Spectral theory for the q-Boson particle system

Alexei Borodin

A physicist's guide to solving the Kardar-Parisi-Zhang equation

$$\frac{\partial U}{\partial t} = \frac{1}{2}\frac{\partial^2 U}{\partial x^2} + \left(\frac{\partial U}{\partial x}\right)^2 + \dot{W}$$

space-time white noise

1. Think of the Cole-Hopf transform instead: $Z = e^U$ solves the SHE

$$\frac{\partial Z}{\partial t} = \frac{1}{2}\frac{\partial^2 Z}{\partial x^2} + \dot{W}\cdot Z$$

2. Look at the moments $\langle Z(t,x_1)\cdots Z(t,x_n)\rangle$. They are solutions of the quantum delta Bose gas evolution [Kardar '87], [Molchanov '87].

$$\frac{\partial}{\partial t}\langle Z(t,x_1)\cdots Z(t,x_n)\rangle = \frac{1}{2}\left(\sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2} + \sum_{i\neq j}\delta(x_i - x_j)\right)\langle Z(t,x_1)\cdots Z(t,x_n)\rangle$$

3. Use Bethe ansatz to solve it [Lieb-Liniger '63], [McGuire '64], [Yang '67-68].

4. Reconstruct the solution using the known moments:

The replica trick.

Possible mathematician's interpretation. Be wise – discretize!

1. Start with a good discrete system that formally converges to KPZ. This should give a solution that we ought to care about.

2. Find `moments' that would solve an integrable autonomous system of equations.

3. Reduce it to a direct sum of 1d eq's + boundary cond's and use Bethe ansatz to solve it, for arbitrary initial conditions.

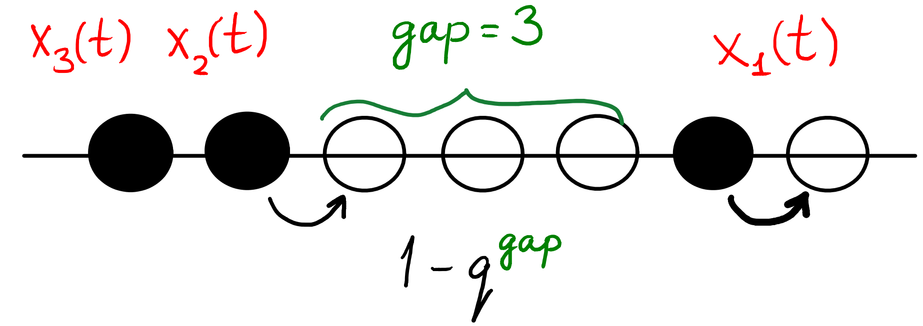
4. Reconstruct the solution using the known `moments' and take the limit to KPZ/SHE.

We can do 1–3 for two systems, **q-TASEP** and **ASEP**.

So far we can do 4 only for very special initial conditions.

q-TASEP [B-Corwin '11]

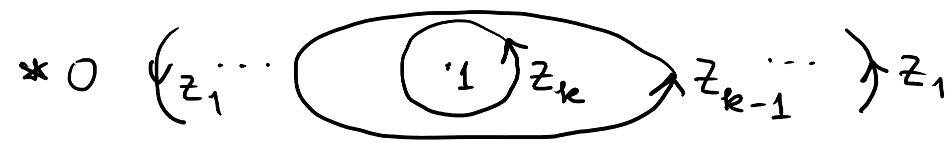
Patricles jump by one to the right.
Each particle has an independent exponential clock of rate $1-q^{gap}$, where `gap' is the number of empty spots ahead.



Theorem [B-Corwin '11], [B-C-Sasamoto '12], [B-C-Gorin-Shakirov '13]
For the q-TASEP with step initial data $\{x_n(0)=-n\}_{n\geq 1}$

$$\mathbb{E}\, q^{(x_{N_1}(t)+N_1)+\ldots+(x_{N_k}(t)+N_k)} = \frac{(-1)^k q^{\frac{k(k-1)}{2}}}{(2\pi i)^k} \oint \cdots \oint \prod_{A<B} \frac{z_A - z_B}{z_A - q z_B} \prod_{j=1}^{k} \frac{e^{(q-1)t z_j}}{(1-z_j)^{N_j}} \frac{dz_j}{z_j}$$

$(N_1 \geq N_2 \geq \ldots \geq N_k)$



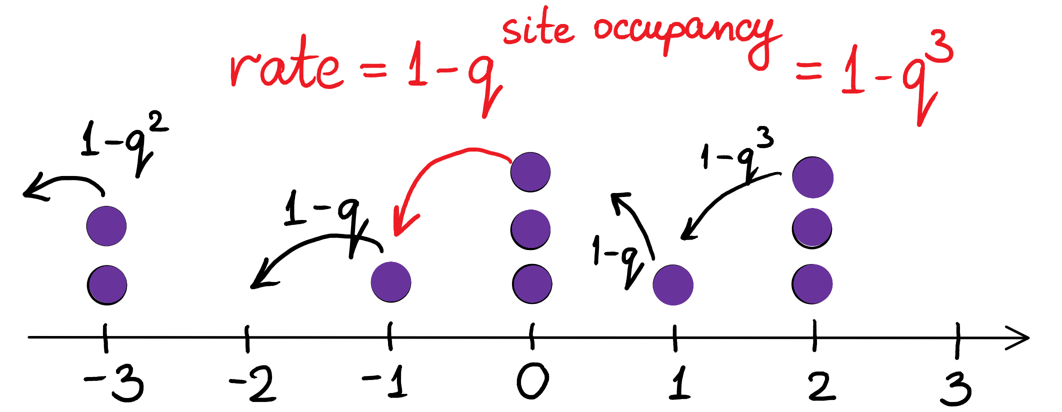
The original proof involved Macdonald processes. A simpler one?

q-Boson stochastic particle system [Sasamoto-Wadati '98]

Top particles at each location jump to the left by one indep. with rates $1-q^{\#\text{ of particles at the site}}$.

The generator is $\left(\vec{n}_j^- = (\cdots, n_j - 1, \cdots)\right)$

$$(Hf)(\vec{n}) = \sum_{\text{clusters } i} \left(1-q^{c_i}\right)\left(f\left(\vec{n}^-_{c_1+\dots+c_i}\right) - f(\vec{n})\right)$$



particles at $\vec{n} = (2,2,2,1,0,-1,-3,-3)$

clusters $\vec{c} = (3,1,3,1,2)$

Proposition [B-Corwin-Sasamoto '12] For a q-TASEP with finitely many particles on the right, $f(t,\vec{n}) = \mathbb{E}\left[\prod_{j=1}^{k} q^{x_{n_j}(t)+n_j}\right]$ is the unique solution of

$$\frac{d}{dt} f(t,\vec{n}) = (Hf)(t,\vec{n}), \qquad f(0,\vec{n}) = \mathbb{E}\left[\prod_{j=1}^{k} q^{x_{n_j}(0)+n_j}\right].$$

q-TASEP and q-Boson particle system are dual with respect to f.

q-TASEP gaps also evolve as a q-Boson particle system.

Solving q-Boson system means finding q-TASEP q-moments.

Coordinate integrability of the q-Boson system

The generator of k free (distant) particles is

$$(\mathcal{L}u)(\vec{n}) = (1-q)\sum_{i=1}^{k}(\nabla_i u)(\vec{n}),$$

∇_i is $(\nabla f)(x) = f(x-1) - f(x)$ acting in n_i

Define the boundary conditions as

$$\left(\nabla_i - q\nabla_{i+1}\right)u\Big|_{n_i = n_{i+1}} = 0 \qquad \text{for all } 1 \le i \le k-1$$

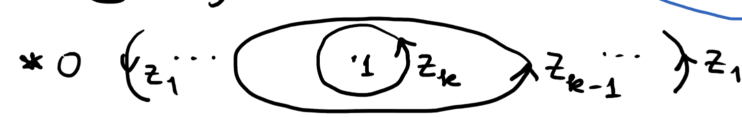
Proposition [B-Corwin-Sasamoto '12] If $u : \mathbb{Z}^k \times \mathbb{R}_{\ge 0} \to \mathbb{C}$ satisfies the free evolution equation $\frac{d}{dt}u = \mathcal{L}u$ and boundary conditions, then its restriction to $\{n_1 \ge \ldots \ge n_k\}$ satisfies the q-Boson system evolution equation $\frac{d}{dt}u = Hu$.

This suffices to re-prove the nested integral formula

$$\mathbb{E}\, q^{(x_{N_1}(t)+N_1)+\ldots+(x_{N_k}(t)+N_k)} = \frac{(-1)^k q^{\frac{k(k-1)}{2}}}{(2\pi i)^k} \oint \cdots \oint \prod_{A<B} \frac{z_A - z_B}{z_A - q z_B} \prod_{j=1}^{k} \frac{e^{(q-1)t z_j}}{(1-z_j)^{N_j}} \frac{dz_j}{z_j}$$

free evolution

boundary conditions



Algebraic integrability of the q-Boson system

[Sasamoto-Wadati '98] showed that periodic H is the image of a q-Boson Hamiltonian

$$\mathcal{H} = -\sum_{j=1}^{M} \left(B_{j-1}^{\dagger} - B_{j}^{\dagger} \right) B_j , \qquad \overbrace{[B_i, B_j^{\dagger}] = q^{-2N_i} \delta_{ij} , \quad [N_i, B_j] = -B_j \delta_{ij} , \quad [N_i, B_j^{\dagger}] = B_j^{\dagger} \delta_{ij} ,}^{q\text{-Boson algebra}}$$

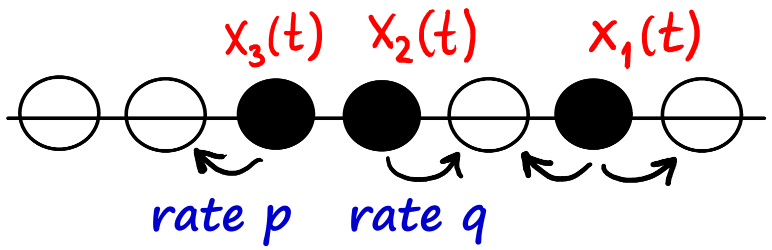
under (ν_j is the number of particles at site j)

$$(B_j f)(\vec{\nu}) = \frac{1-q^{\nu_j}}{1-q} f(\dots, \nu_j - 1, \dots), \quad (B_j^{\dagger} f)(\vec{\nu}) = f(\dots, \nu_j + 1, \dots), \quad (N_j f)(\vec{\nu}) = \nu_j f(\vec{\nu}),$$

and that $\mathcal{H}$ arises from the monodromy matrix of a quantum integrable system with trigometric R-matrix, same as in XXZ/ASEP.

Actually, ASEP has a parallel story.

The ASEP story (briefly)



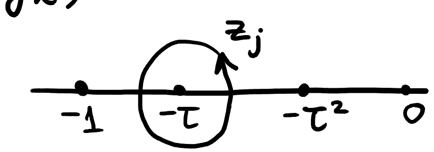
$0<p<q<1, \quad p+q=1$

Set $\tau = p/q < 1$, $\quad n_y(t) = \#\{m \geq 1 : x_m(t) \geq y\}$, $\quad Q_y = \dfrac{\tau^{n_y} - \tau^{n_{y-1}}}{\tau - 1}$.

Theorem [B-C-Sasamoto, 2012] For ASEP with step initial data $\{x_n(0) = -n\}_{n\geq 1}$

$$\mathbb{E}\left[Q_{y_1}(t)\cdots Q_{y_k}(t)\right] = \frac{\tau^{k(k-1)/2}}{(2\pi i)^k} \oint \cdots \oint \prod_{A<B} \frac{z_A - z_B}{z_A - \tau z_B}$$

$(y_1 > y_2 > \ldots > y_k)$



$$* \prod_{j=1}^{k} e^{-\frac{z_j (p-q)^2 t}{(1+z_j)(p+qz_j)}} \left(\frac{1+z_j/\tau}{1+z_j}\right)^{y_j+1} \frac{dz_j}{\tau + z_j}$$

The dual of ASEP is another ASEP [Schutz '97], which is also integrable in both coordinate and algebraic sense. [Tracy-Widom '08+] used Bethe ansatz approach to study ASEP's transition probabilities.

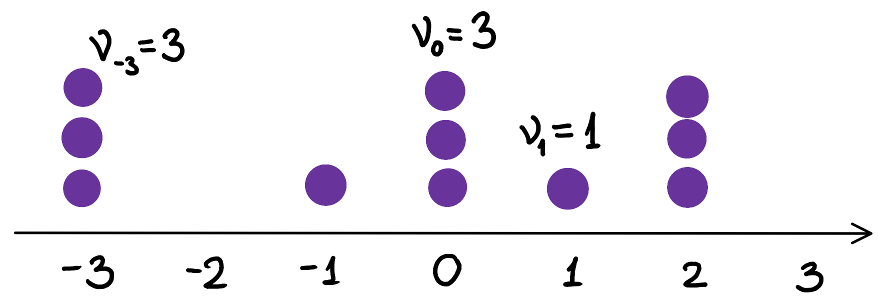
PT-invariance

To be able to solve q-Boson system (thus q-TASEP) for general initial conditions, we want to diagonalize H.

It is not self-adjoint, but PT-invariance (under joint space reflection and time inversion) effectively replaces self-adjointness:

Let μ be an invariant product measure

$$\mu(d\nu) = \bigotimes_{n\in\mathbb{Z}} \mu_0(d\nu_n), \qquad \mu_0(k) = \text{const}\cdot \begin{cases} \alpha^k / k_q!, & k \geq 0, \\ 0, & k < 0. \end{cases}$$



Then $H = PH^*P^{-1}$ in $L^2(\{\nu_n\}_{n\in\mathbb{Z}}, \mu)$

with $(Pf)(\{\nu_n\}_{n\in\mathbb{Z}}) = f(\{\nu_{-n}\}_{n\in\mathbb{Z}})$ (parity transformation).

Coordinate Bethe ansatz [Bethe '31]

(Algebraic) eigenfunctions for a sum of 1d operators

$$(\mathcal{L}\Psi)(\vec{x}) = \sum_{i=1}^{k} (L_{x_i}\Psi)(\vec{x}), \qquad \vec{x} = (x_1, \dots, x_k) \in \mathcal{X}^k,$$

that satisfy boundary conditions

$$B_{x_i x_{i+1}} \Psi\Big|_{x_i = x_{i+1}} = 0, \qquad 1 \le i \le k-1, \qquad B : \{\text{functions on } \mathcal{X}^2\} \circlearrowleft$$

Example: $B(g(x,y)) = (\nabla_x - q\nabla_y) g(x,y)$

can be found via

1. Diagonalizing 1d operator $\quad L\psi_z = \lambda_z \psi_z, \quad \psi_z : \mathcal{X} \to \mathbb{C}$

2. Taking linear combinations $\quad \Psi_{\vec{z}}(\vec{x}) = \sum_{\sigma \in S(k)} A_\sigma(\vec{z}) \, \psi_{z_{\sigma(1)}}(x_1) \cdots \psi_{z_{\sigma(k)}}(x_k)$

3. Choosing

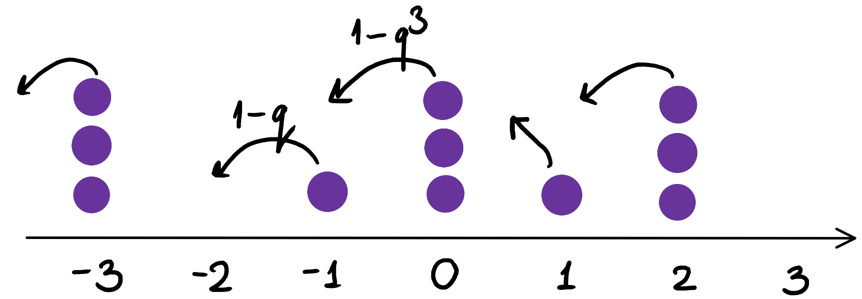
$$A_\sigma(\vec{z}) = \operatorname{sgn}(\sigma) \prod_{a > b} \frac{S(z_{\sigma(a)}, z_{\sigma(b)})}{S(z_a, z_b)}, \qquad S(z_1, z_2) = \frac{B(\psi_{z_1}(x)\psi_{z_2}(y))\big|_{y=x}}{\psi_{z_1}(x)\,\psi_{z_2}(x)}$$

No quantization of spectrum (Bethe equations) in infinite volume.

Left and right eigenfunctions

For q-Boson gen. $(Hf)(\vec{n}) = \sum_{\text{clusters } i} (1-q^{c_i})\left(f(\vec{n}^{\,-}_{c_1+\dots+c_i}) - f(\vec{n})\right)$ that reduces to

$$(\mathcal{L}u)(\vec{n}) = (1-q)\sum_{i=1}^{k} (\nabla_i u)(\vec{n}), \qquad (\nabla_i - q\nabla_{i+1})u\big|_{n_i=n_{i+1}} = 0,$$



particles at $\vec{n} = (2,2,2,1,0,-1,-3,-3)$

clusters $\vec{c} = (3,1,3,1,2)$

Bethe ansatz yields ($z_1,\dots,z_k \in \mathbb{C}\setminus\{1\}$)

$$\Psi^{\ell}_{\vec{z}}(\vec{n}) = \sum_{\sigma\in S(k)} \prod_{a>b} \frac{z_{\sigma(a)} - q z_{\sigma(b)}}{z_{\sigma(a)} - z_{\sigma(b)}} \prod_{j=1}^{k} \frac{1}{(1-z_{\sigma(j)})^{n_j}}$$

$$\Psi^{r}_{\vec{z}}(\vec{n}) = \frac{1}{C_q(\vec{n})} \sum_{\sigma\in S(k)} \prod_{a>b} \frac{z_{\sigma(a)} - q^{-1} z_{\sigma(b)}}{z_{\sigma(a)} - z_{\sigma(b)}} \prod_{j=1}^{k} (1-z_{\sigma(j)})^{n_j}$$

with $C_q(\vec{n}) = (-1)^k q^{-k(k-1)/2} (c_1)!_q (c_2)!_q \cdots$ and

$$H\Psi^{\ell}_{\vec{z}} = (1-q)(z_1+\dots+z_k)\Psi^{\ell}_{\vec{z}}, \qquad H^{\text{transpose}}\Psi^{r}_{\vec{z}} = (1-q)(z_1+\dots+z_k)\Psi^{r}_{\vec{z}}.$$

Direct and inverse Fourier type transforms

Let

$$\mathcal{W}^k = \{ f : \{ n_1 \geq \dots \geq n_k \mid n_j \in \mathbb{Z} \} \to \mathbb{C} \text{ of compact support} \}$$

$$\mathcal{C}^k = \mathbb{C}\left[(z_1 - 1)^{\pm 1}, \dots, (z_k - 1)^{\pm 1} \right]^{S(k)} = \text{symmetric Laurent poly's in } (z_j - 1),\ 1 \leq j \leq k.$$

Direct tranform: $\mathcal{F} : \mathcal{W}^k \to \mathcal{C}^k$

$$\mathcal{F} : f \longmapsto \sum_{n_1 \geq \dots \geq n_k} f(\vec{n}) \cdot \Psi^r_{\vec{z}}(\vec{n}) =: \langle f, \Psi^r_{\vec{z}} \rangle_{\mathcal{W}}$$

Inverse transform: $\mathcal{J} : \mathcal{C}^k \to \mathcal{W}^k$

$$\mathcal{J} : G \longmapsto (q-1)^k q^{-\frac{k(k-1)}{2}} \frac{1}{(2\pi i)^k k!} \oint_{\substack{|w_j| = R > 1 \\ j = 1, \dots, k}} \cdots \oint \det \left[\frac{1}{q w_i - w_j} \right]_{i,j=1}^k \prod_{j=1}^k \frac{w_j}{1 - w_j} \Psi^\ell_{\vec{w}}(\vec{n}) G(\vec{w}) \, d\vec{w}$$

$$=: \langle G, \Psi^\ell(\vec{n}) \rangle_{\mathcal{C}}$$

Contour deformations

Inverse transform: $\mathcal{J} : \mathcal{C}^k \to \mathcal{W}^k$

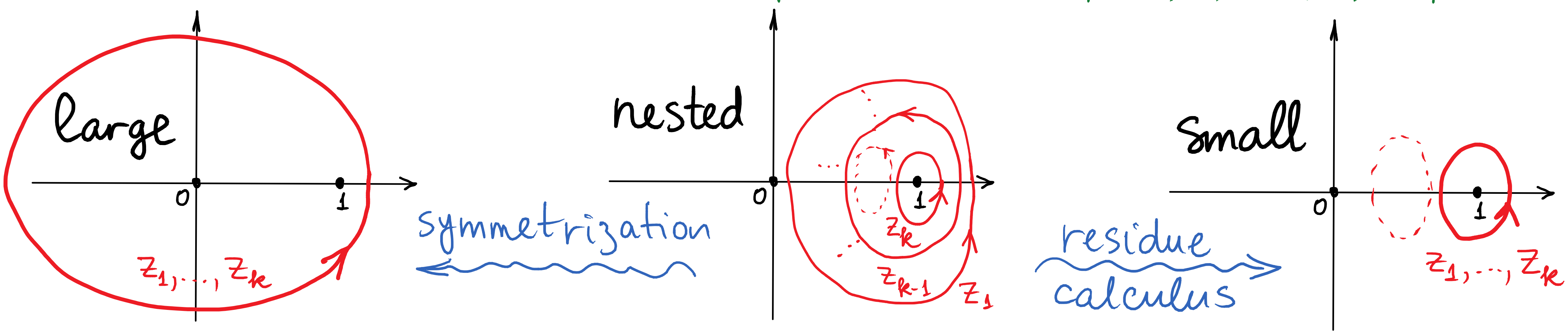
$$\mathcal{J} : G \longmapsto (q-1)^k q^{-\frac{k(k-1)}{2}} \frac{1}{(2\pi i)^k k!} \underbrace{\oint \cdots \oint}_{\text{large}} \det\left[\frac{1}{qw_i - w_j}\right]_{i,j=1}^{k} \prod_{j=1}^{k} \frac{w_j}{1-w_j}\, \Psi^{\ell}_{\vec w}(\vec n)\, G(\vec w)\, d\vec w$$

$$= \frac{1}{(2\pi i)^k} \underbrace{\oint \cdots \oint}_{\text{nested}} \prod_{a<b} \frac{z_a - z_b}{z_a - q z_b} \prod_{j=1}^{k} \frac{1}{(1-z_j)^{n_j+1}}\, G(\vec z)\, d\vec z =$$

different sets of Bethe states

$$= \sum_{\substack{\lambda = (\lambda_1 \geq \dots \geq \lambda_\ell \geq 0) \\ \lambda_1 + \dots + \lambda_\ell = k \quad \lambda = 1^{m_1} 2^{m_2} \dots}} \frac{(q-1)^k q^{-k^2/2}}{m_1! m_2! \cdots} \frac{1}{(2\pi i)^\ell} \underbrace{\oint \cdots \oint}_{\text{small}} \det\left[\frac{1}{q^{\lambda_i} w_i - w_j}\right]_{i,j=1}^{\ell} \prod_{j=1}^{\ell} \frac{w_j\, q^{\lambda_j^2/2}}{(1-w_j)\cdots(1-q^{\lambda_j - 1} w_j)}\, \Psi^{\ell}_{\vec w \circ \lambda}(\vec n)\, G(\vec w \circ \lambda)\, dw_1 \cdots dw_\ell$$

$$\vec w \circ \lambda = (w_1, q w_1, \dots, q^{\lambda_1 - 1} w_1, w_2, q w_2, \dots, q^{\lambda_2 - 1} w_2, \dots, w_\ell, q w_\ell, \dots, q^{\lambda_\ell - 1} w_\ell)$$



Plancherel isomorphism theorem

Theorem [B-Corwin-Petrov-Sasamoto '13] On spaces $\mathcal{W}^k$ and $\mathcal{C}^k$, operators $\mathcal{F}$ and $\mathcal{J}$ are mutual inverses of each other.

Isometry:

$$\langle f, g\rangle_{\mathcal{W}} = \langle \mathcal{F}f, \mathcal{F}g\rangle_{\mathcal{C}} \quad \text{for} \quad f, g \in \mathcal{W}^k$$

$$\langle F, G\rangle_{\mathcal{C}} = \langle \mathcal{J}F, \mathcal{J}G\rangle_{\mathcal{W}} \quad \text{for} \quad F, G \in \mathcal{C}^k$$

Biorthogonality:

$$\langle \Psi^{\ell}_{\bullet}(\vec{m}), \Psi^{r}_{\bullet}(\vec{n})\rangle_{\mathcal{C}} = \delta_{\vec{m},\vec{n}}$$

in a certain weak sense

$$\langle \Psi^{\ell}_{\vec{z}}(\cdot), \Psi^{r}_{\vec{w}}(\cdot)\rangle_{\mathcal{W}} = \frac{1}{k!} \prod_{a\neq b} \frac{z_a - qz_b}{z_a - z_b} \prod_{j=1}^{k} \frac{1}{1-z_j} \det\left[\delta(z_i - w_j)\right]_{i,j=1}^{k}$$

This diagonalizes the generator of the q-Boson stochastic system and proves completeness of the Bethe ansatz for it.

Back to the q-Boson particle system

Corollary The (unique) solution of the q-Boson evolution equation

$$\frac{d}{dt} f(t,\vec{n}) = (Hf)(t,\vec{n}), \qquad f(0,\vec{n}) = f_0, \qquad (Hf)(\vec{n}) = \sum_{\text{clusters } i} (1-q^{c_i})\left(f(\vec{n}^{\,-}_{c_1+\dots+c_i}) - f(\vec{n})\right)$$

has the form

$$f(t,\vec{n}) = \mathcal{J}\Big(e^{\overbrace{t(q-1)(z_1+\dots+z_k)}^{\text{eigenvalue of } H \text{ corr. to } \Psi_{\vec{z}}}} \mathcal{F} f_0\Big) = \frac{1}{(2\pi i)^k} \underbrace{\oint \cdots \oint}_{\text{nested}} \prod_{a<b} \frac{z_a - z_b}{z_a - q z_b} \prod_{j=1}^{k} \frac{e^{t(q-1)z_j}}{(1-z_j)^{n_j+1}} \langle f_0, \Psi^r_{\vec{z}} \rangle_{\mathcal{W}} \, d\vec{z}$$

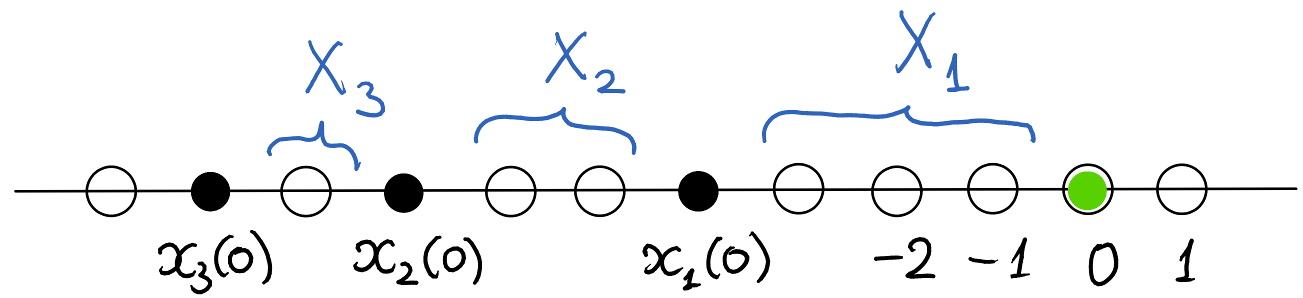
The computation of $\mathcal{F} f_0$ can still be difficult. It is, however, automatic if $f_0 = \mathcal{J} G \Rightarrow \mathcal{F} f_0 = \mathcal{F}\mathcal{J} G = G$.

In the case of q-TASEP's step initial condition

$$f_0(\vec{n}) = \mathbb{1}_{\{n_i \geq 1,\ 1 \leq i \leq k\}}, \qquad G(\vec{z}) = q^{k(k-1)/2} \prod_{j=1}^{k} \frac{z_j - 1}{z_j}.$$

Half-equilibrium initial condition

For q-TASEP, we define



$$x_1(0) = -1 - X_1, \quad x_2(0) = -1 - x_1(0) - X_2, \quad \dots, \quad x_n(0) = -1 - x_{n-1}(0) - X_n, \quad \dots$$

where $X_1, X_2, \dots$ are i.i.d. with $\mathrm{Prob}\{X = k\} = \mathrm{const}_\alpha \cdot \begin{cases} \dfrac{\alpha^k}{(1-q)\cdots(1-q^k)}, & k \geq 0, \\ 0, & k < 0. \end{cases}$

Then

$$f_0(\vec{n}) = \prod_{j=1}^{k} \left(1 - \frac{\alpha}{q^j}\right)^{-n_j} \mathbb{1}_{n_j > 0} = \mathcal{J}\left(q^{\frac{k(k-1)}{2}} \prod_{j=1}^{k} \frac{z_j - 1}{z_j - \alpha/q} \right).$$

Hence

$$\mathbb{E}\, q^{(x_{N_1}(t)+N_1)+\dots+(x_{N_k}(t)+N_k)} = \frac{(-1)^k q^{\frac{k(k-1)}{2}}}{(2\pi i)^k} \underbrace{\oint \cdots \oint}_{\text{nested}} \prod_{A<B} \frac{z_A - z_B}{z_A - q z_B} \prod_{j=1}^{k} \frac{e^{(q-1)t z_j}}{(1-z_j)^{N_j}} \frac{dz_j}{z_j - \alpha/q}.$$

Large time asymptotics of q-TASEP and KPZ in [B-Corwin-Ferrari '12].

Extension to equilibrium: [Imamura-Sasamoto '12] via replica, [BCF-Veto '14]

Other systems

1. A very similar story takes place for ASEP/XXZ in infinite volume [B-Corwin-Petrov-Sasamoto '14]. Analogous results are contained in [Babbitt-Thomas '77] for SSEP/XXX, [Babbitt-Gutkin '90], yet complete proofs seem to be inaccessible.
2. Our Plancherel theorem nontrivially degenerates to two different discrete versions of the delta Bose gas (one of them was treated by [Van Diejen '04], [Macdonald '71]), and further down to the standard continuous delta Bose gas (where we recover results of [Yang '68], [Oxford '79], [Heckman-Opdam '97]).

Different degenerations require different form of $\mathcal{J}$!

Degenerations of wave functions

$$\sum_{\sigma\in S(k)} \sigma\left[\prod_{a>b}\frac{z_a-qz_b}{z_a-z_b}\prod_{j=1}^{k}\frac{1}{(1-z_j)^{n_j}}\right]$$

q-Boson particle system

$z_j \gg 1$

$q=e^{-\varepsilon}\to 1$

$1-z_j=O(\varepsilon)$

$$\sum_{\sigma\in S(k)} \sigma\left[\prod_{a>b}\frac{z_a-qz_b}{z_a-z_b}\prod_{j=1}^{k} z_j^{-n_j}\right]$$

Hall-Littlewood polynomials

$$\sum_{\sigma\in S(k)} \sigma\left[\prod_{a>b}\frac{z_a-z_b-1}{z_a-z_b}\prod_{j=1}^{k} z_j^{-n_j}\right]$$

semi-discrete Brownian polymer

$q=e^{-\varepsilon}\to 1$

$1-z_j=O(\varepsilon)$

rescale near crit. pt. of e^{tz}/z^{n} with $t\sim\sqrt{n}\to\infty$ (time is involved)

$$\sum_{\sigma\in S(k)} \sigma\left[\prod_{a>b}\frac{z_a-z_b-1}{z_a-z_b}\prod_{j=1}^{k} e^{x_j z_j}\right]$$

continuous delta Bose gas / KPZ

Noncommutative harmonic analysis

Plancherel theorems often ride on top of noncommutative harmonic analysis statements via imposing additional symmetry.

<u>A classical example</u> [Frobenius, 1896] Let K be a finite group. Its double $G=K\times K$ acts on $L^2(K)$ by left and right argument shifts:

$$((g,h)\cdot F)(x) = F(g^{-1}xh).$$

Decomposition on irreducibles has the form

$$L^2(K) = \bigoplus_{\pi\in \mathrm{Irr}(K)} \pi \otimes \pi^*.$$

For functions that are inv. wrt conjugation this gives Plancherel:

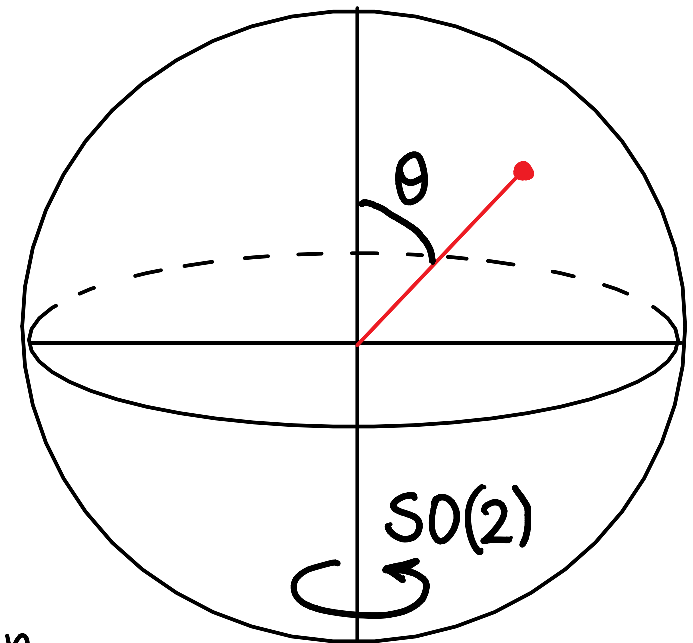
$$F(x) = \sum_{\pi\in\mathrm{Irr}(K)} \underbrace{\frac{\dim^2\pi}{|K|}}_{\text{Plancherel measure}} \frac{\chi^{\pi}(x)}{\dim\pi} \cdot \underbrace{\left\langle F, \frac{\chi^{\pi}}{\dim\pi}\right\rangle_{L^2(K)}}_{\text{direct transform}}$$

characters (traces) of irr. representations

Harmonic analysis on symmetric spaces

For a Lie group G and its subgroup K, G acts in $L^2(G/K)$. Plancherel theorem for K-inv functions captures the decomposition on irreps of G and diagonalizes K-invariant part of the Laplacian on G/K.

$G=SO(3)$, $K=SO(2)$, $G/K=S^2$



$$\Delta_{inv} = \frac{\partial^2}{\partial\theta^2} + \frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\theta}$$

Eigenfunctions $P_\ell(\cos\theta)$, with Legendre poly's $\{P_\ell\}_{\ell\geq 0}$

$$\int_0^\pi P_m(\cos\theta)P_n(\cos\theta)\sin\theta\, d\theta = \int_{-1}^{1} P_m(x)\,P_n(x)\,dx = c_n\,\delta_{mn}$$

For real semi-simple G and maximal compact K, this is the celebrated theory of [Gelfand-Naimark, 1946+] and [Harish-Chandra, 1947+].

In the case of the continuous delta Bose gas, $H = \frac{1}{2}\left(\Delta + \sum_{i \neq j} \delta(x_i - x_j)\right)$, the Plancherel theorem rides on top of harmonic analysis for the degenerate (or graded) Hecke algebra of type A, that is generated by permutations and $\left\{\frac{\partial}{\partial x_i}\right\}_{i=1}^{n}$ subject to relations

$$r_j \cdot \frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_{r_j(i)}} \cdot r_j = \begin{cases} -1, & i=j \\ 1, & i=j+1 \\ 0, & \text{otherwise} \end{cases} \qquad r_j = (j\ j+1) \in S(n).$$

Its representation in $C^\infty(\mathbb{R}^n)$ is given by [Yang '67], [Gutkin '82]

$$\big(Q(r_j)f\big)(x_1,\dots,x_n) = f(\dots, x_{j+1}, x_j, \dots) - \int_0^{x_j - x_{j+1}} f(\dots, x_j - t, x_{j+1}+t, \dots)\,dt, \quad Q\left(\frac{\partial}{\partial x_i}\right) = \frac{\partial}{\partial x_i}.$$

$C^\infty(\mathbb{R}^n)$ is embedded into continuous functions satisfying correct boundary conditions via $f \mapsto f_+$, $f_+(\sigma^{-1} \cdot x) = \big(Q(\sigma)f\big)(x)$, $x_1 \geq \dots \geq x_n$, $\sigma \in S(n)$.

Restricting the harmonic analysis to symmetric functions gives the Plancherel theorem [Heckman-Opdam '97].

- One of the discretizations of the delta Bose gas that we obtain, for which the wave functions are the Hall-Littlewood polynomials, is connected to the harmonic analysis on G/K, where

$$G = GL(n, F), \qquad K = GL(n, \mathcal{O})$$

F is a non-archimedean local field (like $\mathbb{Q}_p$ or $\mathbb{F}_p((t))$),

$\mathcal{O}$ is its ring of integers ($\mathbb{Z}_p$ or $\mathbb{F}_p[[t]]$), $q = p^{-1}$. [Macdonald '71]

- The other discretization corresponds to $H = \sum_{i=1}^{k} \nabla_{n_i} + \sum_{i<j} \delta_{n_i = n_j}$ that arises from moments of the semi-discrete Brownian polymer. First in this case, and later in the q-case, [Takeyama '12, '14] constructed a representation of a rational twist of the affine Hecke algebra, but so far there is no harmonic analysis.

Mysterious connection to Macdonald polynomials

Macdonald polynomials $P_\lambda(x_1,\dots,x_N) \in \mathbb{Q}(q,t)[x_1,\dots,x_N]^{S(N)}$ labelled by partitions $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_N \geq 0)$ form a basis in symmetric polynomials in N variables over $\mathbb{Q}(q,t)$. They diagonalize

$$\mathcal{D}_1^{(N)} = \sum_{i=1}^{N} \left(\prod_{a<b} (x_a - x_b)^{-1} \, T_{t,x_i} \prod_{a<b} (x_a - x_b) \right) T_{q,x_i} = \sum_{i=1}^{N} \prod_{j \neq i} \frac{t x_i - x_j}{x_i - x_j} \, T_{q,x_i}$$

$(T_q f)(z) = f(qz)$

with (generically) pairwise different eigenvalues

$$\mathcal{D}_1^{(N)} P_\lambda = \left(q^{\lambda_1} t^{N-1} + q^{\lambda_2} t^{N-2} + \dots + q^{\lambda_N} \right) P_\lambda .$$

Proposition [B-Corwin '13] Assume t=0. Then

$$\left[\left(\mathcal{D}_1^{(N)} \right)^k, \text{ multiplication by } (x_1 + \dots + x_N) \right] = (1 - q^k)\, x_N \left(\mathcal{D}_1^{(N-1)} - \mathcal{D}_1^{(N)} \right) \left(\mathcal{D}_1^{(N)} \right)^{k-1}.$$

Mysterious connection to Macdonald polynomials

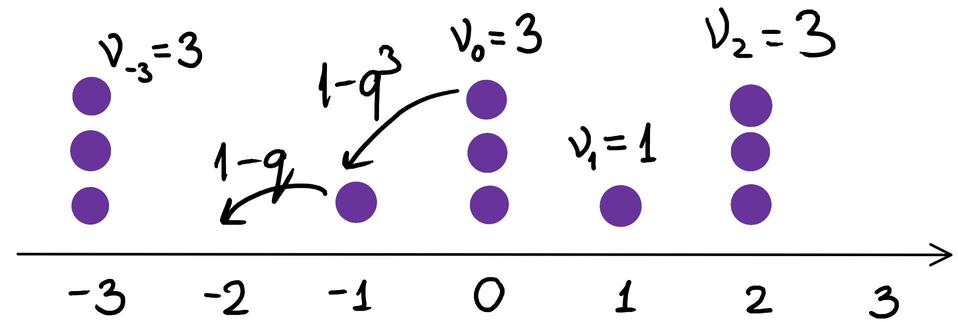
$$\left[\left(\mathcal{D}_1^{(N)}\right)^k, \text{ multiplication by } (x_1+\dots+x_N)\right] = (1-q^k)\, x_N \left(\mathcal{D}_1^{(N-1)} - \mathcal{D}_1^{(N)}\right)\left(\mathcal{D}_1^{(N)}\right)^{k-1}.$$

Corollary For any symmetric analytic $F(x_1, \dots, x_N)$

$$f(t, \vec{\nu}) = \begin{cases} 0, & \text{if at least one } \nu_{-m} > 0,\ m \geq 0 \\ e^{-t(x_1+\dots+x_N)} \left(\mathcal{D}_1^{(1)}\right)^{\nu_1} \left(\mathcal{D}_1^{(2)}\right)^{\nu_2} \cdots \left(\mathcal{D}_1^{(N)}\right)^{\nu_N} e^{t(x_1+\dots+x_N)} F(x_1, \dots, x_N)\Big|_{x_1=\dots=x_N=1} & \text{otherwise} \end{cases}$$

solves the evolution equation of the q-Boson system $\frac{d}{dt} f(t, \vec{n}) = (Hf)(t, \vec{n})$, where H is the generator.

q-TASEP's step initial condition corresponds to $F(x_1, \dots, x_N) \equiv 1$.



How does this relate to Plancherel theory?

Summary

- The wish to analyze q-TASEP for arbitrary initial conditions lead to new Plancherel theory of Bethe type.
- Its degenerations include that for quantum delta Bose gas, and q-TASEP moments do not suffer from intermittency, thus can be used for rigorous replica like computations.
- Similar Plancherel theory exists for ASEP.
- The connection to Macdonald processes is apparent but remains somewhat mysterious.
- More work needed to turn the algebraic advances into new analytic results.