Spectral properties of random Markov matrices

P. Caputo, with C. Bordenave and D. Chafai, and D. Piras

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Outline

A natural model for a random Markov matrix:
stochastic matrix K with random entries

$$K_{i,j} = \frac{U_{i,j}}{\sum_k U_{i,k}} \qquad U_{i,j} \geqslant 0 \quad \text{i.i.d.}$$

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Reversible case: $U_{i,j} = U_{j,i}$ (random conductances)

Non-reversible case: $U_{i,j}$ i.i.d. (weighted oriented graph)

Bulk behavior: convergence of empirical spectral density of K

1. Finite second moment: semi-circular law, circular law
2. Heavy tails: $\mathbb{P}(U_{i,j} > t) \sim t^{-\alpha}$, $\alpha \in (0,2)$, new invariance principles

Random *reversible* stochastic matrix

$G = (V, E)$: complete graph over n vertices with self–loops
$V = \{1, \dots, n\}$, $E = \{\{i, j\}, \ i, j \in V\}$.

Random network $(G, \mathbf{U})$:

$$\mathbf{U} = (U_{ij})_{1 \leqslant i \leqslant j \leqslant n}$$

i.i.d. RV's with law $\mathcal{L}$ on $[0, \infty)$.

Symmetry (undirected graph): $U_{ji} = U_{ij}$, $j > i$.

Random walk on $(G, \mathbf{U})$:

$$K_{ij} = \frac{U_{ij}}{\rho_i}, \qquad \rho_i = \sum_{j=1}^{n} U_{ij} .$$

K is a reversible stochastic matrix: $\rho_i \, K_{ij} = \rho_j \, K_{ji}$.

Eigenvalues of K

K is a.s. irreducible and aperiodic with eigenvalues:

$$-1 < \lambda_n \leqslant \lambda_{n-1} \leqslant \cdots \leqslant \lambda_2 < \lambda_1 = 1\,.$$

Empirical spectral distribution (ESD):

$$\mu_K = \frac{1}{n} \sum_{i=1}^{n} \delta_{\lambda_i}\,.$$

Moments: $p_\ell(i)$ return probability at i after ℓ steps

$$\int_{-1}^{1} x^\ell \, \mu_K(dx) = \frac{1}{n} \operatorname{Tr}(K^\ell) = \frac{1}{n} \sum_{i=1}^{n} p_\ell(i)\,.$$

Convergence of ESD μ_K (after scaling if necessary) ?

Finite variance, reversible case

Suppose $\mathbb{E}[U_{ij}^2] < \infty$,
$\mathbb{E}[U_{ij}] = 1$ (no loss of generality), $\sigma^2 = \mathbb{E}[(U_{ij} - 1)^2]$.

Theorem

If $\sigma^2 \in (0, \infty)$, then almost surely

$$\mu_{\sqrt{n}\,K} \xrightarrow[n\to\infty]{w} \mathcal{W}_{2\sigma}\,,$$

where $\mathcal{W}_{2\sigma}$ is Wigner's Semi-circle law:

$$\mathcal{W}_{2\sigma}(dx) = \frac{1}{2\pi\sigma^2}\sqrt{4\sigma^2 - x^2}\,1_{[-2\sigma,2\sigma]}(x)\,dx.$$

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Idea of proof (perturbation argument):

Uniform strong LLN: $\rho_i \sim n\,\mathbb{E}[U_{ij}] = n\,,\; K_{ij} \sim n^{-1}\,U_{ij}.$

$$\delta_n := \max_{i=1,\dots,n} |\rho_i/n - 1| = o(1)\,,\; a.s.\ (n \to \infty)$$

Heavy tails

For $\alpha > 0$, we say that $\mathcal{L} \in \mathcal{H}_\alpha$, or simply $U_{ij} \in \mathcal{H}_\alpha$, if

$$G(t) = \mathbb{P}(U_{ij} > t) = L(t)\, t^{-\alpha}\,,$$
$$\lim_{t\to\infty} \frac{L(x\,t)}{L(t)} = 1\,,\ x > 0\,. \quad \text{(slow variation)}$$

$\alpha \in (0,2) \Rightarrow \mathbb{E}[U_{ij}^2] = \infty$, U_{ij} in domain of attract. of α-stable law.

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Scaling: $a_n = n^{1/\alpha}\,\ell(n)$, with $\ell(n)$ slowly varying,

$$n\, G(a_n t) \to t^{-\alpha} \quad \text{as} \quad n \to \infty.$$

Example: $X = Unif[0,1]$ then $X^{-1/\alpha} \in \mathcal{H}_\alpha$, with $a_n = n^{1/\alpha}$.

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Recall:

a_n^{-1}(order statistics of n i.i.d. RVs in $\mathcal{H}_\alpha$) $\sim (\Gamma_1^{-1/\alpha}, \ldots, \Gamma_n^{-1/\alpha})$

where $\Gamma_k = \sum_{i=1}^k E_i$, and E_i are i.i.d. Esp(1), i.e. $PPP(\alpha x^{-\alpha-1})$.

Heavy tails: i.i.d. case

Symmetric i.i.d. matrix $A = (A_{ij})$, with $|A_{ij}| \in \mathcal{H}_\alpha$, $\alpha \in (0,2)$, with $\lim_{t\to\infty} \frac{\mathbb{P}(A_{i,j}>t)}{\mathbb{P}(|A_{i,j}|>t)} = \theta \in [0,1]$.

Theorem

For $\alpha \in (0,2)$, there exists a symmetric probability μ_α on $\mathbb{R}$ depending only on α such that, a.s.

$$\mu_{a_n^{-1}A} = \frac{1}{n}\sum_{i=1}^{n} \delta_{\lambda_i(a_n^{-1}A)} \xrightarrow[n\to\infty]{w} \mu_\alpha \,.$$

Moreover μ_α is a.c. with bounded density and $\mu_\alpha([t,\infty)) \sim \frac{1}{2}\, t^{-\alpha}$.

Bouchaud–Cizeau (PRE 1994), Zakharevich (CMP 2006), Ben Arous–Guionnet (CMP 2008). Belinschi-Dembo-Guionnet (CMP 2009). Resolvent method (Stieltjes transform).

Our approach gives an alternative proof.

Heavy tails: Markov matrix

Stochastic matrix $K_{ij} = U_{ij}/\rho_i$, with $U_{ij} \in \mathcal{H}_\alpha$. μ_α as above.

Theorem

Suppose $\alpha \in [1, 2)$. Then, a.s.

$$\mu_{\kappa_n K} = \frac{1}{n} \sum_{i=1}^{n} \delta_{\lambda_i(\kappa_n K)} \xrightarrow[n\to\infty]{w} \mu_\alpha \,.$$

where $\kappa_n = n w_n a_n^{-1}$, $w_n = \mathbb{E}[U_{ij}\,\chi(U_{ij} \leqslant a_n)]$.

Theorem

Suppose $\alpha \in (0, 1)$. Then, there exists a probability measure $\widetilde{\mu}_\alpha$ on $[-1, 1]$ depending only on α such that, a.s.

$$\mu_K = \frac{1}{n} \sum_{i=1}^{n} \delta_{\lambda_i(K)} \xrightarrow[n\to\infty]{w} \widetilde{\mu}_\alpha \,.$$

Scaling, $\alpha \in (0,2)$:

$$a_n^{-1}(\rho_i - nw_n) \xrightarrow[n\to\infty]{d} s_\alpha \quad \alpha\text{-stable}$$

$\alpha \in (1,2)$: $w_n \to \mathbb{E}[U_{ij}] = 1$ and $\rho_i/n \to 1$ a.s.

$$\kappa_n K \sim n a_n^{-1} K \sim a_n^{-1} A, \qquad \text{where } A \text{ has i.i.d. entries.}$$

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$\alpha = 1$, then $\rho_i / nw_n \to 1$ in probability

[example: $U = (Unif[0,1])^{-1}$, then $\kappa_n = w_n = \log n$]

$$\kappa_n K = n w_n a_n^{-1} K \sim a_n^{-1} A, \quad A \text{ i.i.d. entries.}$$

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$$\kappa_n K = n w_n a_n^{-1} K \sim a_n^{-1} A, \quad A \text{ i.i.d. entries.}$$

$\alpha \in (0,1)$, then $a_n^{-1}\rho_i \xrightarrow[n\to\infty]{d} s_\alpha^+$

each row of K converges to Poisson-Dirichlet(α): $\Gamma_k = \sum_{i=1}^k E_i$

$$Z = \left(\sum_{n=1}^\infty \Gamma_n^{-\frac{1}{\alpha}}\right)^{-1} \left(\Gamma_1^{-\frac{1}{\alpha}}, \Gamma_2^{-\frac{1}{\alpha}}, \dots\right).$$

Some ideas of the proof

Start with symmetric i.i.d. matrix $A_{ij} = U_{ij}$ as a weighted graph:

Convergence of resolvents from

local convergence of graphs

[Bordenave, Lelarge]
Objective method (Aldous-Steele '04)
Limiting graph is a random infinite rooted tree $(\mathcal{T}_\alpha; o)$:

Recall that a_n^{-1}(order stat. of row 1) $\sim PPP(\alpha x^{-\alpha-1})$

This convergence can be extended to local convergence to the Poisson weighted infinite tree **PWIT** (Aldous '92)

Define **PWIT**(m_α)$= \mathcal{T}_\alpha$, for

$$m_\alpha(dx) = \alpha\, x^{-1-\alpha}\, dx\,, \quad \text{on } (0,\infty)\,.$$

Start from the root o, with $\mathbb{N}$ offsprings. Each edge (o,k) is given a mark ξ_k where $\xi_1 > \xi_2 > \cdots$ is a realization of PPP(m_α). The distance of offspring k from o is defined by ξ_k^{-1}. Repeat this *independently* at each offspring to obtain an infinite ∞-ary tree with Poissonian marks (PWIT).

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Convergence: $(G,\mathbf{U})$ rooted at 1. The vector $(a_n^{-1}\, U_{1,j})_j$ converges in distribution to PPP(m_α), (via order statistics). In the *local* sense

$$(G,\mathbf{U};1) \to (\mathcal{T}_\alpha; o)\,.$$

(again via order statistics. Small weights correspond to points far away from the root.) This holds for any $\alpha > 0$.

Similar result for $A = (A_{ij})$, with $|A_{ij}| = U_{ij} \in \mathcal{H}_\alpha$. *Signed marks*.

Key point: for $\alpha \in (0,2)$, this convergence is sufficient to establish convergence (in distribution) of resolvent diagonal entries.
Hilbert space is $\ell^2(\mathcal{V})$, $\mathcal{V}$ the vertices of the tree:

$$\langle \delta_1, (a_n^{-1}A - z)^{-1}\delta_1 \rangle \to \langle \delta_o, (\mathbf{T} - z)^{-1}\delta_o \rangle$$

where $\mathbf{T}$ is the limiting operator associated to $a_n^{-1}A$:

$$\langle \delta_u, \mathbf{T}\delta_v \rangle = \xi_{u,v} \quad \text{mark across edge } (u,v) \text{ in } \mathcal{T}_\alpha\,.$$

$\mathbf{T}$ is symmetric in $\ell^2(\mathcal{V})$. Note: If $\mathbf{T}$ is self adjoint then $\exists\, \mu_{\mathbf{T}}$

$$\langle \delta_o, (\mathbf{T} - z)^{-1}\delta_o \rangle = \int_{\mathbb{R}} \frac{\mu_{\mathbf{T}}(dx)}{x - z}\,, \qquad z \in \mathbb{C}_+$$

Taking expectation we have $\mathbb{E}\mu_{a_n^{-1}A} \to \mu_\alpha := \mathbb{E}[\mu_{\mathbf{T}}]$, since

$$\int_{\mathbb{R}} \frac{\mathbb{E}[\mu_{a_n^{-1}A}](dx)}{x - z} = \mathbb{E}[\langle \delta_1, (a_n^{-1}A - z)^{-1}\delta_1 \rangle] \to \int_{\mathbb{R}} \frac{\mathbb{E}[\mu_{\mathbf{T}}](dx)}{x - z}$$

Technical point: must show $\mathbf{T}$ is essentially self-adjoint; no solution $\varphi \neq 0$ of $\mathbf{T}^*\varphi = \pm i\varphi$ (exploit tree structure).

Almost sure convergence $\mu_{a_n^{-1}A} \to \mu_\alpha$ follows from concentration properties of ESD of i.i.d. matrices.

This gives an alternative proof in the i.i.d. case
[see Ben Arous–Guionnet 08 for an earlier different proof]

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Properties of μ_α.

Recursive Distributional Equation: $h(z) = \langle \delta_o, (\mathbf{T} - z)^{-1} \delta_o \rangle$ satisfies

$$h(z) \stackrel{d}{=} - \left(z + \sum_k \xi_k \, h_k(z) \right)^{-1}$$

where ξ_k is $\mathrm{PPP}(m_{\alpha/2})$ and $h_k(z)$ are i.i.d. copies of $h(z)$.

The Markov matrix case: again network convergence.

$\alpha \in (0,1)$: the PPP(m_α) satisfies $\sum_i \xi_i < \infty$ a.s.

$(K_{1,j})_j \sim$ $PD(\alpha)$ *Poisson–Dirichlet law* (Pitman-Yor '97).

$$Z = \left(\sum_{n=1}^{\infty} \Gamma_n^{-\frac{1}{\alpha}}\right)^{-1} \left(\Gamma_1^{-\frac{1}{\alpha}}, \Gamma_2^{-\frac{1}{\alpha}}, \dots\right) \qquad \Gamma_k = \sum_{i=1}^{k} E_i \,.$$

Limit operator $\mathbf{K}$ describes Random Walk on **PWIT** $\mathcal{T}_\alpha$

$$\mathbf{K}_{u,v} = \frac{\xi_{u,v}}{\rho_u}\,, \quad \rho_u = \sum_{v \in \mathcal{V}:\, v \sim u} \xi_{u,v}\,.$$

Here, for every $u \in \mathcal{V}$: $\{\xi_{u,v}\,,\ v \in \mathcal{V} :\ v \text{ child of } u\}$ is PPP(m_α).

Note: limit operator $\mathbf{K}$ is a non-trivial generalization of Poisson-Dirichlet law (dependecies!).

$\mathbf{K}$ is *bounded* self adjoint operator in $\ell^2(\mathcal{V},\rho)$ and
$\mathbb{E}\mu_K \to \widetilde{\mu}_\alpha = \mathbb{E}[\mu_{\mathbf{K}}]$, since

$$\int_{\mathbb{R}} \frac{\mathbb{E}[\mu_K](dx)}{x-z} = \mathbb{E}[\langle \delta_1, (K-z)^{-1}\delta_1\rangle] \to \int_{\mathbb{R}} \frac{\mathbb{E}[\mu_{\mathbf{K}}](dx)}{x-z}$$

$\mu_{\mathbf{K}}$ spectral measure of $\mathbf{K}$ at the root vector δ_o.

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Moments of $\mu_{\mathbf{K}}$ are return probabilities for RW on $\mathcal{T}_\alpha$.

Shape of $\widetilde{\mu}_\alpha$: Beta-like law on $[-1, 0]$ and $[0, 1]$.

Tail of $\widetilde{\mu}_\alpha$ at edge: $\widetilde{\mu}_\alpha(1 - \varepsilon, 1) \sim \varepsilon^\alpha$.

$$\widetilde{\mu}_\alpha \to \tfrac{1}{4}\,\delta_{-1} + \tfrac{1}{2}\,\delta_0 + \tfrac{1}{4}\,\delta_1, \quad \alpha \downarrow 0$$

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$$\widetilde{\mu}_\alpha \to \tfrac{1}{4}\, \delta_{-1} + \tfrac{1}{2}\, \delta_0 + \tfrac{1}{4}\, \delta_1, \quad \alpha \downarrow 0$$

Spectral gap: $1 - \lambda_2 = O(n^{-1/\alpha})$ (up to SV corrections).

Reversible invariant measure of RW on **PWIT** $\mathcal{T}_\alpha$.

$\widetilde{\rho}_1 \geqslant \widetilde{\rho}_2 \geqslant \cdots \geqslant \widetilde{\rho}_n$ ranked values of invariant vector

$$(\rho_1 + \cdots + \rho_n)^{-1} (\rho_1, \ldots, \rho_n) \ .$$

Theorem

1. If $\alpha \in (0,1)$, then

$$\widetilde{\rho} \xrightarrow[n\to\infty]{d} \frac{1}{2} \left(V_1, V_1, V_2, V_2, \ldots \right) ,$$

where $V_1 > V_2 > \cdots$ is a Poisson–Dirichlet $\mathrm{PD}(\alpha, 0)$ random vector.

2. If $\alpha \in [1,2)$, then

$$\kappa_{n(n+1)/2} \, \widetilde{\rho} \xrightarrow[n\to\infty]{d} \frac{1}{2} \left(\xi_1, \xi_1, \xi_2, \xi_2, \ldots \right) ,$$

where $\xi_1 > \xi_2 > \cdots$ is PPP(m_α), and $\kappa_n = n a_n^{-1} w_n$.

Further investigations and open problems:

- More details on the measures μ_α, $\widetilde{\mu}_\alpha$
- Analysis of stochastic the process associated to the limiting operator $\mathbf{K}$
- Extremal eigenvalues: Poisson statistics ? (known for i.i.d. matrix Soshnikov 2004, Auffinger-Ben Arous-Peche 2008)

Non-reversible Markov matrix

$G = (V, E)$: complete *oriented* graph over n vertices with self–loops $V = \{1, \ldots, n\}$, $E = \{(i,j), \ i,j \in V\}$.

Random network $(G, \mathbf{U})$:

$$\mathbf{U} = (U_{ij})_{1 \leqslant i,j \leqslant n}$$

i.i.d. RV's with law $\mathcal{L}$ on $[0, \infty)$. No symmetry.

Random walk on $(G, \mathbf{U})$:

$$K_{ij} = \frac{U_{ij}}{\rho_i}, \qquad \rho_i = \sum_{j=1}^{n} U_{ij}.$$

Eigenvalues: $|\lambda_1(K)| \geqslant \cdots \geqslant |\lambda_n(K)|$. $\quad \mu_K = \frac{1}{n} \sum_{i=1}^{n} \delta_{\lambda_i(K)}$

Circular law theorem

From works of Girko (1984) ... Bai (1997) ... Tao-Vu (2009).

$$\mathcal{U}_\sigma(dz) = \frac{1}{\pi\sigma^2} 1_{\{|z| \leqslant \sigma\}} \, dz$$

Theorem

If $X_{i,j}$ are i.i.d. with variance $\sigma^2 \in (0, \infty)$ then a.s.

$$\mu_{\frac{1}{\sqrt{n}} X} \xrightarrow[n\to\infty]{w} \mathcal{U}_\sigma.$$

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In our case we prove

Theorem

If $U_{i,j}$ has variance $\sigma^2 \in (0,\infty)$ and its law is a.c. with bounded density then a.s.

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idea: as before $|\rho_i/n - 1| \to 0$ uniformly and therefore $K_{ij} \sim \frac{U_{ij}}{n}$, but here there is no easy perturbation argument

The logarithmic potential $U_\mu(z) = -\int_{\mathbb{C}} \log|z' - z|\mu(dz')$ determines the distribution μ:

$$\Delta U_\mu = -2\pi\,\mu\,, \quad \text{in } \mathcal{D}'(\mathbb{C})\,.$$

For any $n \times n$ matrix A:

$$\begin{aligned} U_{\mu_A}(z) &= -\frac{1}{n}\sum_{i=1}^{n} \log|\lambda_i(A) - z| = -\frac{1}{n}\log|\det(A - z)| \\ &= -\frac{1}{n}\log\det\left(\sqrt{(A-z)(A-z)^*}\right) \end{aligned}$$

Let ν_A denote the ESD of *singular spectrum* $\nu_A = \frac{1}{n}\sum_{i=1}^{n}\delta_{\sigma_i(A)}$, where $s_i(A) = \lambda_i(\sqrt{AA^*})$. Then

$$U_{\mu_A}(z) = -\int_0^\infty \log(t)\,\nu_{A-z}(dt)$$

A is normal, i.e. $AA^* = A^*A$, iff $|\lambda_i(A)| = s_i(A)\ \forall\, i$. In general

$$\prod_{i=1}^{n}|\lambda_i(A)| = \prod_{i=1}^{n}\sigma_i(A)\,,\ \ |\lambda_1(A)| \leqslant s_1(A)\,,\ \ |\lambda_n(A)| \geqslant s_n(A)\,.$$

Lemma (Girko's hermitization strategy)

Let $(A_n)_{n \geqslant 1}$ be a sequence of $n \times n$ matrices. We have

$$U_{\mu_{A_n}}(z) = -\int_0^\infty \log(t)\, \nu_{A_n - z}(dt)\,.$$

Suppose that for a.a. $z \in \mathbb{C}$, there is a probability ν_z on $[0, \infty)$ such that

(i) $\nu_{A_n - z} \to \nu_z$ *weakly as* $n \to \infty$

(ii) $\log(\cdot)$ *is uniformly integrable fo* $\nu_{A_n - z}$

Then there exists a probability μ on $\mathbb{C}$ such that $\mu_{A_n} \to \mu$ weakly as $n \to \infty$, and $U_\mu(z) = -\int_0^\infty \log(t)\, \nu_z(dt)$.

[In the random case, one can use this lemma for a.a. realizations.]

For our matrices K: We prove

Theorem (singular values)

If $\sigma^2 \in (0, \infty)$, then almost surely

$$\nu_{\sqrt{n}K} \xrightarrow[n\to\infty]{w} \mathcal{Q}_\sigma \, .$$

where $Q_\sigma(dt) = \frac{1}{\pi\sigma^2}\sqrt{4\sigma^2 - t^2}\, 1_{\{0<t<2\sigma^2\}}\, dt$. Moreover, for a.a. $z \in \mathbb{C}$, $\nu_{\sqrt{n}K-z} \xrightarrow[n\to\infty]{w} \nu_z$ with ν_z satisfying

$$U_{\mathcal{U}_\sigma}(z) := -\int_{\mathbb{C}} \log|z'-z|\mathcal{U}_\sigma(dz') = -\int_0^\infty \log(t)\, \nu_z(dt).$$

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$$\nu_{\sqrt{n}K} \xrightarrow[n\to\infty]{w} \mathcal{Q}_\sigma \, .$$

where $Q_\sigma(dt) = \frac{1}{\pi\sigma^2}\sqrt{4\sigma^2 - t^2}\,1_{\{0<t<2\sigma^2\}}\,dt$. *Moreover, for a.a.* $z \in \mathbb{C}$, $\nu_{\sqrt{n}K-z} \xrightarrow[n\to\infty]{w} \nu_z$ *with* ν_z *satisfying*

$$U_{\mathcal{U}_\sigma}(z) := -\int_{\mathbb{C}} \log|z' - z| \mathcal{U}_\sigma(dz') = -\int_0^\infty \log(t)\,\nu_z(dt).$$

Theorem (uniform integrability)

If $\sigma^2 \in (0, \infty)$, then almost surely, for a.a. $z \in \mathbb{C}$, $\log(\cdot)$ is uniformly integrable w.r.t. $\nu_{\sqrt{n}K-z}$, i.e. for all $\epsilon > 0$:
$\lim_{\beta\to\infty} \mathbb{P}\left(\sup_n \left|\int_{|\log(\cdot)|>\beta} \log(t)\nu_{\sqrt{n}K-z}(dt)\right| > \epsilon\right) \to 0.$

Ideas of proof

Singular values: use perturbation for hermitian matrices and known results for convergence of $\nu_{\frac{1}{\sqrt{n}}X-z}$ (Pan-Zhou, Bai, ...).

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Following Tao-Vu we need two facts:

1) Smallest singular value bound: for every $a, C > 0$ there exists $b > 0$ such that for any $z \in \mathbb{C}$ with $|z| \leqslant C$

$$\mathbb{P}(s_n(\sqrt{n}K - z) \leqslant n^{-b}) \leqslant n^{-a}.$$

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Extensions: it's possible to remove the bdd density assumtpion [refined estimate for s_n following Rudelson-Vershynin, Götze-Tikhomirov]. One can also treat sparse graphs: $U_{ij} \mapsto \varepsilon_{ij} U_{ij}$ with ε_{ij} iid Bernoulli($p(n)$), $np(n)(\log n)^{-6} \to \infty$, $p(n) \to 0$

Non-hermitian i.i.d. heavy tailed matrices

$A = (A_{ij})_{1 \leqslant i,j \leqslant n}$, i.i.d. with law $|A_{ij}| \in \mathcal{H}_\alpha$, $\alpha \in (0,2)$, and $\lim_{t\to\infty} \frac{\mathbb{P}(A_{i,j}>t)}{\mathbb{P}(|A_{i,j}|>t)} = \theta \in [0,1]$. Assume also bounded density of A_{ij}.

Theorem

There exists an isotropic probability μ_α on $\mathbb{C}$ depending only on α such that, a.s.

$$\mu_{a_n^{-1}A} = \frac{1}{n}\sum_{i=1}^{n} \delta_{\lambda_i(a_n^{-1}A)} \xrightarrow[n\to\infty]{w} \mu_\alpha \,.$$

Moreover μ_α is a.c. with bounded density $\mu_\alpha(dz) = \varphi(|z|)\,dz$ satisfying

$$\varphi(t) \sim t^{2(\alpha-1)}\, e^{-\frac{\alpha}{2}\, t^{\alpha}}\,, \quad t \to \infty\,.$$

[No heavy tails. Shrinking of the spectrum w.r.t. singular values]

Ideas of proof I

From Girko's hermitization: need to establish a) Singular values convergence and b) Uniform integrability.

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Theorem (singular values)

There exists ν_z depending on α and $z \in \mathbb{C}$, such that for a.a. $z \in \mathbb{C}$, almost surely,

$$\nu_{a_n^{-1}A-z} \xrightarrow[n\to\infty]{w} \nu_z \,.$$

[For $z = 0$ already in Belinschi-Dembo-Guionnet 2009].

We prove it using again PWIT technology. Need a *bipartized* version of **PWIT**. Note: ν_z has heavy tails e.g. at $z = 0$!

Ideas of proof II

Theorem (uniform integrability)

For a.a. $z \in \mathbb{C}$, almost surely, $\log(\cdot)$ *is uniformly integrable w.r.t.* $\nu_{a_n^{-1}A-z}$

As before, for the proof we need:

1) Smallest singular value bound: here OK by bdd density assumption

$$\mathbb{P}(s_n(a_n^{-1}A - z) \leqslant n^{-b}) \leqslant n^{-a}.$$

2) Control of $s_{n-i}(a_n^{-1}A - z)$ for $n^{1-\varepsilon} < i < n$:

Here we cannot have $s_{n-i}(a_n^{-1}A - z) \geqslant c\,\frac{i}{n}$.

There is not enough concentration.

We establish weaker estimates that are still sufficient.

Non-reversible Markov matrix: heavy tailed weights

[Work in progress with D. Piras]

$G = (V, E)$: complete *oriented* graph over n vertices with self–loops $V = \{1, \ldots, n\}$, $E = \{(i, j), \ i, j \in V\}$.

Random network $(G, \mathbf{U})$:

$$\mathbf{U} = (U_{ij})_{1 \leqslant i,j \leqslant n}$$

i.i.d. RV's with law $\mathcal{L} \in \mathcal{H}_\alpha$, $\alpha \in (0, 1)$. No symmetry.

As before we consider the Random walk on $(G, \mathbf{U})$:

$$K_{ij} = \frac{U_{ij}}{\rho_i}, \qquad \rho_i = \sum_{j=1}^{n} U_{ij} .$$

Expect convergence of ESD $\mu_K = \frac{1}{n} \sum_{i=1}^{n} \delta_{\lambda_i(K)}$ without scaling.

Main result

Theorem

Assume bdd density for the law $\mathcal{L}$. For any $\alpha \in (0,1)$, there exists a radial probability $\hat{\mu}_\alpha$ on $\mathbb{D} = \{z \in \mathbb{C} : |z| \leqslant 1\}$ depending only on α such that, a.s.

$$\mu_K = \frac{1}{n} \sum_{i=1}^{n} \delta_{\lambda_i(K)} \xrightarrow[n\to\infty]{w} \hat{\mu}_\alpha \,.$$

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Key steps:

1. Convergence of singular value spectrum ν_{K-z}, for all $z \in \mathbb{C}$.

2. Uniform integrability of $\log(\cdot)$ for ν_{K-z} for almost all $z \in \mathbb{C}$.

Note: for $z = 1$, the matrix $K - z$ is singular with probability 1 !

The singular values

Theorem (Bulk)

There exists a probability measure $\hat{\nu}_{\alpha,z}$ on $\mathbb{R}_+$, depending on α and $z \in \mathbb{C}$, such that for a.a. $z \in \mathbb{C}$, a.s.,

$$\nu_{K-z} \xrightarrow[n\to\infty]{w} \hat{\nu}_{\alpha,z} \,.$$

The measure $\hat{\nu}_{\alpha,z}$ has unbounded support with exponential tails.

Theorem (Invertibility)

For any $\delta > 0$ there exists $r > 0$ such that for all $|z| < \delta^{-1}$ and $|z-1| > \delta$ one has almost surely

$$\lim_{n\to\infty} n^r s_n(K-z) = +\infty.$$

Modified PWIT

Key observations:

1. order stat. of first row

   $= \rho_1^{-1}$(order statistics of n i.i.d. RVs in $\mathcal{H}_\alpha$)

   $\sim PD(\alpha) = \left(\frac{\xi_1}{\sum_{i=1}^\infty \xi_i}, \frac{\xi_2}{\sum_{i=1}^\infty \xi_i}, \dots \right)$

2. order stat. of first column

   $= \left(\text{order statistics of } \frac{U_{i,1}}{\rho_i} \right) \sim \left(\frac{\xi_1}{a+\xi_1}, \frac{\xi_2}{a+\xi_2}, \dots \right)$

where $a > 0$, and $\{\xi_i\} = \{\Gamma_i^{-1/\alpha}\}$ is $PPP(\alpha x^{-\alpha-1} dx)$.
Call $\mathcal{T}_\alpha^{\pm}$ the PWIT obtained by alternating $PD(\alpha)$ generations with $\left\{ \frac{\xi_i}{a+\xi_i} \right\}$-generations.

We obtain that bipartized matrix $\begin{pmatrix} 0 & K \\ K^* & 0 \end{pmatrix}$ converges locally to the random rooted tree: $\mathcal{T}_\alpha^{+}$ with prob. $\frac{1}{2}$ and $\mathcal{T}_\alpha^{-}$ with prob. $\frac{1}{2}$.

Bipartized matrix

Example: $n = 2$

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \Rightarrow B = \begin{pmatrix} 0 & A_{11} & 0 & A_{12} \\ \bar{A}_{11} & 0 & A_{21} & 0 \\ 0 & \bar{A}_{21} & 0 & A_{22} \\ \bar{A}_{12} & 0 & \bar{A}_{22} & 0 \end{pmatrix}$$

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Moreover, B similar to

$$\widetilde{B} = \begin{pmatrix} 0 & 0 & A_{11} & A_{12} \\ 0 & 0 & A_{21} & A_{22} \\ \bar{A}_{11} & \bar{A}_{21} & 0 & 0 \\ \bar{A}_{12} & \bar{A}_{22} & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & A \\ A^* & 0 \end{pmatrix}$$

Bipartized matrix

In general,

$$B = (B_{ij}), \text{ with } B_{ij} = \begin{pmatrix} 0 & A_{ij} \\ \bar{A}_{ji} & 0 \end{pmatrix} \text{ is } 2 \times 2 \text{ matrix.}$$

Since B similar to $\widetilde{B} = \begin{pmatrix} 0 & A \\ A^* & 0 \end{pmatrix}$,

$$\mu_B = \frac{1}{2n} \sum_{i=1}^{n} \left(\delta_{\sigma_i(A)} + \delta_{-\sigma_i(A)}\right).$$

New resolvents: $z \in \mathbb{C}$, $\eta \in \mathbb{C}_+$:

$$R(U) = (B - U \otimes I_n)^{-1}, \qquad U = U(z, \eta) = \begin{pmatrix} \eta & z \\ \bar{z} & \eta \end{pmatrix}$$

Then $R(U)_{kk} = \begin{pmatrix} a_k(z, \eta) & b_k(z, \eta) \\ \bar{b}_k(z, \eta) & c_k(z, \eta) \end{pmatrix}$

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Crucial relations: a random matrix A with exchangeable entries satisfies, in $\mathcal{D}'(\mathbb{C})$

$$\mathbb{E}\mu_A = -\frac{1}{4\pi}(\partial_x - i\partial_y)\mathbb{E}\, b_1(\cdot, 0) = \lim_{t\downarrow 0} -\frac{1}{4\pi}(\partial_x - i\partial_y)\mathbb{E}\, b_1(\cdot, it) \qquad (*)$$

To prove properties of $\mu_{a_n^{-1}A}$: establish convergence to bipartized PWIT and use relations like $(*)$ together with recursive characterizations of $\mathbb{E}b_1(\cdot, it)$ on PWIT.