About random matrices

Joint works with
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Probability Seminar
Stanford University
Monday March 29, 2021

Outline

Historical roots

High dimensional phenomena

Beyond Girko matrices

Spectrum edge

Ginibre

Random matrices

- **Wishart** ≈ 1930
  Empirical covariance matrices
  Mathematical statistics

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  Condition number of linear systems
  Computer science and numerical analysis

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- **Voiculescu** $\approx$ 1990
  Freeness as a high dimensional phenomenon
  Operator algebra

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Girko matrices

$$X = \begin{pmatrix} X_{11} & \cdots & X_{1n} \\ & \vdots & \\ X_{n1} & \cdots & X_{nn} \end{pmatrix}$$

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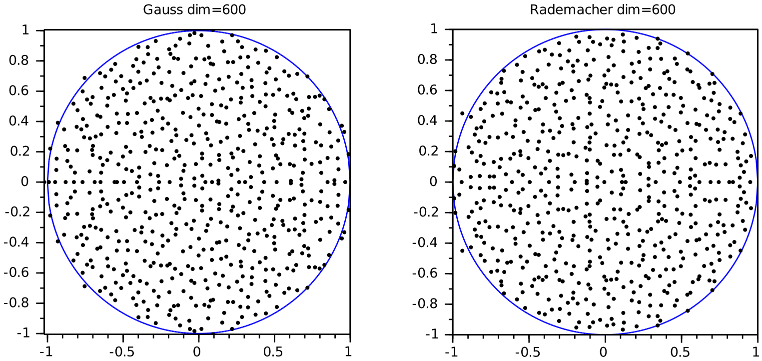
- Spectrum multiset and empirical spectral distribution

$$\{\lambda_1(A), \ldots, \lambda_n(A)\} = \{z \in \mathbb{C} : \det(A - zI) = 0\}$$

$$\mu_A = \frac{1}{n} \sum_{i=1}^{n} \delta_{\lambda_i(A)}$$

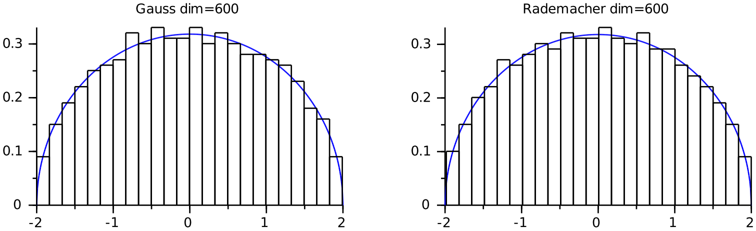
Universality $\frac{X}{\sqrt{n}}$ (Girko)



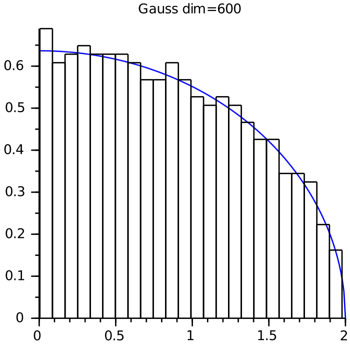

```
julia> eigvals(sign.(randn(dim,dim))/sqrt(dim))
```

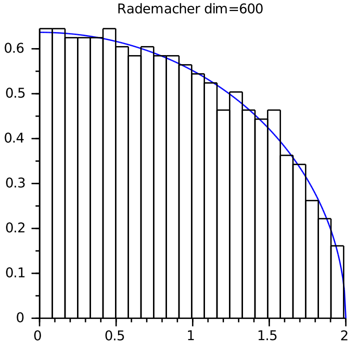
Random but not independent - Collective phenomenon

Universality $\frac{X+X^*}{\sqrt{2n}}$ (Wigner)



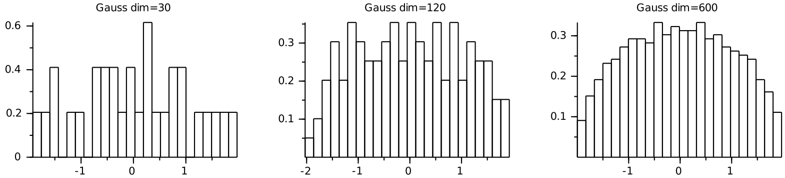
Universality $\frac{\sqrt{XX^*}}{\sqrt{n}}$ (Marchenko – Pastur)





High dimensional phenomenon

- High dimensional phenomenon (NLLLN, here in Wigner case)



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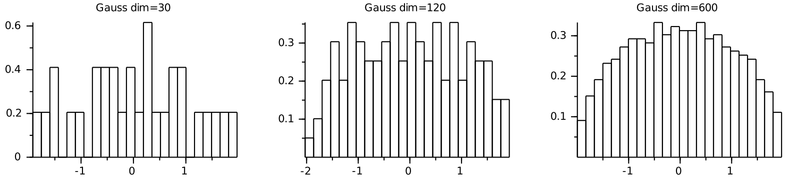

- Why this $\frac{1}{\sqrt{n}}$ normalization? Law of large numbers!

$$\frac{X}{\sqrt{n}} = \begin{pmatrix} \frac{X_{11}}{\sqrt{n}} & \cdots & \frac{X_{1n}}{\sqrt{n}} \\ & \vdots & \\ \frac{X_{n1}}{\sqrt{n}} & \cdots & \frac{X_{nn}}{\sqrt{n}} \end{pmatrix} \approx \text{Unitary}$$

Basic theorems

- Wigner

$$\mu_{\frac{X+X^*}{\sqrt{2n}}} \underset{n\to\infty}{\longrightarrow} \frac{\sqrt{4-x^2}}{2\pi}\mathbf{1}_{x\in[-2,2]}\mathrm{d}x$$

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- O1 : Bai – Silverstein (1980 – 2010), O2 : Erdős – Yau (2005 –)

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  - Inversion

$$\mu_A = \frac{\Delta}{2\pi}(\log|\cdot| * \mu_A)$$

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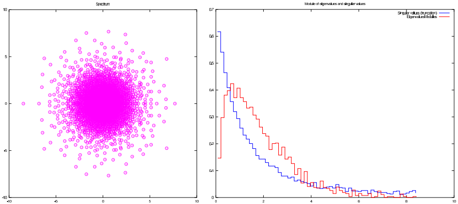
Outline

Girko – Lévy matrices : heavy tailed entries



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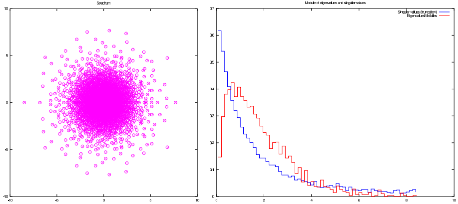

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$$\mu_{\frac{X}{\sqrt[\alpha]{n}}} \underset{n\to\infty}{\longrightarrow} \mu_{\boldsymbol{\alpha}} = \frac{\Delta_z}{2\pi}\int_0^\infty \log(s)\mathrm{d}\nu_{z,\alpha}(s)$$

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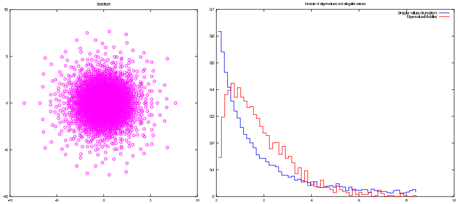

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- μ_α isotropic and not heavy tailed with density f_α

$$f_\alpha(z) \underset{|z|\to\infty}{\sim} c_\alpha \, |z|^{2(\alpha-1)} \, \mathrm{e}^{-\frac{\alpha}{2}|z|^\alpha}.$$

Operator convergence to Poisson Weighted Infinite Tree (Aldous)

Random regular graphs and free probability

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  - Voiculescu Free Central Limit Theorem

$$\frac{u_1 + \cdots + u_d}{\sqrt{d}} \xrightarrow[d\to\infty]{\text{law}} \frac{\mathbf{1}_{|z|\leq 1}}{\pi}$$

Outline

Girko matrices: spectrum edge

- Light tails and heavy tails, outliers and point processes

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  - $\mathbb{E}[|X_{11}|^4] < \infty$: uniform law of large numbers $\max_i \sum_j |X_{ij}|^2$

Girko matrices : spectral radius

- Optimal theorem **without** any extra condition (2018, 2020)

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- Reciprocal characteristic polynomial : on $\{z \in \mathbb{C} : |z| < 1\}$

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  - $\alpha = \mathbb{E}[X_{11}^2]$, $|\alpha| \le \mathbb{E}[|X_{11}|^2] = 1$ thus no zeros on $\{z \in \mathbb{C} : |z| < 1\}$

Proof outline

- Hurwitz phenomenon for zeros of random analytic functions

$$\inf_{|z|<1} \left| \det\left(1 - z\frac{X}{\sqrt{n}}\right) \right| > 0 \quad \text{as } n \to \infty$$

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- Hurwitz phenomenon for zeros of random analytic functions

$$\inf_{|z|<1}\left|\det\left(1-z\frac{X}{\sqrt{n}}\right)\right|>0 \quad \text{as } n\to\infty$$

- Tightness via orthogonal decomposition (→ Basak–Zeitouni)

$$\det\left(1-z\frac{X}{\sqrt{n}}\right)=1+\sum_{k=1}^{n}(-z)^n\sum_{\substack{I\subset\{1,\dots,n\}\\|I|=k}}\frac{\det(X_{I,I})}{\sqrt{n}^k}$$

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$$\det\left(1 - z\frac{X}{\sqrt{n}}\right) = \exp\left(- \sum_{k=1}^{\infty} \frac{\mathrm{Trace}(X^k)}{\sqrt{n}^k} \frac{z^k}{k} \right)$$

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- Joint CLT for fixed traces via moments ($\to$ Janson – Nowicki)

Proof outline

- Hurwitz phenomenon for zeros of random analytic functions

$$\inf_{|z|<1} \left| \det\left(1 - z\frac{X}{\sqrt{n}}\right) \right| > 0 \quad \text{as } n \to \infty$$

- Tightness via orthogonal decomposition (→ Basak – Zeitouni)

$$\det\left(1 - z\frac{X}{\sqrt{n}}\right) = 1 + \sum_{k=1}^{n} (-z)^n \sum_{\substack{I\subset\{1,\ldots,n\} \\ |I|=k}} \frac{\det(X_{I,I})}{\sqrt{n}^k}$$

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- Prefactor $\sqrt{1 - \alpha z^2}$ due to centering in CLT

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$$U^{\mu}(z) = (-\log|\cdot| * \mu)(z)$$

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- CLT for linear statistics (Rider–Silverstein, Erdős et al)

$$n(U^{\mu_A}(z) - U^{\mu_\bullet}(z)) \xrightarrow[n\to\infty]{\mathrm{d}} G(z)$$

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- Convergence to Gaussian Analytic Function for $|z| > 1$

$$|\det(1-z^{-1}A)| = \mathrm{e}^{-n(U^{\mu_A}(z)-U^{\mu_\bullet}(z))} \xrightarrow[n\to\infty]{\mathrm{d}} \mathrm{e}^{-G(z)}.$$

Complex analytic link with CLT for linear statistics

- f analytic in a neighborhood of closed unit disc

$$\int f(\lambda)\mu_A(\mathrm{d}z) = \frac{1}{2\pi \mathrm{i}} \int \Big(\oint \frac{f(z)}{z-\lambda} \mathrm{d}z \Big) \mu(\mathrm{d}\lambda)$$
$$= \frac{1}{2\pi \mathrm{i}} \oint f(z) \Big(\int \frac{\mu(\mathrm{d}\lambda)}{z-\lambda} \Big) \mathrm{d}z$$
$$= \frac{1}{2\pi \mathrm{i}} \oint f(z)(\log\det(z-A))' \mathrm{d}z.$$

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- Compare with usual approach

$$\overline{\partial}\partial(\log|z| * \mu) = \Delta(\log|z| * \mu) = 2\pi\mu$$

$$\overline{\partial}(\tfrac{1}{z} * \mu) = \pi\mu$$

Elliptic ensemble and Girko–Wigner interpolation

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- Work in progress : characteristic polynomial convergence

Outline

Historical roots

High dimensional phenomena

Beyond Girko matrices

Spectrum edge

Ginibre

Ginibre$_{\mathbb{C}}$ matrices : exactly solvable model

$$X = \begin{pmatrix} X_{11} & \cdots & X_{1n} \\ & \vdots & \\ X_{n1} & \cdots & X_{nn} \end{pmatrix}$$

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- Unitary invariance and Maxwell – Mehta characterization

$$\propto \mathrm{e}^{-\mathrm{Trace}(XX^*)}$$

Ginibre$_{\mathbb{C}}$: spectrum density

- Schur unitary decomposition $X = U(D+N)U^*$ gives

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- Planar Coulomb gas (two-dimensional one-component plasma)

$$\propto \mathrm{e}^{-n\sum_{i=1}^n |\lambda_i|^2 - \frac{1}{2}\sum_{i\neq j} \log \frac{1}{|\lambda_i - \lambda_k|}}$$

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- Mehta : Convergence of density of $\mathbb{E}\mu_n = \mathbb{E}\frac{1}{n}\sum_{k=1}^{n}\delta_{\lambda_k}$

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- Kostlan : moduli (here $Z_i \sim \mathrm{Gamma}(i,1)$ are independent)

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$$n(\mu_n(f) - \mu_\bullet(f)) \xrightarrow[n\to\infty]{\mathrm{d}} \mathcal{N}(0, \frac{1}{4\pi}\|f\|^2_{\mathrm{H}^1(D)} + \frac{1}{2}\|f\|^2_{\mathrm{H}^{1/2}(\partial D)})$$

- Kostlan : moduli (here $Z_i \sim \mathrm{Gamma}(i, 1)$ are independent)

$$\{|\lambda_1|, \ldots, |\lambda_n|\} \overset{\mathrm{d}}{=} \{Z_1, \ldots, Z_n\}$$

- Rider : spectral radius $(\gamma_n = \log(n/(2\pi)) - 2\log(\log(n)))$

$$\sqrt{4n\gamma_n}\left(\max_{1\leq i\leq n}|\lambda_i| - 1 - \sqrt{\frac{\gamma_n}{4n}}\right) \xrightarrow[n\to\infty]{\mathrm{d}} \mathrm{Gumbel} = -\log \mathrm{Exp}$$

Ginibre$_{\mathbb{C}}$: Coulomb gas point of view

- Coulomb gas encoded with empirical measure $\mu_n = \frac{1}{n}\sum_{i=1}^{n} \delta_{\lambda_i}$

$$\propto \mathrm{e}^{-n^2 \mathcal{E}_{\neq}(\mu_n)}$$

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- LDP – CLT linked by Hessian. Universality classes for LDP?

Spectral radius fluctuations : universality conjecture

- Rider : Ginibre$_{\mathbb{C}}$ with $\gamma_n = \log \frac{n}{2\pi} - 2\log\log n$

$$\sqrt{4n\gamma_n}\left(\frac{\rho(X)}{\sqrt{n}} - 1 - \sqrt{\frac{\gamma_n}{4n}}\right) \xrightarrow[n\to\infty]{\text{law}} \text{Gumbel} = -\log(\text{Expo})$$

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    True for determinantal case $\beta = 2$ via Kostlan and Laplace

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  - Girko matrices with centered entries of unit variance
    Completely open!

Thank you!