Yang Mills, unitary Brownian bridge and potential theory under constraint

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- Some concluding remarks

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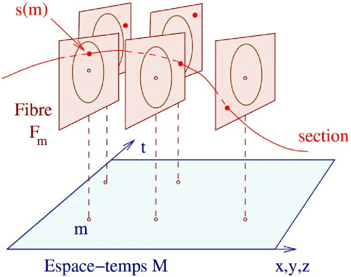

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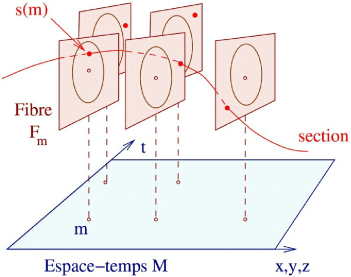

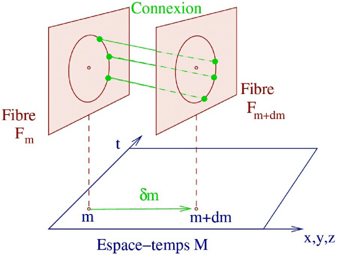

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A lot of results concerning Yang-Mills on a cylinder or a sphere (Douglas-Kazakov, Gross-Matytsin (circa 1995)), in particular

Some properties of large N two-dimensional Yang-Mills theory
[Nucl.Phys. B437 (1995)]

Unitary Brownian motion

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For $N \geq 1$, this can be generalized as follows :

$$dU_N(t) = dK_N(t)U_N(t) - \frac{1}{2}U_N(t)dt,$$

with K_N a Brownian motion on $\mathfrak{u}(N)$ eqquipped with $(X, Y)_{\mathfrak{u}(N)} = N\mathrm{Tr}(X^* Y)$.

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Ex : $p_2 := \sum x_i^2 = \sum_{i\leq j} x_i x_j - \sum_{i<j} x_i x_j = s_{(2)} - s_{(1,1)}$

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$$\text{and} \qquad \int_{\mathcal{U}(N)} \overline{s_\alpha(U)} s_\beta(U) dm_N(U) = \delta_{\alpha,\beta}\mathbf{1}_{\ell(\alpha)\le N}.$$

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For any $t > 0$, we denote by ν_t the probability measure on $\mathbb{U}$ such that, for all $n \geq 0$, $\int z^{-n} d\nu_t(z) = \int z^n d\nu_t(z) = c_n(t)$.

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From harmonic analysis, we get that

$$Z_{N,T} = C_{N,T} \sum_{\ell} e^{-N^2 I_T(\hat{\mu}_\ell)},$$

with

$$I_T(\mu) := -\iint \ln|x-y| d\mu(x) d\mu(y) + \int \frac{T}{2} x^2 d\mu(x)$$

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Proposition

For all $T > 0$,

$$\lim_{N\to\infty} \frac{1}{N^2} \ln Z_{N,T} = \frac{T}{24} + \frac{3}{2} - \inf \; I_T(\mu),$$

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Tools : large deviations results.

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- *If $T \leq \pi^2$, the density of μ_T^* with respect to Lebesgue measure is given by*

$$\frac{d\mu_T^*(x)}{dx} = \frac{T}{2\pi}\sqrt{\frac{4}{T} - x^2}\mathbf{1}_{\left[-\frac{2}{\sqrt{T}},\frac{2}{\sqrt{T}}\right]}(x),$$

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Consequence : The function F is of class $\mathcal{C}^2$ on $\mathbb{R}_+^*$ and of class $\mathcal{C}^\infty$ on $\mathbb{R}_+^* \setminus \{\pi^2\}$. At π^2, $F^{(3)}$ has a discontinuity of first kind.

Potential theory under constraint

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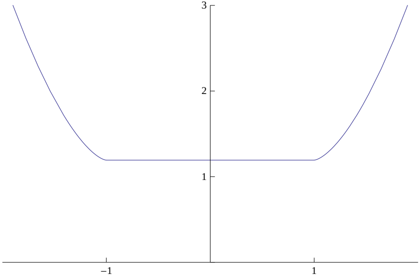
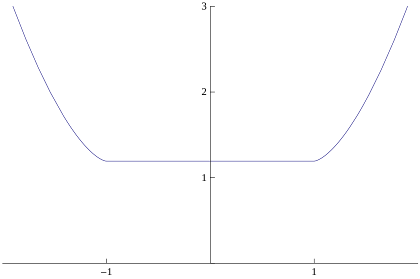


$U^{\mu} + Q \geq C$
$U^{\mu} + Q = C$ on the support

Potential theory under constraint



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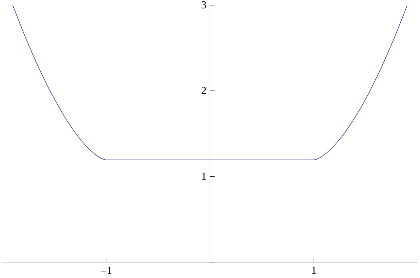

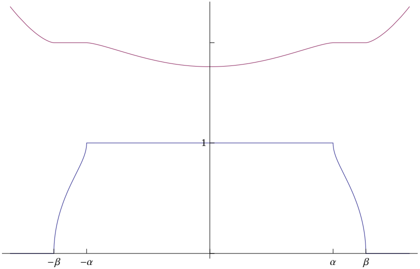

$U^\mu + Q \geq C$
$U^\mu + Q = C$ on the support

$U^\mu + Q \geq C$ outside the support
$U^\mu + Q = C$ on the "free" part
$U^\mu + Q \leq C$ where it saturates

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- Fascinating model for which everything can be computed explicitely
- In a recent work of Liechty and Wang, μ_T^* appears as the equilibrium measure associated to orthogonal poynomials for a discrete gaussian measure (also linked with Unitary brownian bridge)
- for some parameters (t, T), the asymptotic spectral measure of uBb is known and related to the family μ_T^* in a way which is still to be understood in details (work in progress with T. Lévy).