad z_j \in \mathbb{C}$$

where w is a Meijer G-function

$$w(r) = G^{k,0}_{0,k}\left(\begin{matrix} - \\ 0, \dots, 0 \end{matrix} \,\middle|\, r^2 \right)$$

$$= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(s)^k \, r^{-2s} ds, \qquad c > 0$$

Normal matrix model

Normal matrix model

- **Probability measure on $n \times n$ complex matrices**

$$\frac{1}{Z_n} e^{-\frac{n}{t_0} \mathrm{Tr}(MM^* - V(M) - \overline{V}(M^*))} dM, \qquad t_0 > 0,$$

where

$$V(M) = \sum_{k=1}^{\infty} \frac{t_k}{k} M^k.$$

- **Model depends on parameters**

$$t_0 > 0, \qquad t_1, t_2, \ldots$$

- **For $t_1 = t_2 = \cdots = 0$ this is the Ginibre ensemble.**

Eigenvalues

- **Eigenvalues of M have joint p.d.f.**

$$\frac{1}{Z_n}\prod_{j<k}|z_j - z_k|^2 \prod_{j=1}^{n} e^{-\frac{n}{t_0}\mathcal{V}(z_j)} \qquad \mathcal{V}(z) = |z|^2 - 2\,\mathrm{Re}\,V(z)$$

- **Logarithmic energy in external field**

$$\iint \log\frac{1}{|z-w|}d\mu(z)d\mu(w) + \frac{1}{t_0}\int\left(|z|^2 - 2\,\mathrm{Re}\,V(z)\right)d\mu(z)$$

- **Minimizer is**

$$d\mu_{\mathcal{V}}(z) = \frac{1}{\pi t_0} 1_{z\in\Omega}\, dA(z)$$

2D Lebesgue measure restricted to domain $\Omega = \Omega(t_0)$

with

$$\textbf{area}(\Omega) = \pi t_0$$

Laplacian growth

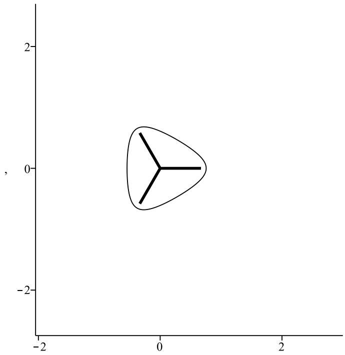
- Ω **is characterized by area**$(\Omega) = \pi t_0$ **and**

$$t_k = -\frac{1}{\pi} \iint_{\mathbb{C}\setminus\Omega} \frac{dA(z)}{z^k}, \qquad k \geq 1,$$

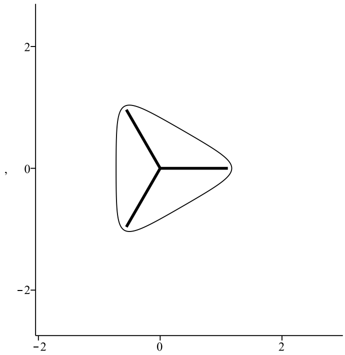
- **As a function of** t_0**, the boundary of** Ω **evolves according to the model of Laplacian growth**

Wiegmann-Zabrodin (2000)
Teoderescu-Bettelheim-Agam-Zabrodin-Wiegmann (2005)

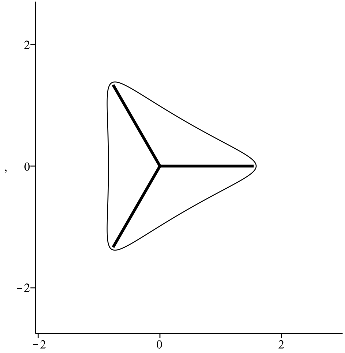
Cubic case



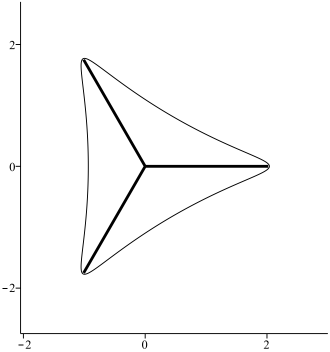
Cubic case



Cubic case



Cubic case



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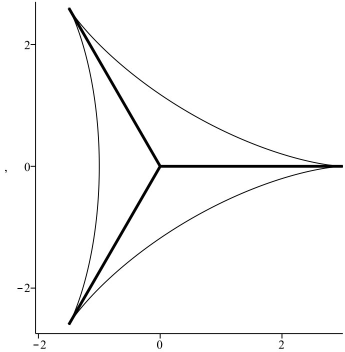

Orthogonal polynomials

- **Average characteristic polynomial**

$$P_n(z) = \mathbb{E}\left[zI_n - M\right]$$

is an orthogonal polynomial for scalar product

$$\langle f, g\rangle = \iint_{\mathbb{C}} f(z)\overline{g(z)}e^{-\frac{n}{t_0}(|z|^2 - V(z) - \overline{V(z)})} dA(z)$$

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- **The zeros of P_n do not fill out the domain Ω, but accumulate along a contour Σ.**
- **In the cubic case** $V(z) = \frac{1}{3}z^3$ **the contour is a three-star**

$$\Sigma = [0, z_1] \cup [0, \omega z_1] \cup [0, \omega^2 z_1], \qquad \omega = e^{2\pi i/3}.$$

Mathematical problem

Mathematical problem

- **Normal matrix model**

$$\frac{1}{Z_n} e^{-\frac{n}{t_0} \operatorname{Tr}(MM^* - V(M) - \overline{V}(M^*))} dM, \qquad t_0 > 0,$$

is **not well-defined** **if** V **is a polynomial of degree** ≥ 3

- **The integral defining the scalar product**

$$\iint_{\mathbb{C}} f(z)\overline{g(z)} e^{-\frac{n}{t_0}(|z|^2 - 2\operatorname{Re} V(z))} dA(z)$$

does not converge if f **and** g **are polynomials.**

Mathematical problem

- **Normal matrix model**

$$\frac{1}{Z_n} e^{-\frac{n}{t_0} \operatorname{Tr}(MM^* - V(M) - \overline{V}(M^*))} dM, \qquad t_0 > 0,$$

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- **The integral defining the scalar product**

$$\iint_{\mathbb{C}} f(z)\overline{g(z)} e^{-\frac{n}{t_0}(|z|^2 - 2 \operatorname{Re} V(z))} dA(z)$$

does not converge if f and g are polynomials.

- **No convergence problem for**

$$V(x) = -\log|z-a|$$

Balogh, Bertola, Lee, McLaughlin (arXiv 2012)

Cut-off approach

- **Elbau and Felder (2005) use a cut-off domain. They restrict to matrices with eigenvalues in some bounded domain D.**
- **Then probability measure on eigenvalues is a log-gas on D.**

$$\frac{1}{Z_n}\prod_{j<k}|z_j - z_k|^2 \prod_{j=1}^{n} e^{-\frac{n}{t_0}\mathcal{V}(z_j)}, \qquad z_j \in D$$

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$$\frac{1}{Z_n} \prod_{j<k} |z_j - z_k|^2 \prod_{j=1}^{n} e^{-\frac{n}{t_0}\mathcal{V}(z_j)}, \qquad z_j \in D$$

- **Eigenvalues fill out a domain Ω that evolves according to Laplacian growth if t_0 is small enough.**

Different approach

Different approach

- **Scalar product**

$$\langle f, g\rangle = \iint_{\mathbb{C}} f(z)\overline{g(z)}e^{-\frac{n}{t_0}(|z|^2 - V(z) - \overline{V(z)})} dA(z)$$

satisfies (due to Green's theorem)

$$n\langle zf, g\rangle = t_0\langle f, g'\rangle + n\langle f, V'g\rangle$$

Different approach

- **Scalar product**

$$\langle f, g\rangle = \iint_{\mathbb{C}} f(z)\overline{g(z)}e^{-\frac{n}{t_0}(|z|^2 - V(z) - \overline{V(z)})}dA(z)$$

satisfies (due to Green's theorem)

$$n\langle zf, g\rangle = t_0\langle f, g'\rangle + n\langle f, V'g\rangle$$

- **We look for other scalar product satisfying this structure relation, and also the Hermitian form condition**

$$\langle g, f\rangle = \overline{\langle f, g\rangle}.$$

Double integral representations

Theorem (Bleher-Kuijlaars, Bertola (2003))

If $\deg V = r + 1$ **then the space of Hermitian forms satisfying**

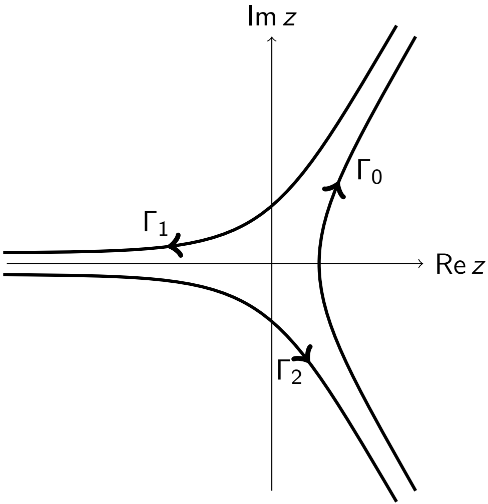
$$n\langle zf, g\rangle = t_0\langle f, g'\rangle + n\langle f, V'g\rangle$$

is r^2 **dimensional.**
Any such Hermitian form is of the form

$$\sum_{j,k=0}^{r} C_{j,k} \int_{\Gamma_j} dz \int_{\overline{\Gamma}_k} dw\, f(z)\overline{g}(w) e^{-\frac{n}{t_0}\left(zw - V(z) - \overline{V}(w)\right)}$$

where $(C_{j,k})_{j,k=0,\ldots,r}$ **is a Hermitian matrix with zero row and column sums, and** $\Gamma_0, \ldots, \Gamma_r$ **are unbounded contours along which the integrals converge.**

Contours Γ_j for cubic potential



- **Contours Γ_0, Γ_1, Γ_2 for $V(z) = \frac{1}{3}z^3$ with $t_3 > 0$.**

Hermitian form

$$\sum_{j,k=0}^{r} C_{j,k} \int_{\Gamma_j} dz \int_{\overline{\Gamma}_k} dw\, f(z)\overline{g}(w) e^{-\frac{n}{t_0}\left(zw - V(z) - \overline{V}(w)\right)}$$

- **Problem: Analyze the OPs for this Hermitian form and prove that**

$*$ **Zeros accumulate on Σ with limiting measure μ^*.**

$*$ **Domain Ω exists such that**

$$\frac{1}{\pi t_0} \iint_{\Omega} \log|z-x| dA(x) = \int_{\Sigma} \log|z-s| d\mu^*(s), \qquad z \in \mathbb{C} \setminus \Omega$$

and $\partial\Omega$ evolves according to Laplacian growth.

Cubic model $V(z) = \frac{1}{3}z^3$

Theorem (Bleher-Kuijlaars)

In cubic model, there is a choice for the Hermitian form, such that for

$$0 < t_0 < t_{0,crit} = \frac{1}{8}$$

the following hold.

(a) **The orthogonal polynomial P_n exists for n large.**

(b) **The zeros of P_n accumulate on**

$$\Sigma = \bigcup_{j=0}^{2} [0, \omega^j z_1], \quad z_1 = \frac{3}{4}\left(1 - \sqrt{1 - 8t_0}\right)^{2/3}.$$

with a limiting density μ^*

Laplacian growth

Theorem (continued)

(c) **The equation**

$$z^2 + t_0 \int \frac{d\mu^*(s)}{z-s} = \bar{z}$$

defines a simple closed curve $\partial\Omega$ that is the boundary of a domain Ω that evolves according to Laplacian growth.

(d) **In addition**

$$\frac{1}{\pi t_0} \iint_\Omega \log|z-x| dA(x) = \int_\Sigma \log|z-s| d\mu^*(s), \qquad z \in \mathbb{C} \setminus \Omega$$

(e) μ^* **is minimizer for a vector equilibrium problem.**

Extension to $V(z) = \frac{z^d}{d}$ **with** $d \geq 3$**,**

Kuijlaars-López Gárcia (arxiv 2014)

Supercritical regime

Schwarz function in cubic model

- **The function**

$$\xi(z) = z^2 + t_0 \int \frac{d\mu^*(s)}{z-s}$$

is the Schwarz function for Ω.

- **It satisfies an algebraic equation (spectral curve)**

$$\xi^3 - z^2\xi^2 - (1+t_0)z\xi + z^3 + A = 0$$

with

$$A = \frac{1}{32}(1 + 20t_0 - 8t_0^2 - (1-8t_0)^{3/2})$$

Boutroux condition

- **What happens for $t_0 > t_{0,crit} = \frac{1}{8}$?**
- **For $t_0 < t_{0,crit}$ the number $A = A(t_0)$ is chosen such that**

$$\xi^3 - z^2\xi^2 - (1 + t_0)z\xi + z^3 + A = 0$$

  defines a Riemann surface of genus zero
- **For $t_0 > t_{0,crit}$ we choose it such that**

$$\oint_\gamma \xi dz \qquad \text{is purely imaginary}$$

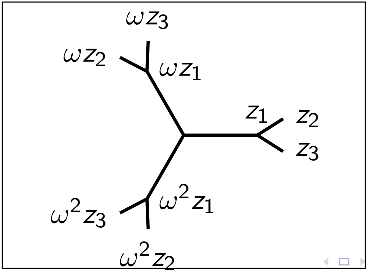
  for all cycles γ on the Riemann surface.
- **This is Boutroux condition.**

Polynomials in supercritical regime

Theorem (Kuijlaars-Tovbis)

For $t_{0,crit} < t_0 < \widehat{t}_{0,crit}$ we can find $A = A(t_0) > 0$ such that the Boutroux condition is satisfied.
The OPs P_n exist for infinitely many n, and their zeros accumulate with limiting measure μ^* on

$$\Sigma = \bigcup_{j=0}^{2} \left([0, \omega^j z_1] \cup [\omega^j z_1, \omega^j z_2] \cup [\omega^j z_1, \omega^j z_3]\right)$$

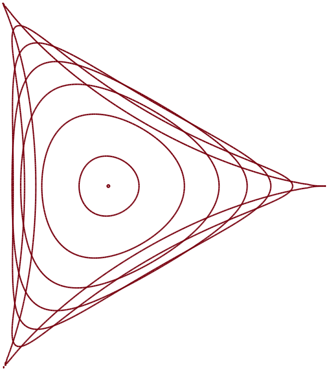


Domain Ω

- There is a domain with boundary

$$\partial\Omega(t_0): \quad \xi(z) = \bar{z}$$

- Domain shrinks as t_0 increases, and completely **disappears** at the second critical value.



Thank you for your attention.