Integrated Covolatility Matrix Estimation for High Dimensional Diffusion Processes in the Presence of Microstructure Noise

Xinghua Zheng

Department of ISOM, HKUST

http://ihome.ust.hk/∼xhzheng/

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Based on Joint Work with Ningning Xia

Outline

Background: Integrated Covariance Matrix

- $\mathbf{X}_t = (X_t^{(1)}, \ldots, X_t^{(p)})^T$ denotes a p-dimensional log price process
- Model:

$$d\mathbf{X}_t = \boldsymbol{\mu}_t dt + \boldsymbol{\Theta}_t d\mathbf{W}_t, \quad t \in [0, 1]$$

  where
  1. $\boldsymbol{\mu}_t$ is a p-dimensional drift process;
  2. $\boldsymbol{\Theta}_t$ is a $p \times p$ matrix-valued covolatility process;
  3. $\mathbf{W}_t$ is a p-dimensional standard Brownian motion.
- Both μ_t and Θ_t can be *stochastic*, *discontinuous*, and *dependent on* $\mathbf{W}_t$
- The integrated covariance matrix (ICV):

$$\Sigma^{ICV} := \int_0^1 \Theta_t \Theta_t^T dt.$$

- ICV is the key in risk management and portfolio optimization

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Estimate ICV: Realized Covariance (RCV) Matrix

- Suppose one observes $(\mathbf{X}_t)$ at times $0 = t_0^n < t_1^n < \ldots < t_n^n = 1$
- The realized covariance matrix (RCV):

$$\mathbf{\Sigma}^{RCV} := \sum_{i=1}^{n} \Delta\mathbf{X}_i (\Delta\mathbf{X}_i)^T$$

  where $\Delta\mathbf{X}_i = \mathbf{X}_{t_i^n} - \mathbf{X}_{t_{i-1}^n}$.
- When dimension p is fixed and observation frequency n tends to infinity,

$$\|\mathbf{\Sigma}^{RCV} - \mathbf{\Sigma}^{ICV}\| \xrightarrow{p} 0,$$

  where $\|\cdot\|$ can be any matrix norm.
- In practical applications, p is often comparable with n
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Limiting Behaviour of RCV Matrix

- If $\mu_t \equiv 0, \Theta_t \equiv \Theta$, $t_i^n = i/n$, then $\Sigma^{ICV} = \Theta\Theta^T$, and

$$\Delta \mathbf{X}_i = \int_{(i-1)/n}^{i/n} \Theta d\mathbf{W}_t \stackrel{d}{=} \frac{1}{\sqrt{n}} \mathbf{Y}_i,$$

  where $\mathbf{Y}_i \stackrel{i.i.d.}{\sim} N(0, \Sigma^{ICV})$
- Hence

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Marčenko-Pastur Theorem

- n i.i.d. p-dim observations $\mathbf{Y}_1, \cdots, \mathbf{Y}_n$, with mean 0 and covariance matrix $\mathbf{\Sigma}$
- Sample covariance matrix

$$S = \frac{1}{n}\sum_{i=1}^{n} \mathbf{Y}_i (\mathbf{Y}_i)^T$$

- If (1) the empirical spectral distribution (ESD) of $\mathbf{\Sigma}$, $F^{\mathbf{\Sigma}}$, converges to H, and (2) $p/n \to y \in (0, \infty)$, then the ESD of $\mathbf{S}$ converges to a nonrandom limit F, whose Stieltjes transform $m_F(\cdot)$ relates to H through

$$m_F(z) = \int_{\tau\in\mathbb{R}} \frac{dH(\tau)}{\tau(1 - y - yzm_F(z))}, \quad \forall z \in \mathbb{C}^+.$$

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Implications

- If indeed $d\mathbf{X}_t = \Theta d\mathbf{W}_t$, then based on the M-P theorem,
  - Knowing limit of ESD (LSD) of $\mathbf{\Sigma}^{ICV}$, one can "predict" the ESD(LSD) of $\mathbf{\Sigma}^{RCV}$
  - Starting from the observable ESD of $\mathbf{\Sigma}^{RCV}$, one can "recover" the ESD of $\mathbf{\Sigma}^{ICV}$ ([Bai, Chen, and Yao(2010)], [El Karoui(2008)], [Mestre(2008)], $\cdots$).
- However, in practice, $\mu_t \neq 0, \Theta_t \neq \Theta$, and

$$\Delta\mathbf{X}_i = \int_{(i-1)/n}^{i/n} (\mu_t dt + \Theta_t d\mathbf{W}_t)$$

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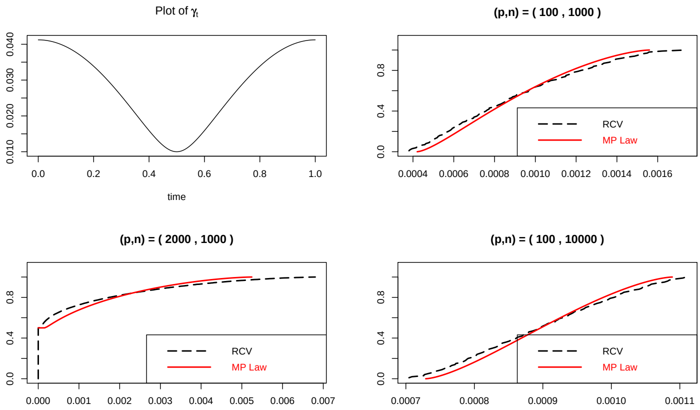
Illustration: ESD of Σ^{RCV} depends on time variability of Θ_t

An example of ESD of RCV with a time varying γ_t:



Yet Another Challenge

- In practice, another challenge is that the observations are contaminated:

$$\mathbf{Y}_{t_i} = \mathbf{X}_{t_i} + \varepsilon_i$$

- Fundamental questions: in the high-dimensional setting, with the *noisy* high-frequency observations $(\mathbf{Y}_{t_i})$,
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Consequence of Microstructure Noise

- Suppose $\boldsymbol{\mu}_t \equiv 0, \boldsymbol{\Theta}_t \equiv \mathbf{I}$, $t_i^n = \dfrac{i}{n}$ for $i = 0, 1, \cdots, n$, and

$$\mathbf{Y}_{t_i} = \mathbf{X}_{t_i} + \varepsilon_i, \quad \varepsilon_i \overset{i.i.d.}{\sim} (0, \sigma^2 \mathbf{I})$$

- Then $\Delta \mathbf{Y}_i := \Delta \mathbf{X}_i + \Delta \varepsilon_i$, where

$$\begin{aligned} \Delta \mathbf{X}_i &= \mathbf{X}_{t_i} - \mathbf{X}_{t_{i-1}} \overset{d}{=} \frac{1}{\sqrt{n}} \mathbf{Z}_i = O_p\left(\frac{1}{\sqrt{n}}\right) \\ \Delta \varepsilon_i &= \varepsilon_i - \varepsilon_{i-1} \overset{d}{=} \sqrt{2}\sigma \mathbf{e}_i \quad = O_p(1) \end{aligned}$$

$$\mathbf{Z}_i \overset{i.i.d.}{\sim} N(0, \mathbf{I}), \qquad \mathbf{e}_i \overset{i.i.d.}{\sim} (0, \mathbf{I}).$$

- Noise dominates signal!

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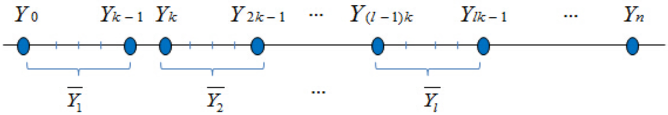

Define moving averages

$$\overline{\mathbf{Y}}_\ell = \frac{1}{k}\sum_{j=(\ell-1)k}^{\ell k-1} \mathbf{Y}_{t_j} = \overline{\mathbf{X}}_\ell + \bar{\varepsilon}_\ell \quad \ell = 1, 2, \cdots, [n/k].$$

- Main intuition: averaging reduces the variance of the noise in $\overline{\mathbf{Y}}_\ell$ by a factor of $1/k$.
- RCV based on $(\overline{\mathbf{Y}}_\ell)$ may be more relevant.

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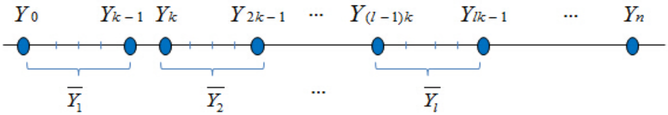

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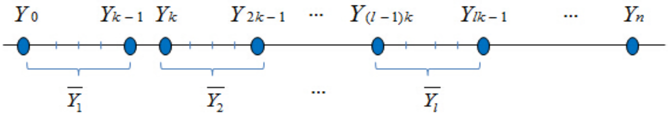

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PA-RCV

- Choose a $\theta \in (0, \infty)$ and let moving window length be $k = [\theta\sqrt{n}]$.
- The observations $(\mathbf{Y}_t)$ can be grouped into $m = [n/(2k)]$ pairs of non-overlapping windows.
- Define the PA-RCV matrix as

$$\begin{aligned}\Sigma^{PARCV} &:= \sum_{\ell=1}^{m} (\Delta_{2\ell}\bar{\mathbf{Y}})(\Delta_{2\ell}\bar{\mathbf{Y}})^T \\ &= \sum_{\ell=1}^{m} (\Delta_{2\ell}\bar{\mathbf{X}} + \Delta_{2\ell}\bar{\varepsilon})(\Delta_{2\ell}\bar{\mathbf{X}} + \Delta_{2\ell}\bar{\varepsilon})^T,\end{aligned}$$

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- The matrix $\mathbf{\Sigma}^{PARCV}$ can be viewed as the sample covariance matrix based on *noisy* observations $\Delta_{2i}\bar{\mathbf{X}} + \Delta_{2i}\bar{\varepsilon}$;
- [Dozier and Silverstein(2007)] consider such information-plus-noise-type sample covariance matrices as

$$\mathbf{S}_n = \frac{1}{n}(\mathbf{A}_n + \sigma\varepsilon_n)(\mathbf{A}_n + \sigma\varepsilon_n)^T,$$

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- This relationship shows how the LSD of $\mathbf{S}_n$ depends on that of $\mathcal{A}_n$.
- In practice, we are often more interested in making inference about signals $\mathbf{A}_n$ based on noisy observation $\mathbf{A}_n + \sigma\varepsilon_n$.
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Inversion Theorem

Theorem (1)

[Xia and Zheng(2014)] Under the assumptions above and if F admits a bounded density over a finite interval and possibly a point mass at 0, then $m_{\mathcal{A}}(z)$ is determined by F in that it uniquely solves the following equation

$$m_{\mathcal{A}}(z) = \int \frac{dF(\tau)}{\frac{\tau}{1 - y\sigma^2 m_{\mathcal{A}}(z)} - z(1 - y\sigma^2 m_{\mathcal{A}}(z)) + \sigma^2(y-1)}. \quad (2.1)$$

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The Class $\mathcal{C}$

- Say that $(\mathbf{X}_t)$ belongs to Class $\mathcal{C}$ if its covolatility process $(\boldsymbol{\Theta}_t)$ has the form

$$\boldsymbol{\Theta}_t = \gamma_t \boldsymbol{\Lambda}$$

where $(\gamma_t) \in D([0,1];\mathbb{R})$ and $\boldsymbol{\Lambda}$ is a $p \times p$ matrix.

LSD of PA-RCV for Class $\mathcal{C}$

Theorem (2)

Suppose that

- $(\mathbf{X}_t)$ *belongs to Class* $\mathcal{C}$ *with a covolatility process* $\mathbf{\Theta}_t = \gamma_t \mathbf{\Lambda}$
- *Observe* $\mathbf{Y}_{i/n} = \mathbf{X}_{i/n} + \varepsilon_i$ *where* (ε_i) *are i.i.d. with* $\mathrm{E}(\varepsilon_i) = 0$ *and* $\mathrm{cov}(\varepsilon_i) = \sigma_e^2 \mathbf{I}_p$.
- $\breve{\mathbf{\Sigma}}_p = \mathbf{\Lambda}\mathbf{\Lambda}^T$ *with an LSD* $\breve{H}$.

Assume $k = [\theta\sqrt{n}]$ *for some* $\theta \in (0, \infty)$ *and* $m = [n/(2k)]$ *satisfies that* $\lim_{p\to\infty} p/m = y > 0$. *Then as* $p \to \infty$,

- *ESDs of* $\mathbf{\Sigma}^{ICV}$ *and* $\mathbf{\Sigma}^{PARCV}$ *converge to* H *and* F, *respectively, where*

$$H(x) = \breve{H}(x/\zeta), \quad \text{for all } x \geq 0 \quad \textit{and} \quad \zeta = \lim \int_0^1 (\gamma_t)^2 \, dt.$$

- *Moreover, if* F *admits a bounded density over a finite interval and possibly a point mass at 0, then we have the following relationships*

$$m_A(z) = -\frac{1}{z} \int \frac{\zeta}{\tau M(z) + \zeta} \, dH(\tau), \qquad (2.2)$$

LSD of PA-RCV for Class $\mathcal{C}$

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LSD of PA-RCV for Class $\mathcal{C}$, *ctd*

- *where* $m_{\mathcal{A}}(z)$ *denotes the Stieltjes transform of the LSD of*

$$\sum_{i=1}^{m} \Delta_{2i}\bar{\mathbf{X}}(\Delta_{2i}\bar{\mathbf{X}})^T,$$

and is the unique solution to equation

$$m_{\mathcal{A}}(z) = \int \frac{dF(\tau)}{\frac{\tau}{1 - y\theta^{-2}\sigma_e^2 m_{\mathcal{A}}(z)} - z(1 - y\theta^{-2}\sigma_e^2 m_{\mathcal{A}}(z)) + \theta^{-2}\sigma_e^2(y-1)} \tag{2.3}$$

- *and* $M(z)$, *together with another function* $\widetilde{m}(z)$, *uniquely solve the following equations in* $\mathbb{C}^+ \times \mathbb{C}^+$

$$\begin{cases} M(z) = -\frac{1}{z}\int_0^1 \frac{(1/3)(\gamma_s^*)^2}{1 + y\widetilde{m}(z)(1/3)(\gamma_s^*)^2}\,ds, \\ \widetilde{m}(z) = -\frac{1}{z}\int \frac{\tau}{\tau M(z) + \zeta}\,dH(\tau). \end{cases} \tag{2.4}$$

Implications of the Convergence

- Σ^{PARCV} and hence F is observable
- Use (2.3) to estimate $m_{\mathcal{A}}$
- Further use (2.2) and (2.4) to estimate H, the LSD of Σ^{ICV}, the object of interest
- One challenge: the process (γ_t) in (2.4) is not observable
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Alternative Estimator: Pre-averaging Time-Variation Adjusted RCV (PA-TVARCV)

- Fix an $\alpha \in (1/2, 1)$ and $\theta \in (0, \infty)$, let $k = [\theta n^{\alpha}]$, $m = [n/(2k)]$.
- Define PA-TVARCV as

$$\boldsymbol{\Sigma}^{PATVARCV} := \frac{\operatorname{tr}(\mathbf{S}_p)}{m} \sum_{i=1}^{m} \frac{\Delta_{2i}\bar{\mathbf{Y}}(\Delta_{2i}\bar{\mathbf{Y}})^T}{|\Delta_{2i}\bar{\mathbf{Y}}|^2},$$

where $\mathbf{S}_p$ is a standard pre-averaging estimator in [Jacod et al.(2009)]

$$\mathbf{S}_p := \frac{12}{\nu\sqrt{n}} \sum_{i=0}^{n-\ell_n+1} \Delta\bar{\mathbf{Y}}_i(\Delta\bar{\mathbf{Y}}_i)^T - \frac{6}{\nu^2 n} \sum_{i=1}^{n} \Delta_i\mathbf{Y}(\Delta_i\mathbf{Y})^T,$$

where $\ell_n = [\nu\sqrt{n}]$ for some $\nu \in (0, \infty)$,

$$\Delta\bar{\mathbf{Y}}_i = \frac{1}{\ell_n}\left(\sum_{j=[\ell_n/2]}^{\ell_n-1} \mathbf{Y}_{(i+j)/n} - \sum_{j=0}^{[\ell_n/2]-1} \mathbf{Y}_{(i+j)/n}\right), \ \Delta_i\mathbf{Y} = \mathbf{Y}_{i/n} - \mathbf{Y}_{(i-1)/n}.$$

LSD of PA-TVARCV for Class $\mathcal{C}$

Theorem (3)

Suppose that

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- *Observe* $\mathbf{Y}_{i/n} = \mathbf{X}_{i/n} + \boldsymbol{\varepsilon}_i$ *where* $(\boldsymbol{\varepsilon}_i)$ *are i.i.d. with* $\mathrm{E}(\boldsymbol{\varepsilon}_i) = 0$ *and* $\mathrm{cov}(\boldsymbol{\varepsilon}_i) = \mathrm{diag}(d_1^2, \cdots, d_p^2)$;
- p *and* m *satisfy* $p/m \to y \in (0, \infty)$ *as* $p \to \infty$.

Then the LSD of $\mathbf{\Sigma}^{PATVARCV}$ *is uniquely determined by that of ICV through Stieltjes transforms via the standard M-P equation*

$$m_F(z) = \int_{\tau\in\mathbb{R}} \frac{dH(\tau)}{\tau(1 - y(1 + zm_F(z)) - z)}, \quad \forall z \in \mathbb{C}^+.$$

- No (γ_t) involved
- More importantly, noise is also eliminated!

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Simulation Studies

- To compare the ESDs of PA-RCV, PA-TVARCV matrices and reference matrix

$$\mathbf{S}_p := \frac{1}{m}(\boldsymbol{\Sigma}^{ICV})^{1/2}\mathbf{Z}_m\mathbf{Z}_m^T(\boldsymbol{\Sigma}^{ICV})^{1/2},$$

- According to Theorem 3, the ESDs of PA-TVARCV and the reference matrix should be similar
- In contrast, according to Theorem 2, that of PA-RCV should be distinguishably different from theirs
- For different p's, with
$n = 23400, \quad k = 250 \left(\approx 1.63\sqrt{n} \approx n^{0.55}\right), \quad m = [n/(2k)],$

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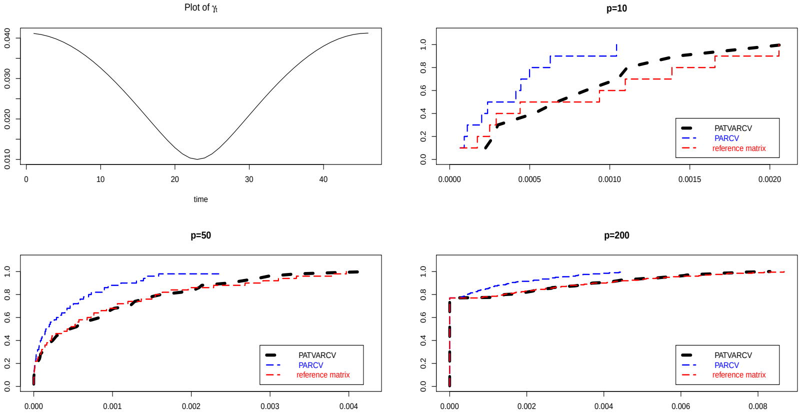
Simulation Studies, *ctd*

ESDs of PA-RCV and PA-TVARCV, with a continuous (γ_t)



Summary

1. Under high-dimensional noisy setting, we propose PA-RCV estimator and PA-TVARCV estimator, both of which can be used to recover the ESD of ICV matrix.
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