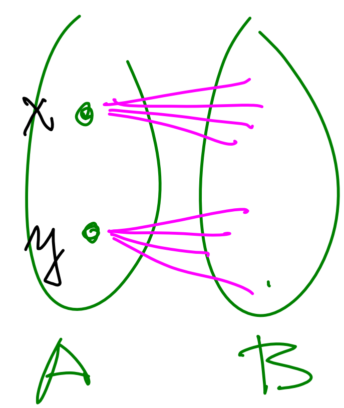
E.g. show (must be true):

$\boxed{r=3}$ a.s. $\nexists$ max cut (A,B) s.t.

$x, y \in A$ with $\underline{d_B(x,y)} = 0$

$p > C n^{-1/2} \sqrt{\log n}$

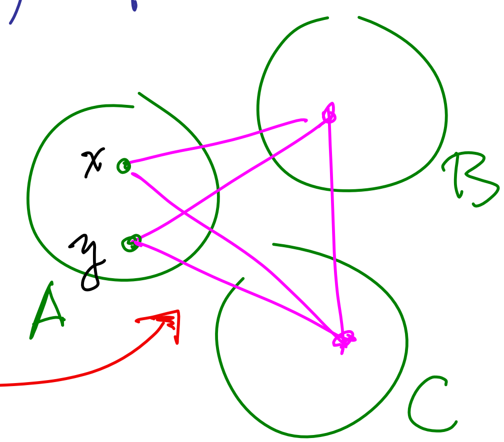
should be $\approx np^2/2$



$\boxed{r=4}$ a.s. $\nexists$ max "cut" (A,B,C) s.t.

$x, y \in A$ with $\underline{K(x,y,B,C)} = 0$

number of these



(should be $\approx n^2 p^5/a$)

[a.s. $\left.\begin{array}{l}\Pi = (A,B) \text{ M.C.} \\ x, y \in A\end{array}\right] \Longrightarrow d_B(x,y) \neq 0$ (*)]

Pf (−1) a.s. $d(v) \approx np \ \forall v$ (→ WMA)

[Fix x, y]

(0) $\left.\begin{array}{l}(A,B) \text{ M.C.} \\ x \in A\end{array}\right] \Longrightarrow d_B(x) \geq d(x)/2$

$(\approx np^2/2)$

⟶ ETS (*) for

$\Pi \in \mathcal{C} = \{(A,B) : x, y \in A, \ \underline{\underline{d_B(x) \gtrsim np/2}}\}$

(1) <u>Choose G:</u>

(a) $N(x)$

(b) rest of E <u>except</u> $\nabla(y, N(x))$

(c) $\nabla(y, N(x))$

$\left.\begin{array}{l}\text{(a), (b)}\end{array}\right\} G'$

$\Pi^* = (S, T)$:

max* among cuts <u>of G'</u> in $\mathcal{C}$

$x, y \in S.$

$d_T(x) \gtrsim np/2$

* <u>first</u> max ...

(a) $N(x)$

(b) rest <u>except</u> $\nabla(y, N(x))$

(c) $\nabla(y, N(x))$

G'

$x, y \in S$

$d_T(x) \gtrsim \frac{np}{2}$

$\Pi^* = (S, T)$: max <u>for G'</u> among cuts in G

$p > C\sqrt{\log n / n}$ $\Longrightarrow$ <u>in G</u>:

p <u>w.v.h.p.</u> $\underbrace{\left|N(y) \cap N_T(x)\right|}_{d_T(x,y)} \gtrsim np^2/2$

WMA.

In G': $\Pi^* = (S, T)$: max in $\boxed{\mathcal{C}}$

in G: $d_T(x,y) \gtrsim np^2/2$

In G: $\Pi = (A, B) \in \mathcal{C}$:

$$|\Pi_G| = |\Pi_{G'}| + d_B(x,y)$$

$$\wedge|$$

$$|\Pi^*_G| = |\Pi^*_{G'}| + \boxed{d_T(x,y)} \gtrsim np^2/2$$

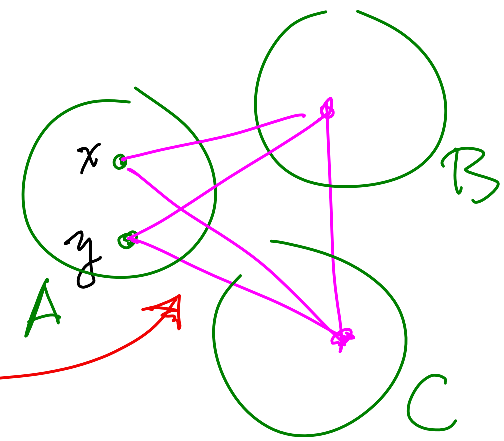
Π M.C. $\longrightarrow$ $d_B(x,y) \geq d_T(x,y)$ $(\gtrsim np^2/2)$

(more than we asked)

($r=4$) a.s. $\nexists$ (A,B,C) MC & $x,y \in A$

$\overline{\omega}$ $\underline{K(x,y,B,C)} = 0$

number of these

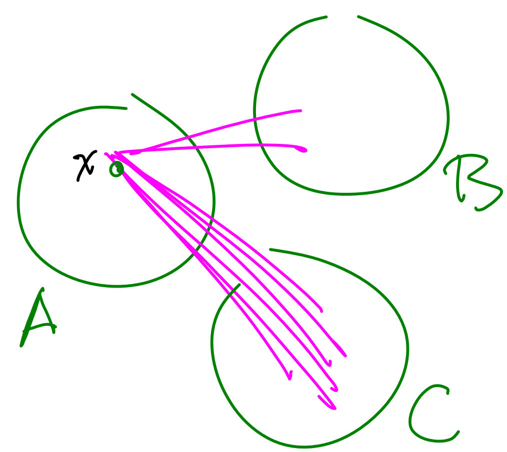


uhh ... how about:

a.s. $\nexists$ (A,B,C) MC & $x \in A$ $\overline{\omega}$

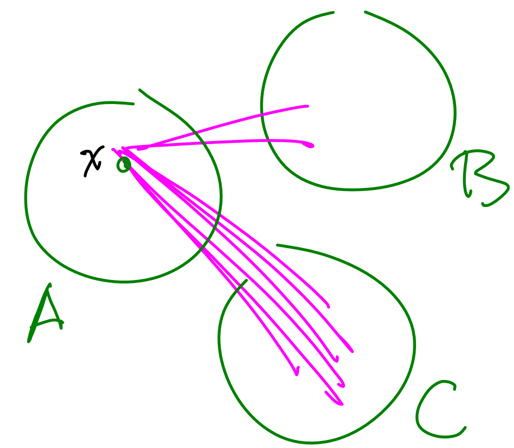
$$\circledast \quad d_B(x) = \begin{cases} 0 \\ \text{small} \end{cases}$$

??



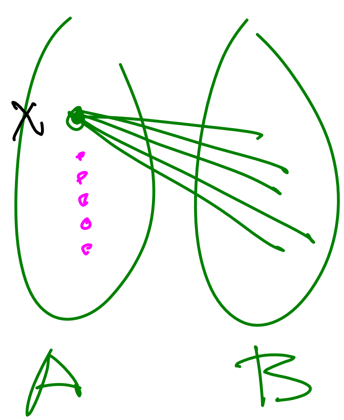
a.s. $\nexists$ (A, B, C) MC & $x \in A$ $\overline{\omega}$

$(*)$ $d_B(x) = \begin{cases} 0 \\ \text{small} \end{cases}$



??

BTW <u>don't</u> know $(r=3)$:

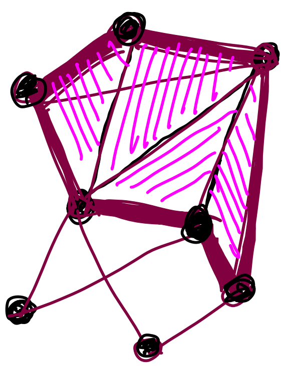


a.s. $\nexists$ MC (A, B) &, $x \in A$

$\overline{\omega}$ $d_A(x) = 0$

3rd "group" (briefly): a topological q.

$\mathcal{F} = \{$ every cycle $=$ mod 2 sum of Δ's $\}$



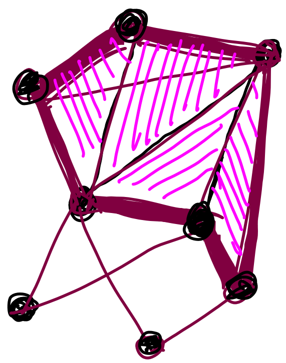
a.k.a. $H_1(\Delta(G), \mathbb{Z}_2) = 0$

$\{$cliques of $G\}$

D-Hamm-K (Conj of M. Kahle):

$$p > (1+\varepsilon)\sqrt{\frac{3}{2}\frac{\ln n}{n}} \Longrightarrow G \models \mathcal{F} \text{ a.s.}$$

$\mathcal{F} = \{$ every cycle $=$ mod 2 sum of Δ's $\}$



thm $p > (1+\varepsilon)\sqrt{\frac{3}{2}\frac{\ln n}{n}}$

$\Longrightarrow G \models \mathcal{F}$ a.s.

real <u>statement</u>:

$Q := \{$every edge in a $\Delta\}$

$\forall p \quad \Pr(G_{n,p} \models Q \wedge \overline{\mathcal{F}}) = o(1)$

Conj (M. Kahle) $p > (1+\varepsilon)\, p_c(k,n)^*$

$\Longrightarrow$ w.h.p. $H_k(\Delta(G), R) = 0$

anything

* $p_c(k,n) \longleftrightarrow$ every K_{k+1} in a K_{k+2}

Kahle: true if $R = \mathbb{Q}$

(DHK: $k=1$, $R = \mathbb{Z}_2$)

all other cases open

Back to

general increasing $\mathcal{F} \subseteq 2^X$

and "GAP"

(what drives p_c ?)

(pre-) <u>Conj</u> (K-Kalai 07)

$\underline{\underline{\forall \mathcal{F}}}$ $\exists$ trivial l.b. $q(\mathcal{F})$ on $p_c(\mathcal{F})$ w/

$$p_c(\mathcal{F}) < K\, q(\mathcal{F}) \cdot \log|X|$$

$\boxed{\mathcal{F} \subseteq 2^X \text{ incr.}}$

const (e.g. $K = 1$?)

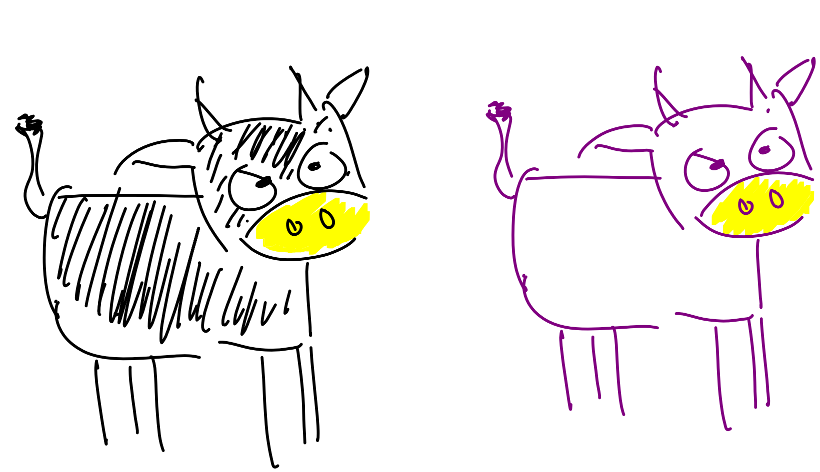
e.g. $G_{n,p}$, $\mathcal{F} = \mathcal{F}_H$ (H = something)

conj $\longrightarrow$ if $\mathbb{E}_p[\# F\text{'s}] > 1 \quad \forall F \subseteq H$

then $p_c(\mathcal{F}_H) < K p \log n$

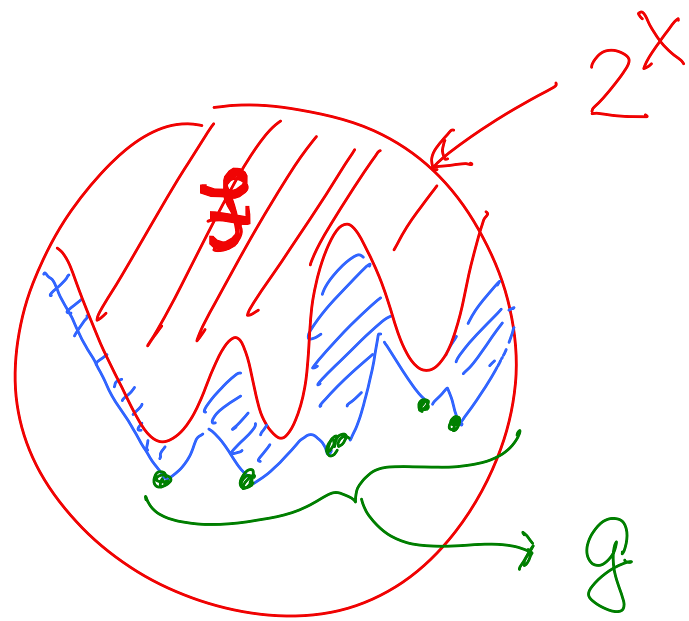
— and if $\boxed{\mathcal{F} \neq \mathcal{F}_H \,?}$

— <u>Conj</u>: <u>also</u>



<u>trivial proof</u> that $p_c(\mathcal{F}) > p$:

$\exists\, \mathcal{G} \subseteq 2^X \ni$



① $\boxed{\forall B \in \mathcal{F}\ \ \exists A \in \mathcal{G},\ A \subseteq B}$

$\mathcal{F} \subseteq \langle \mathcal{G} \rangle := \{ B \subseteq X : \exists A \in \mathcal{G},\ A \subseteq B \}$

② $\sum_{A \in \mathcal{G}} p^{|A|} < \frac{1}{2}$ unimp.

"triv." pf (*) of $p_c(\mathcal{F}) > p$:

$\exists\, \mathcal{G} \subseteq 2^X \ni$ (1) $\langle \mathcal{G} \rangle \supseteq \mathcal{F}$

(2) $\sum_{A \in \mathcal{G}} p^{|A|} < \frac{1}{2}$

(*) $\boxed{\Upsilon \sim \mu_p} \Longrightarrow$

$$\frac{1}{2} \overset{(2)}{>} \sum_{A \in \mathcal{G}} \underbrace{p^{|A|}}_{\Pr(\Upsilon \supseteq A)} = \mathbb{E}_p \left| \{ A \in \mathcal{G} : A \subseteq \Upsilon \} \right| \overset{(1)}{\geq} \mu_p(\mathcal{F})$$

$:= \sup\{p : \exists g\} \leq p_c(\mathcal{F})$ triv.

"expectation threshold"

$\boxed{\text{\underline{CONJ} (K-Kalai '07) } \exists K \quad \forall \mathcal{F} \quad p_c(\mathcal{F}) < K\, p_{\mathbb{E}}(\mathcal{F}) \log |X|}$

$$\Rightarrow \begin{cases} \text{tree conj (EX)} \\ \text{Shamir (easier EX)} \\ \cdots\cdots \end{cases}$$

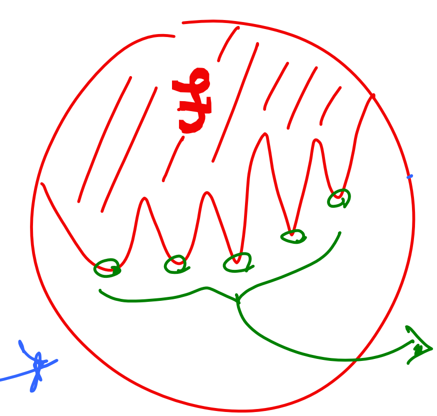
show

$p_{\mathbb{E}} \leq \cdots$

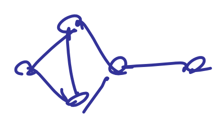
$$p_{\mathbb{E}}(\mathcal{F}) = \sup\left\{p : \exists\, \mathcal{G} \ni \begin{cases} \langle \mathcal{G} \rangle \supseteq \mathcal{F} \\ \sum_{A \in \mathcal{G}} p^{|A|} < \frac{1}{2} \end{cases}\right\}$$

<u>Conj</u> $p_c < K\, p_{\mathbb{E}} \log|X|$

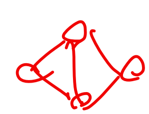
<u>E.g.</u> $\mathcal{G} = \min(\mathcal{F})$?



$\mathcal{G}$?

<u>NO</u>: $H =$ , $\mathcal{F} = \mathcal{F}_H \longrightarrow$

$\mathcal{G} = \min(\mathcal{F}) = \{H\text{'s}\}$ <u>BAD</u>

$\Big(\mathcal{G} = \{$  's$\}$ <u>GOOD</u>$\Big)$

E.g. Ramsey ($\triangle$'s)

(recall $p_c = \Theta(n^{-1/2})$)

SHOW: $p < cn^{-1/2} \Longrightarrow \exists\, g$ [so no gap

we may regard pfs* as:

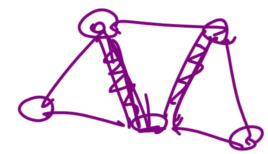
coloring algorithm $\to$ failure finds g

* Łuczak-Ruciński-Voigt,
Rödl-Ruciński, Nenadov-Steger

[coloring algorithm finds g]

repeat until can't: remove edge on $\leq 1\ \Delta$

$\rightarrow$ "core" S must contain one of:

① large "Δ-conn" piece  ...

(unlikely: $\mathbb{E}\,\#$ "children" $\approx 3np^2$)

② H of $\underline{\text{size } O(1)}$ w/ $\underline{e_H > 2v_H}$

$\boxed{\text{unlikely}}$

(nontriv) $\Rightarrow$ $e_H \leq 2v_H \implies H \not\rightarrow (K_3)_2$

$\Rightarrow$

But in general "what's g?"

— or "how do we find g" —

is ??

(more on this soon)

Talagrand

another triv. pf.* of $p_c(\mathcal{F}) > p$:

$\exists\ \alpha : 2^X \to \mathbb{R}^+ \ni$

① $B \in \mathcal{F} \Rightarrow \sum_{S \subseteq B} \alpha_S \geq 1$

$\left(\leftrightarrow \langle \mathcal{G} \rangle \supseteq \mathcal{F} \right)$

② $\sum_S \alpha_S p^{|S|} < \frac{1}{2}$

* EX

① $B \in \mathcal{F} \Rightarrow \sum_{S \subseteq B} \alpha_S \geq 1$

② $\sum_S \alpha_S \, p^{|S|} < \frac{1}{2}$

<u>names</u>: $\mathcal{F}$ is

<u>p-small</u> if <u>$\exists\, \mathcal{G}$</u> . . .

<u>weakly-p-small</u> (wps) if <u>$\exists\, \alpha$</u>

(TRIV: p-small $\Rightarrow$ wps)

@ wps: LP relaxation of p-small:

$\boxed{\text{wps:}}$ $\min \sum \alpha_S \, p^{|S|}$

subject to $\alpha_S \geq 0 \quad \forall S$

$$\sum_{S \subseteq A} \alpha_S \geq 1 \quad \forall A \in \mathcal{F}$$

$\boxed{p\text{-S:}}$ same $\bar{\omega}$ $\underline{\alpha_S \in \{0,1\}}$

$p_{\mathbb{E}}^* := \sup\{p : \exists\, \alpha\}$ $\to$

$$p_{\mathbb{E}} \le p_{\mathbb{E}}^* \le p_c$$

<u>Conj*</u> $\forall \mathfrak{f}$ $\quad p_c(\mathfrak{f}) < K p_{\mathbb{E}}^*(\mathfrak{f}) \log|X|$

of course Conj KK $\Rightarrow$ Conj*,

but <u>maybe equiv</u>:

<u>Conj 2</u> $\exists K \ni$

$$p_{\mathbb{E}}(\mathfrak{f}) > p_{\mathbb{E}}^*(\mathfrak{f})/K \quad \forall \mathfrak{f}$$

Big (?) advantage of Conj*: duality

Def $\nu : 2^X \to \mathbb{R}^+$ is p-spread if

$$\underbrace{\nu(\langle I \rangle)}_{\sum_{A \supseteq I} \nu_A \,=\, \Pr(A \supseteq I) \text{ if } A \sim \nu} \le p^{|I|} \quad \forall I \subseteq X$$

Thm $p^*_{\mathbb{E}} < p \xleftrightarrow{\text{ess.}}$

[$p_E^* < p \overset{\text{ess.}}{\Longleftrightarrow} \exists$ p-sp. prob. meas. on $\mathcal{F}$]

TRIVIAL DIRECTION:

ν p-sp. prob. m. on $\mathcal{F}$
α as in $\boxed{\text{wps}}$ $\Longrightarrow$

$$1 = \sum_{A \in \mathcal{F}} \nu_A \le \sum_{A \in \mathcal{F}} \nu_A \sum_{S \subseteq A} \alpha_S \le \sum_S \alpha_S p^{|S|}$$

$\longrightarrow$ $p_E^* < p$ ☒

▶ $\underline{\text{Conj}^*}$ (again) If $\mathcal{F}$ supports a p-spread prob. meas. then

Remk Usually in threshold problems we try to show $p_c <$ natural guess.

For KK Conj. this is problematic:

want: $\boxed{p_{\mathbb{E}} \le q} \Longrightarrow p_c < K q \log|X|$

[nonexistence of q for $q' > q$]

but don't know how to use hypothesis

$\longrightarrow$ start w p_c & try to show

$p \lesssim p_c / \log|X| \Longrightarrow \exists\, q$ [more below

GTeH <u>Conj*</u>: start w p-spread $\mathcal{V}$ (as above)

<u>triv</u>: WMA $\operatorname{supp}(\nu) \subseteq \mathcal{H} := \min \mathcal{F}$

<u>E.g.</u> <u>if</u> $\mathcal{H}$ <u>symmetric</u> ($\operatorname{Aut}(\mathcal{H})$ trans. on $\mathcal{H}$)

($\Longrightarrow \mathcal{H} \subseteq \binom{X}{m}$ some m)

then (EX) WMA ν = <u>unif. meas.</u> on $\mathcal{H}$

$\xrightarrow[\text{EX}]{\text{e.g.}}$ ① $\nu(\langle x \rangle) = \dfrac{m}{|X|} \quad \forall x \in X$

② $\nu(\langle A \rangle) = \nu_A = \dfrac{1}{|\mathcal{H}|} \quad \forall A \in \mathcal{H}$

$\Longrightarrow |\mathcal{H}| \geq p^{-m}$

E.g. <u>Shamir</u> $\left[X = \binom{[n]}{3} \right.$ $(3 \mid n)$

$\mathcal{H} = \{\text{p.m.'s}\}, \quad m = n/3, \quad |\mathcal{H}| \approx \left(\frac{n^2}{2e^2}\right)^m \left.\right]$

EX for now

ν : unif. meas. on $\mathcal{H}$, $\underline{\underline{\mathbb{A} \sim \nu}}$

$(\to \Pr(\mathbb{A} \supseteq S) = \nu(\langle S \rangle))$

$$\to \begin{cases} \Pr(x \in \mathbb{A}) = \dfrac{m}{|X|} \approx \dfrac{2}{n^2} \\ \Pr(S \subseteq \mathbb{A}) \leq \left(\dfrac{2e^2}{n^2}\right)^{|S|} \quad \forall S \subseteq X \end{cases}$$

EX

$\mathbb{A} \sim \nu =$ unif. meas. on $\mathcal{X}$ $(= \{\text{p.m.'s}\})$

$$\Pr(x \in \mathbb{A}) = \frac{m}{|X|} \approx \frac{2}{n^2}$$

$$\Pr(S \subseteq \mathbb{A}) \leq \left(\frac{2e^2}{n^2}\right)^{|S|}$$

$$\text{“}\approx\text{”} \prod_{x \in S} \Pr(x \in \mathbb{A})$$

$\Rightarrow$ Conj* says this should be all we need for $p_c(\mathcal{F}) < K \frac{\log n}{n^2}$ (as in JKV)

[still Talagrand, aiming for something stronger]

recall (?): $\mu, \lambda \in \mathcal{M}(2^X)$ $:= \{$prob. meas's on $2^X\}$

μ <u>stochastically dominates</u> λ $(\mu \succcurlyeq \lambda)$

if $\mu(f) \geq \lambda(f)$ $\forall$ incr. f

[$\mu \succcurlyeq \lambda$: $\mu(\mathcal{F}) \geq \lambda(\mathcal{F})$ $\forall$ incr. $\mathcal{F}$]

ν p-spread $\equiv$ $\left[\mu_p(\langle S \rangle) \geq \nu(\langle S \rangle) \;\; \forall S\right]$

maybe $\mu_p \succcurlyeq \nu$?

– NO (stupid):

<u>e.g.</u> $\nu(\{\phi\}) = 0 < \mu_p(\{\phi\})$

<u>e.g.</u> $\nu(\mathcal{F}) = 1 < \mu_p(\mathcal{F})$

ν <u>supp'd</u> on $\mathcal{F}$

<u>BUT</u>:

Def $\nu \in \mathcal{M}(2^X)$, $\alpha \in [0,1]$ $\longrightarrow$

$\lambda_\alpha(\nu) \in \mathcal{M}(2^X)$: law of r.v. Z :

$$Y \sim \nu, \quad Z = Y_\alpha$$

recall...

Conj A $\exists\, \alpha > 0 \ni$

ν p-spread $\longrightarrow$ $\lambda_\alpha(\nu) \preccurlyeq \mu_p$

- would have important consequences

- Conj A $\overset{\text{EX}}{\subseteq}$ Conj B :

<u>Conj B</u> $\exists\, L \ni$

ν q-spread & $\alpha = 1-(1-p)^{\frac{1}{Lq}}$

$\Longrightarrow \lambda_\alpha(\nu) \preceq \mu_p$

(1) ν q-spread on $\mathcal{F}$,

$p = Lq \ln|X|$, α as above $\xrightarrow{\text{Conj B}}$

$\mu_p(\mathcal{F}) \geq \lambda_\alpha(\nu)(\mathcal{F}) \geq \alpha^{|X|} \gtrsim 1/2$

$\longrightarrow$

better:

$$\|\mathcal{F}\| := \max\{|A| : A \in \min \mathcal{F}\}$$

above arg. gives (assuming Conj. B)

$$p_c(\mathcal{F}) \lesssim L_q \ln \|\mathcal{F}\| \longrightarrow$$

Conj $\forall \mathcal{F}$ $\quad p_c(\mathcal{F}) < K\, p_{\mathbb{E}} \log \|\mathcal{F}\|$

[Remk: true if $\|\mathcal{F}\| = O(1)$

(pf similar to Lyons)]

back to Conj 2 $\left(p_{\mathbb{E}} > p^*_{\mathbb{E}}/K\right)$

[just an example for flavor]

G graph on X, $|G| = m$ (p = something)

Note $\mathbb{E}\,|G[X_p]| = mp^2$

$$\mathcal{F} := \{ W \subseteq X : |G[W]| \geq \underset{\text{(say)}}{200}\, m p^2 \}$$

$\Big[$ G graph on X, $|G| = m$ $\quad \mathbb{E}\,|G[X_p]| = mp^2$

$\mathcal{F} = \{ W \subseteq X : |G[W]| \geqslant 200\, m p^2 \}$ $\Big]$

<u>OBS</u>: $p^*_{\mathbb{E}}(\mathcal{F}) \geq 10\, p$:

$$\alpha_S = \frac{1}{200 m p^2} \qquad \forall S \in G \quad \left(\subseteq \binom{X}{2}\right)$$

$\longrightarrow$ ① $\quad W \in \mathcal{F} \Rightarrow \displaystyle\sum_{S \subseteq W} \alpha_S \geq 1$

② $\quad \sum \alpha_S (10p)^{|S|} = 1/2$

i.e. $\longrightarrow$

$$\left[\ \mathcal{F} = \{ W \subseteq X : |G[W]| \geq 200\, m p^2 \}\ \right]$$

<u>SHOW:</u> $\exists\, \mathcal{G} \subseteq 2^X \ni$

① $\langle \mathcal{G} \rangle \supseteq \mathcal{F}$

② $\displaystyle\sum_{A \in \mathcal{G}} p^{|A|} < 1/2$

<u>E.g.</u> $|W| = 10\, n p \to W \in \mathcal{F}$ <u>typical</u>

$n \to |X|$

but $\mathcal{G} \supseteq \{\text{such } W\text{'s}\}$ is <u>okay</u>:

contrib (to ②) $\approx \binom{n}{10np} p^{10np} < \left(\frac{enp}{10np}\right)^{10np}$ = tiny

$\left[\ \mathcal{F} = \{ W \subseteq X : |G[W]| \geq 200\, m p^2 \}\ \right]$

more interesting: <u>atypically dense</u> W's

<u>E.g.</u> G d-reg, $W \in \mathcal{F}$, $|W| < 10np$

$\xRightarrow{\text{EX}}$ $\exists\, x \in W : |N_x \cap W| > 10dp$

$\uparrow$ nbhd

$\longrightarrow$ add to $\mathcal{G}$:

all sets $\{x\} \cup Y$ w/ $Y \in \binom{N_x}{10dp}$

$\longrightarrow$ contrib: $n\binom{d}{10dp} p^{10dp+1} < n(e/10)^{10dp}$

— fine if $dp \gtrsim \log n$ ($\nexists$ if not? ...)

[back to KK soon]

back to: how sharp is the threshold?

($\longleftrightarrow$ [illegible])

defs:

$p_\alpha(f)$: p $\overline{w}$ $\mu_p(f) = \alpha$

e.g. $p_c = p_{1/2}$

"threshold interval": $\Theta(f) = [p_{.1}, p_{.9}]$

$$\delta = \delta_f = |\Theta(f)| = p_{.9} - p_{.1}$$

$\left[\ \delta = \delta_{\mathcal{F}} = p_{.9} - p_{.1}\ \right]$

th. is $\begin{cases} \underline{\text{coarse}} & \text{if } \delta / p_c^{*} = \Omega(1) \\ \underline{\text{sharp}} & \text{if } \delta / p_c^{*} = o(1) \end{cases}$

$*$ really $\delta(\mathcal{F}_n) / p_c(\mathcal{F}_n)$

▷ depends (seriously) on .1 ?

– YES, but we ignore

For now just <u>gph properties</u> ($\mathcal{F}_i$ $G_{n,p}$)

E.g. $\mathcal{F} = \mathcal{F}_{\triangle}$: $p_c, \delta = \Theta(1/n)$

(coarse)

more precise: fix λ, $p = \lambda/n$ $\longrightarrow$

$\mathbb{E}_p[\#\triangle\text{'s}] \sim (np)^3/6 = \lambda^3/6$

$\xrightarrow{\approx \text{Poisson}}$ $\mu_p(\mathcal{F}) \approx 1 - \exp[-\lambda^3/6]$

__E.g__ $\mathcal{F} = \{\text{no iso's}\}$ (or $\{\text{conn}\}$, $\{\exists \text{ p.m.}\}$ ...)

Thm $p = \dfrac{\ln n + c}{n} \longrightarrow$

$$\mu_p(\mathcal{F}) \longrightarrow e^{-e^{-c}}$$

[Ex: nat.

$$\longrightarrow \begin{cases} p_c \sim \dfrac{\ln n}{n} \\ \delta = O(1/n) \end{cases}$$

__(sharp)__

Easy: if $\mathcal{G}$ is a fixed set of gphs, then $\mathcal{F}_{\mathcal{G}} := \{\text{contain some } H \in \mathcal{G}\}$ has a coarse threshold.

— is this all?

— wrong question; e.g.:

$\mathcal{F} = \{\underbrace{\text{contain } \Delta}_{A} \text{ or } \underbrace{\text{size} > n \log n}_{B}\}$:

$\to$ coarse th. (B irrelevant), but ...

usual <u>rough statement</u> of Friedgut:

If $\mathcal{F}$ has a coarse th. then

[illegible] ⊛

<u>what's p?</u> <u>not</u> nec. true at p_c $\longrightarrow$

<u>obs:</u> coarse th $\Rightarrow$ $\exists\, p^* \in \Theta(\mathcal{F})$ $\ni$

$$\frac{d}{dp}\mu_p(\mathcal{F})\Big|_{p=p^*} < C/p^*$$

<u>Thm</u> ⊛ holds for $p = p^*$

How it's used:

Thm $\forall\, \varepsilon, \alpha, C \;\; \exists\, B \ni$

$$p^* \frac{d}{dp} \mu_p(\mathcal{F})\Big|_{p^*} < C \;\; \& \;\; \mu_{p^*}(\mathcal{F}) \in (\alpha, 1-\alpha)$$

$\Longrightarrow \;\; \exists$ (nice) $H \ni$

① $|H| < B$

② $\mu_{p^*}(\mathcal{F}_H) \in (1/B,\, 1 - 1/B)$

③ $G = G_{n,p^*}$, $\mathbb{H}$ random copy of H

$$\Longrightarrow \;\; \Pr(G \cup \mathbb{H} \in \mathcal{F}) > 1 - \varepsilon$$

$$\left[\ldots \exists H \ldots \mu_{p^*}(\mathcal{F}_H) \in (1/B, B) \right]$$

<u>**Quick Cor**</u> If $\mathcal{F}$ has a coarse th.

then $p_c(\mathcal{F}) = \Theta(n^{-\gamma})$, some $\underline{\gamma \in \mathbb{Q}}$

$\xrightarrow{\text{(e.g.)}}$

$p_c(\mathcal{F}) = \Theta\left(\frac{\log n}{n}\right) \Rightarrow$ th. is sharp

[nongraphic:]

original (trickier) appl:

here usually work w̄ $M = \#$ of clauses:

(almost*:) $\forall k \ \exists c^* (>0) \ni \forall \varepsilon > 0$

$$M < (c^* - \varepsilon) n \Longrightarrow \text{sat. a.s.}$$

$$M > (c^* + \varepsilon) n \Longrightarrow \underline{\underline{\text{un}}}\text{sat. a.s.}$$

*actually: c^*

OPEN: $\lim c_n^*$ exists

Pf of Friedgut (tiny hint):

Fourier transf:

$f = \mathbb{1}_{\mathcal{F}}, \quad q = 1 - p \quad (p = p^*)$

$$\hat{f}(S) = \sum_{A \in \mathcal{F}} \mu_p(A) \left(-\sqrt{\frac{p}{q}}\right)^{|S \cap A|} \sqrt{\frac{q}{p}}^{\,|S \setminus A|}$$

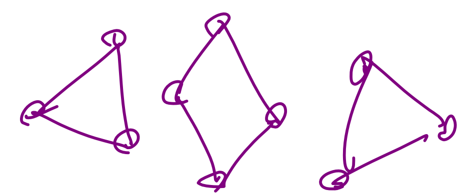
$S \subseteq X$

(just writing, we won't use)

very rough: F.t. finds $\mathcal{G}$:

$\|f\|_2^2$ concentrates on

disjoint U's of members of $\mathcal{G}$.



back to derivative $\left[p^* \cdot \frac{d}{dp} \mu_p(\mathcal{F}) \Big|_{p^*} < C \Rightarrow \ldots \right]$

edge bdry of $\mathcal{F}$:

$$\partial \mathcal{F} = \{ (A, A \setminus e) : A \in \mathcal{F}, A \setminus e \notin \mathcal{F} \ (e \in X) \}$$

OBS (Russo, Margulis):

$$p \cdot \frac{d}{dp} \mu_p(\mathcal{F}) = p \, I_p(\mathcal{F})$$

I_p: "total influence"

$$:= \sum_{A \in \mathcal{F}} \mu_p(A) \left| \{ e : A \setminus e \notin \mathcal{F} \} \right|$$

[meas. of $\partial \mathcal{F}$ if meas. of $(A, A \setminus e)$ is $\mu_p(A)$]

<u>edge-isoperimetric</u> ineq:

$$p\, I_p(\mathcal{F}) \geq \mu_p(\mathcal{F}) \log_p \mu_p(\mathcal{F}) \qquad \text{(I)}$$

- <u>Sharp</u>: $\mathcal{F} = \{\text{contain } [k]\}$

  ($\mu_p(\mathcal{F}) = p^k$; each side of (I) is kp^k)

- $p \in \Theta(\mathcal{F}) \Longrightarrow \mu_p(\mathcal{F}) = \Omega(1)$

  $\Longrightarrow$ r.h.s. of (I) $= \Omega(1)$ $\longrightarrow$

$p = p^*$: $\quad p\, I_p(\mathcal{F}) = p \cdot \frac{d}{dp} \mu_p(\mathcal{F}) < C$

back to KK

$$p\, I_p(\mathcal{F}) \geq \mu_p(\mathcal{F}) \log_p \mu_p(\mathcal{F})$$

isoper

Conj 1 $\forall C \; \exists \varepsilon \ni:$

$$\forall \mathcal{F} \; \exists \; p \in \left[\varepsilon p_c / \log|X| \,,\, p_c\right] \ni$$

$$p\, I_p(\mathcal{F}) < C\, \mu_p(\mathcal{F}) \log \frac{1}{\mu_p(\mathcal{F})}$$

note missing p

Conj 2 $\exists L \ni$

$$p\, I_p(\mathcal{F}) < \mu_p(\mathcal{F}) \log \frac{1}{\mu_p(\mathcal{F})} \implies p_E > p/L$$