Steep Dimers on Rail Yard Graphs

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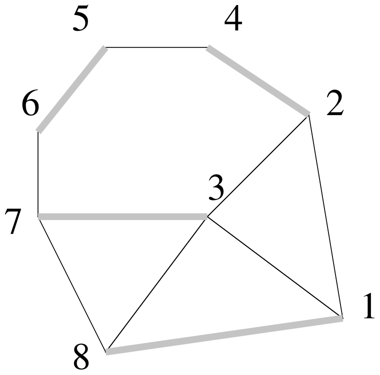
États de la recherche *matrices aléatoires* – 3 décembre 2014

Dimer models

planar graph G

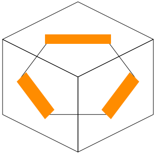


dimer configuration: perfect matching

Several techniques to study these models

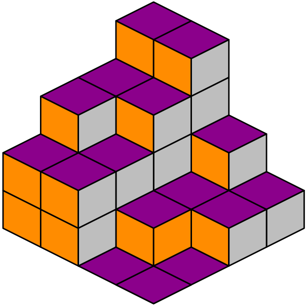
- Kasteleyn theory:
  - partition function: determinant of the Kasteleyn matrix K
  - correlations: minors of K^{-1}
- Non intersecting paths
  - Lindström-Gessel-Viennot
  - orthogonal polynomials

Plane partitions



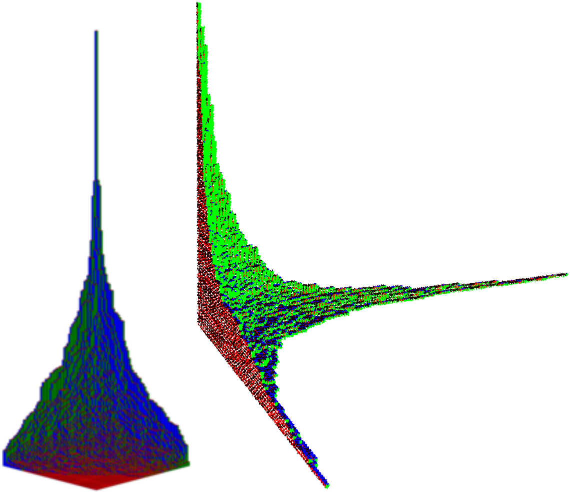
Dimers on the hexagonal lattice: tilings with rhombi
3D interpretation: piles of cubes in the corner of a room.
Partition function: McMahon's formula

$$\sum_{\pi} q^{|\pi|} = \prod_{j=1}^{\infty} \frac{1}{(1-q^j)^j}$$

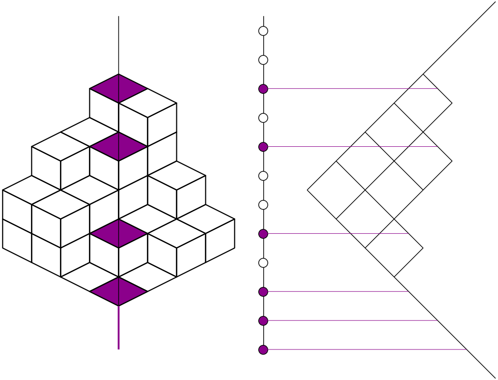


Plane partitions: limit shape and correlations

Limit shape:
Cerf–Kenyon (2001)
Correlations:
Okounkov–Reshetikhin (2003)



Idea: cut the plane partition in vertical slices:



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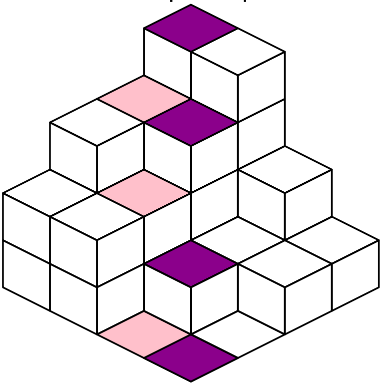


interlacing partitions: $\mu \prec \lambda$

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots$$

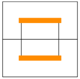
Transfer matrices with nice algebraic properties
Correlations:

$$\mathbb{P}(\text{particles at positions } (t_1, h_1), \dots (t_n, h_n)) = \det K((t_i, h_i), (t_j, h_j))$$

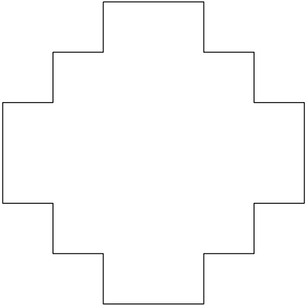
where

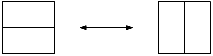
$$K((t,h),(t',h')) = [z^h w^{-h'}] \frac{\Phi(z,t)}{\Phi(w,t')} \frac{\sqrt{zw}}{z-w}$$

Aztec diamond

dimers on the square lattice: dominos 

Aztec diamond of size $n = 3$:



flip accessibility: 

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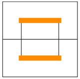
dimers on the square lattice: dominos 

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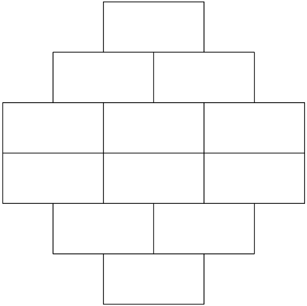
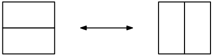


flip accessibility: 

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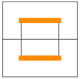
dimers on the square lattice: dominos 

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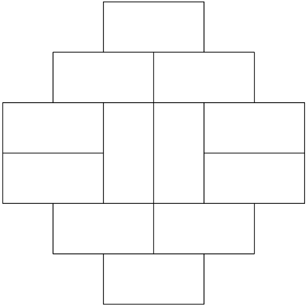
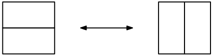


flip accessibility: 

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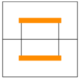
dimers on the square lattice: dominos 

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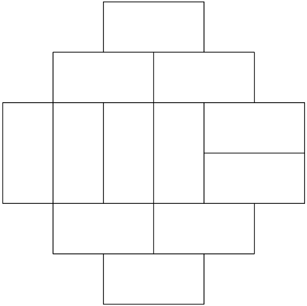
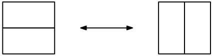


flip accessibility: 

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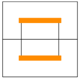
dimers on the square lattice: dominos 

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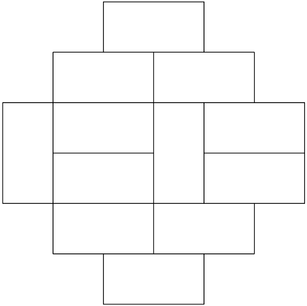
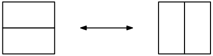


flip accessibility: 

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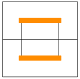
dimers on the square lattice: dominos 

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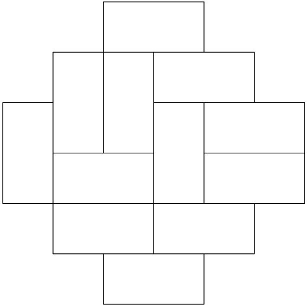
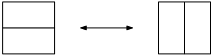


flip accessibility: 

Aztec diamond: partition function

- Number of tilings of size n: $2^{\frac{n(n+1)}{2}}$
- Refined partition function: if $N(T)$ miniminal number of flips to reach T from the horizontal configuration

$$Z(q) = \sum_T q^{N(T)} = \prod_{j=1}^{n} (1 + q^{2j-1})^{n-j+1}$$

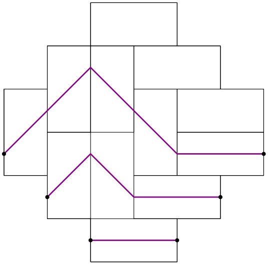
(Elkies Kuperberg Larsen Propp)

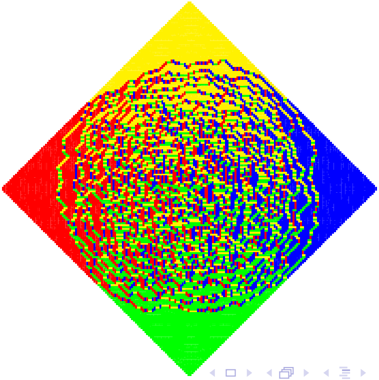
- Stanley

$$Z(q_i) \sum_T \prod q_i^{\#\text{flips on diag i}} = \prod_{1 \leq i \leq j \leq n} (1 + q_{2i-1} \cdots q_{2j-1})$$

Aztec diamond: limit shape

- encode tiling with non intersecting paths
- position of the highest path, Krawtchouk ensemble (Johansson)
- derivation of the *arctic circle theorem* (Jockusch Propp Shore)
- fluctuations aroung the limit shape: Airy process (Johansson)

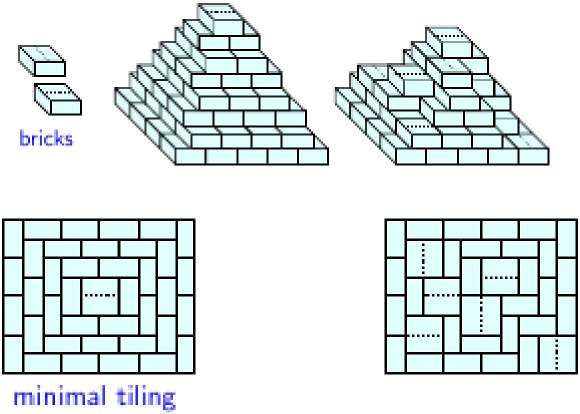




Aztec diamond: correlations

- Correlations between dominos is given by determinants of submatrices of K^{-1} (inverse Kasteleyn matrix)
- In general difficult to compute exactly
- explicit expression for the inverse Kasteleyn matrix (Chhita, Young 2013). No constructive proof.

Pyramid partitions



Pyramid partitions

- partition function (Szendrõi, Kenyon, Young)

$$Z(q) = \prod_{i \geq 1} \frac{(1 + q^{2i-1})^{2i-1}}{(1 - q^{2i})^{2i}}$$

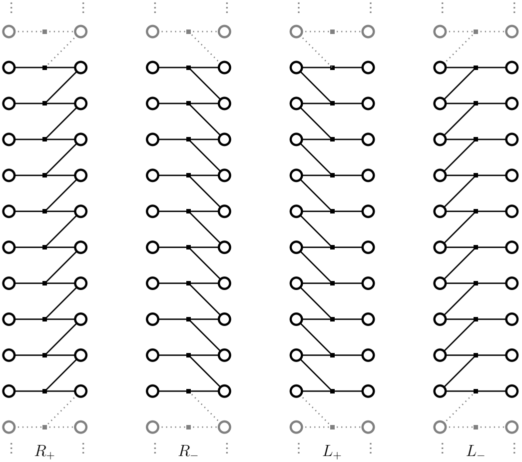
- limit shape (Kenyon-Okounkov):
- local statistics of dominos?

Our goal:

- unified framework to study these three examples (and many more)
- transfer matrix approach to solve these models
  - encode dimer configuration as particles
  - correlations of particles $\leftrightarrow$ (co)interlacing partitions (Schur process)
  - correlations of dimers
- explain the formula obtained by Chhita and Young
- study typical behaviour of such large structures

Elementary Rail Yard Graphs

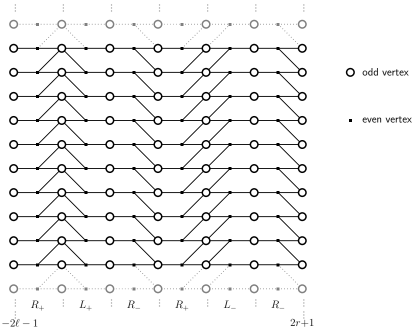
4 elementary graphs.



Can be glued together along columns.

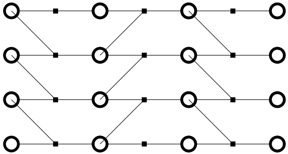
Rail Yard Graphs

Rail yard graphs: sequence of glued elementary graphs.

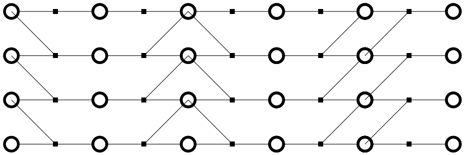


Structure encoded by a word in $L+/L-/R+/R-$.

- If only $L_{\pm}$ are used, faces of degree 6: **hexagonal lattice**



- If alternate $L_{\pm}$ and $R_{\pm}$, faces of degree 4 or degree 8 with vertices of degree 2:



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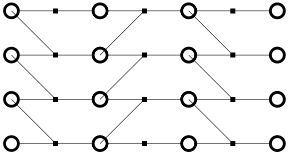
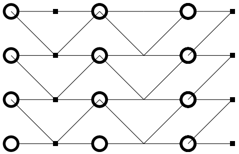


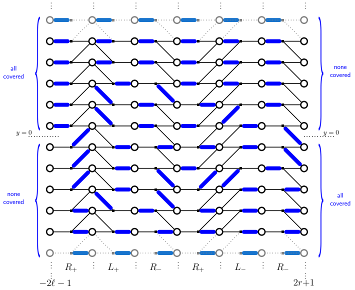
- If alternate $L_\pm$ and $R_\pm$, faces of degree 4 or degree 8 with vertices of degree 2: **square lattice**



Steep dimers on Rail Yard Graphs

boundary conditions: *vacuum*

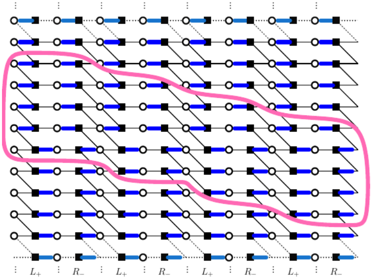
- vertices with negative ordinate on the left, and positive ordinate of the right are left unmatched.
- the other vertices on the boundary are covered by a dimer.



steep configurations: on each column, finite number of diagonal dimers.

Connection to tilings

- Only $L+/L-$: plane partitions / skew plane partitions (Okounkov-Reshetikhin, Borodin,...)
- Alternance $L\pm/R\pm$: steep domino tilings (considered by Bouttier, Chapuy, Corteel)
  ex: L+/R-/L+/R- corresponds to the Aztec diamond



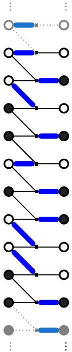
From dimers to Maya diagrams and partitions

From dimers, construct particle configurations $\{\bullet, \circ\}$ (*Maya diagrams*) on columns of odd vertices:

- Put $\bullet$ if vertex matched to the left.
- Put $\circ$ if vertex matched to the right.

For graphs $L+$, $L-$: plane partitions

- dimer configurations $\leftrightarrow$ interlacing $\bullet$ particles.
- number of diagonal edges: total displacement of $\bullet$ particules
- given two Maya diagrams, number of compatible dimer configurations is 1 if $\bullet$ particles interlaced, 0 otherwise.



Transfer matrix

State of *odd* columns encoded by vectors $|\lambda\rangle$ of a Hilbert space.
Transfer operators:

$$\Gamma_{L-}(x)|\lambda\rangle = \sum_{\mu \succ \lambda} x^{|\mu|-|\lambda|}|\mu\rangle, \quad \Gamma_{L+}(y)|\lambda\rangle = \sum_{\mu \prec \lambda} y^{|\lambda|-|\mu|}|\mu\rangle$$

Localisation operators: ψ_k, ψ_k^* create, annihilate particles at position k. $\psi_k \psi_k^*$ projector on diagrams with a particle at site k.
Commutation relations: $\Gamma_{L+}(x)$, $\Gamma_{L-}(y)$, $\Psi(z) = \sum_k \psi_k z^k$ satisfy nice commutation relations:

$$\Gamma_{L+}(y)\Gamma_{L-}(x) = \frac{1}{1-xy}\Gamma_{L-}(x)\Gamma_{L+}(y)$$

$$\Gamma_{L+}(y)\Psi(z) = \frac{1}{1-xz}\Psi(z)\Gamma_{L+}(y)$$

Case of plane partitions:

$$Z(q) = \langle\emptyset|\underbrace{\Gamma_{L+}(q^{m-1/2})\cdots\Gamma_{L+}(q^{1/2})}_{m}\underbrace{\Gamma_{L-}(q^{1/2})\cdots\Gamma_{L+}(q^{n-1/2})}_{n}|\emptyset\rangle$$

$$= \prod_{j=1}^{m}\prod_{k=1}^{n}\frac{1}{1-q^{i+j-1}}$$

Apply as many times as necessary the commutation relation Γ_{L+}/Γ_{L-}.

Graphs R+ and R-

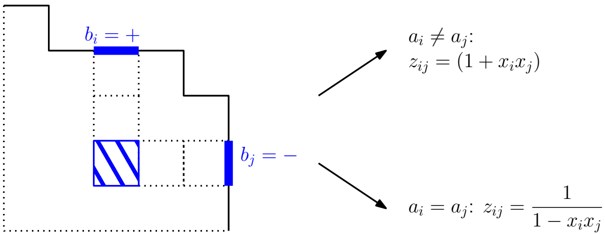
Exchange the role of white/black, left/right.
Now $\circ$ particles are interlacing (the corresponding partitions are cointerlacing).
Two new operators $\Gamma_{R-}(x), \Gamma_{R+}(y)$.

$$\Gamma_{R+}(y)\Gamma_{L-}(x) = (1 + xy)\Gamma_{L-}(x)\Gamma_{R+}(y)$$

Theorem

Let G is a rail yard graph, encoded by $\underline{a} = a_1 \cdots a_n$ and $\underline{b} = b_1 \cdots b_n$, $a_i \in \{L, R\}$, $b_i \in \{+, -\}$. Let $\underline{x} = (x_1, \ldots, x_n)$ the weights per diagonal dimer on each elementary graph. The partition function of the steep dimer configurations on G is

$$Z(\underline{a}, \underline{b}, \underline{x}) = \prod_{\substack{1 \le i < j \le n \\ b_i = +, b_j = -}} z_{ij}; \quad z_{ij} = \begin{cases} 1 + x_i x_j, & \text{if } a_i \neq a_j \\ (1 - x_i x_j)^{-1}, & \text{if } a_i = a_j. \end{cases}$$



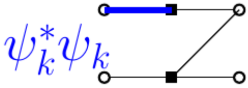
Computing particle probabilities:

$$\mathbb{P}(\bullet \text{ particles at } (t_1, h_1), \dots, (t_k, h_k)) = \frac{1}{Z} \times \langle \emptyset | \underbrace{\Gamma(x_1) \cdots \Gamma(x_{t_1})}_{t_1} \psi_{h_1} \psi^*_{h_1} \underbrace{\cdots}_{t_2 - t_1} \psi_{h_2} \psi^*_{h_2} \cdots | \emptyset \rangle$$

View ψ_{h_j} as some coefficient extraction from $\Psi(z_j)$ and again make use of commutation relations.

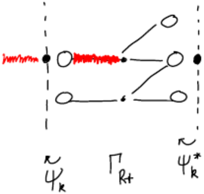
- Map from dimers to particles is local
- Reconstructing the dimer configuration from the Maya diagrams not local.

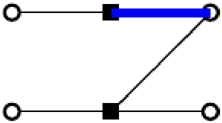
- Easy case: dimers in the *simple columns*



Equivalent to localisation of particles.

Dimers in double column: position not (locally) related to presence of particles. But:





$$\psi_k \Gamma_{R+} \psi_k^*$$

Bijection between configurations inside a "slice" by *rerouting* dimers around central vertices.

Let

$$F_i(z) = \frac{\prod_{m<i/2:R+}(1+x_m z)\prod_{m>i/2:L-}(1-x_m z^{-1})}{\prod_{m<i/2:L+}(1-x_m z)\prod_{m<i/2:R-}(1+x_m z^{-1})}$$

Define the matrix $C_{\alpha\beta}$ indexed by vertices of G (rows are odd vertices/columns are even vertices)

$$C_{\alpha\beta} = [z^{k_\alpha} w^{-k'_\beta}] \frac{F_{i\alpha(z)}}{F_{i'\beta}(w)} \frac{\sqrt{zw}}{z-w}$$

Theorem

The probability that edges $(e_1, \dots, e_n)$, with $e_i = (w_i, b_i)$ belong to the random configuration, is

$$(\textit{product of the weights}) \times \det C_{b_i, w_j}$$

C is an inverse of the Kasteleyn matrix on G.

Applications

- In the particular case of the Aztec diamond:
  - gives a constructive derivation of the formula for the inverse Kasteleyn matrix found by Chhita and Young
  - yet another derivation of the arctic circle theorem, fluctuations...

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    - gives a constructive derivation of the formula for the inverse Kasteleyn matrix found by Chhita and Young
    - yet another derivation of the arctic circle theorem, fluctuations...
- Mixtures of hexagonal/square lattice
- Special case of interest: pyramid partitions