Variance bounds and Superconcentration : a short Survey

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Outline of the Talk

- Introduction
- Semigroup interpolation and hypercontractive arguments.
- Recent improvements of concentration for convex functions.
- Monotone rearrangement and product measures.
- Open questions.

Introduction

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Concentration theory : effective tool in various mathematical areas

- Probability in high dimension
- Probability in Banach spaces
- Empirical process
- Mechanical statistics
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Lack of precision for particular example ?

Standard Gaussian measure

γ_n standard Gaussian measure on $\mathbb{R}^n$, $f : \mathbb{R}^n \to \mathbb{R}$ smooth enough

Poincaré's inequality

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At this level of generality, this inequality is sharp but does not depend on Γ. problem ?

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Poincaré's inequality sub-optimal for some functionals = Superconcentration (Chatterjee)

Branching Random Walk

- $\mathcal{T}$ binary tree with depth n.
- X_e $i.i.d.$ $\mathcal{N}(0,1)$ on each edge e.
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Tools : modified second moment method combined with comparison arguments (very technical proof).

Other examples

- Largest eigenvalue in random matrix theory. [Ledoux, Dallaporta,...].
- First time passage in percolation theory. [Damron, Hanson,...]
- Free energy in Spin Glass theory. [Chen, Panchenko,...].
- Discrete Gaussian Free Field $\mathbb{Z}^2$. [Bramson, Ding, Zeitouni,...]
- Order statistics from an i.i.d. sample. [Boucheron, Thomas,...]
- l_p norm of standard Gaussian vector in $\mathbb{R}^n$. [Paouris, Valettas, Zinn]
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- Each models, ad-hoc methods, sometimes very technicals
- Common properties ? Is it possible, in general, to improve (even slightly) upon classical concentration ?

Some trials to improve concentration of measure

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- Semigroup interpolation and hypercontractive arguments.

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- Inverse, integrated, infinite curvature criterion.
  Attention : few examples available

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Non Gaussian framework

Transporting functional inequalities by monotone rearrangement.

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Transporting functional inequalities by monotone rearrangement.
Attention : limited to product measures

Semigroups arguments. Application to stationary Gaussian sequences

Stationary Gaussian sequences

$(X_n)_{n\geq 0}$ centered stationary Gaussian sequence, with covariance function $\mathbb{E}[X_i X_j] = \phi(|i-j|)$ où $\phi : \mathbb{N} \to \mathbb{R}_+$.

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Extreme theory [Berman]

If $\phi(n)\log n \underset{n\to\infty}{\longrightarrow} 0$ then

$$\sqrt{2\log n}\big(M_n - b_n\big) \xrightarrow{\mathcal{L}} \Lambda_0$$

with $M_n = \max_{i=1,\dots,n} X_i$.

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Gumbel's distribution : $\mathbb{P}(\Lambda_0 \geq t) = 1 - e^{-e^{-t}}$ $\quad$ ($\sim e^{-t}$ for t large enough)

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Tools : variance representation by semigroups and hypercontractivity.

Let us start with an easy case when $\Gamma = I_d$.

Talagrand's inequality : bounding the variance

γ_n standard Gaussian measure on $\mathbb{R}^n$.

Theorem [Talagrand]

$f : \mathbb{R}^n \to \mathbb{R}$ smooth enough

$$\mathrm{Var}_{\gamma_n}(f) \leq C \sum_{i=1}^{n} \frac{\|\partial_i f\|_2^2}{1 + \log \frac{\|\partial_i f\|_2}{\|\partial_i f\|_1}}$$

Improve upon Poincaré's inequality.
Proof ?

Semigroup

Ornstein-Uhlenbeck's semigroup

$$P_t(f)(x) = \int_{\mathbb{R}^n} f(e^{-t}x + \sqrt{1 - e^{-2t}}y)d\gamma_n(y) \quad t \geq 0, x \in \mathbb{R}^n$$

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Hypercontractivity

$$\|P_t f\|_q \leq \|f\|_{p(t)}, \quad p(t) = (q-1)e^{-2t} + 1, t > 0$$

Note : $p(t) < q$ (improve upon Jensen's inequality).

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Interpolation by semigroup

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Hypercontractivity

For $i = 1, \ldots, n$

$$\|P_t(\partial_i f)\|_2 \leq \|\partial_i f\|_{p(t)} \quad p(t) = 1 + e^{-2t},\ t > 0.$$

It implies Talagrand's inequality (after some interpolation arguments based on Hölder's inequality)

Application

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Theorem [Chatterjee]

If $\exists r_0 \geq 0$ and $\exists \mathcal{C}$ a covering of $\{1, \dots, n\}$ such that $\forall i, j \in \{1, \dots, n\}$ if $\mathbb{E}[X_i X_j] = \Gamma_{ij} \geq r_0$ then $\exists D \in \mathcal{C}, \quad i, j \in D$

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$I = \mathrm{argmax_i} X_i$ and $\rho(r_0) = \max_{D \in \mathcal{C}} \mathbb{P}(I \in D)$.
Then

$$\mathrm{Var}(M_n) \leq C\left(r_0 + \frac{1}{\log 1/\rho(r_0)}\right)$$

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One can show that $\rho(r_0) \leq 1/n^\eta$, $0 < \eta < 1$.Finally,

$$\mathrm{Var}(M_n) \leq C\left(\phi(n^\alpha) + \frac{1}{\log n}\right) \leq \frac{C'}{\log n}$$

Chatterjee's Theorem : a sketch of proof

$X \sim \mathcal{N}(0, \Gamma)$, $f : \mathbb{R}^n \to \mathbb{R}$ smooth enough

Variance representation

$$\mathrm{Var}\big(f(X)\big) = 2\int_0^\infty e^{-2t} \sum_{i,j=1}^n \Gamma_{ij}\mathbb{E}[\partial_j f(X) P_t(\partial_i f)(X)]dt.$$

$(P_t)_{t\geq 0}$ generalized Ornstein-Uhlenbeck's semigroup.

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Sketch of proof

- Γ satisfies a « covering »property (which allows one to gather the Γ_{ij} in pack of same « size »).
- $(P_t)_{t\geq 0}$ is hypercontractive, it can be used to control the size (in L^2-norm) of each of these packs.

Stationnary Gaussian sequences

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Recall

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with $\mathbb{P}(\Lambda_0 \geq t) = 1 - e^{-e^{-t}}$.

Non-asymptotic concentration inequality ?

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Non-asymptotic concentration inequality ?

Goal

- $\mathbb{P}(\sqrt{2\log n}(M_n - b_n) \geq t) \leq \psi_1(t), \quad t \geq 0$
- $\mathbb{P}(\sqrt{2\log n}(M_n - b_n) \leq -t) \leq \psi_2(t), \quad t \geq 0$

with ψ_i, $i = 1, 2$ reflecting Gumbel's asymptotics.

Gaussian concentration

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a L-Lipschitz function and $X \sim \mathcal{N}(0, I_d)$ then

Theorem [Borell, Sudakov-Tsirel'son]

$$\mathbb{P}\Big(|f(X) - \mathbb{E}\big[f(X)\big]| \geq t \Big) \leq 2e^{-t^2/2L}$$

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Superconcentration inequality ?

Stationary Gaussian sequences

Assume ϕ is non-increasing and $\phi(n)\log n \to 0$ as $n \to \infty$.

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We would like to obtain (1) for $Z = M_n = \max_{i=1,\dots,n} X_i$ with $K \sim \mathrm{Var}(M_n) \sim C/\log n$.

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Proof : Extension of Chatterjee's Theorem at an exponential level.

Few words on recent results

Other improvements

New concentration results for γ_n the standard Gaussian measure on $\mathbb{R}^n$.

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$$\gamma_n\big(f - E[f] < -t\sqrt{\mathrm{Var}_{\gamma_n}(f)}\big) \leq e^{-ct^2}, \quad t > 1 \tag{2}$$

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Remarks

- (2) obtained thanks to Ehrhard's inequality.

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- (2) obtained thanks to Ehrhard's inequality.
- Valettas also proved, with Ehrhard's inequality, that Borell's inequality is sharp for convex functions which are not superconcentrated.

Superconcentration for product measures by monotone rearrangement

Basic transport of product measures in $\mathbb{R}^n$

Step 1 : choose ν as γ_1 the standard Gaussian measure on $\mathbb{R}$ and μ as the symmetric Exponential measure (with density $g(x) = \frac{1}{2}e^{-|x|}$).

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$$T(x_1, \ldots, x_n) = \big(t(x_1), \ldots, t(x_n)\big) \quad x = (x_1, \ldots, x_n) \in \mathbb{R}^n$$

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Notice : T transports μ^n onto γ_n and $\mathbb{E}_{\gamma_n}(f) = \mathbb{E}_{\mu^n}(f \circ T)$ for $f : \mathbb{R}^n \to \mathbb{R}$ smooth enough

Transporting Poincaré's inequality

Weighted Poincaré's inequality

Poincaré's inequality for the Exponential measure

$$\mathrm{Var}_{\mu^n}(f) \leq 4 \int_{\mathbb{R}^n} |\nabla f|^2 d\mu^n$$

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\mathrm{Var}_{\gamma_n}(f) = \mathrm{Var}_{\mu^n}(f \circ T) &\leq 4 \sum_{i=1}^{n} \int_{\mathbb{R}^n} (\partial_i f)^2 \circ T(x) t'^2(x_i) d\mu^n(x) \\
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Estimate the behavior of $t' \circ t^{-1}$ to bound the variance of f under γ_n

Sample result

Lemma

Under the preceding framework, the following estimates holds

$$|t' \circ t^{-1}(x)| \leq \frac{C}{1+|x|}, \quad x \in \mathbb{R}$$

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Thus,

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  If ν, μ are probability measure on $\mathbb{R}$ with (respectively) density h, g and c.d.f H, G. Then

$$t'(x) = \frac{g(x)}{1 - G(x)} \times \frac{1 - H\big(t(x)\big)}{h\big(t(x)\big)}$$

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Notice : ratio of the so-called hazard rate function associated (respectively) to μ and ν.

Application in Superconcentration

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$f(x) = \max_{i=1,\dots,n} x_i = \sum_{i=1}^n x_i 1_{A_i}$ with $A_i = \{x_i = \max_{j=1,\dots,n} x_j\}$.

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&\leq \frac{C'}{1+\log n}
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Other applications

- For the Gaussian measure : we can study others functionnals (median, l^p-norms) and recover some work of Boucheron-Thomas and Paouris-Valettas-Zinn.

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Note : as far as we know, this can not be obtained by hypercontractive arguments (when $\alpha > 2$).This is also sharp with respect to Extreme Theory.

Extreme Theory and non-asymptotic deviation inequalities

Convergence of Extremes

Recall the following fact, in the Gaussian case,

$$\sqrt{2\log n}(M_n - b_n) \underset{\mathcal{L}}{\longrightarrow} \Lambda_0, \quad n \to \infty$$

with $\mathbb{P}(\Lambda_0 \geq x) = 1 - e^{-e^{-t}}$, $t \in \mathbb{R}$ (Gumbel distribution).

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What about deviation inequalities ?

$$\text{i.e.} \quad \mathbb{P}\left(\sqrt{\log n}(M_n - \mathbb{E}[M_n]) \geq t\right) \leq Ce^{-ct}$$

It should reflect the size of the variance of M_n and the asymptotics of Λ_0 (here on the right tail).

Extension to an exponential level : two further arguments

Lemma

$$\text{If } \mathrm{Var}(e^{\theta Z/2}) \leq \frac{\theta^2}{4} K \mathbb{E}[e^{\theta Z}] \quad \theta > 0 \text{ then}$$

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Goal : obtain (3) with $K \sim \mathrm{Var}(M_n)$.

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Lemma

Let $f : \mathbb{R}^n \to \mathbb{R}$ non-increasing and $g : \mathbb{R}^n \to \mathbb{R}$ non-decreasing, then

$$\mathbb{E}\big[f(X)g(X)\big] \leq \mathbb{E}\big[f(X)\big]\mathbb{E}\big[g(X)\big], \quad X = (X_1, \ldots, X_n)$$

with X_i independent random variables.

Application in the Gaussian case

Standard Gaussian measure

$$\mathrm{Var}_{\gamma_n}(f) \leq C \sum_{i=1}^{n} \int_{\mathbb{R}^n} (\partial_i f(x))^2 \left(\frac{1}{1 + |x_i|} \right)^2 d\gamma_n(x)$$

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$$\mathrm{Var}(e^{\theta M_n/2}) \leq C \frac{\theta^2}{4} \mathbb{E}\left[e^{\theta M_n} \frac{1}{1 + (M_n)^2} \right]$$

(we used again the fact $(A_i)_{i=1,\ldots,n}$ is a partition).

Application in the Gaussian case

Step 2 : $(x_1, \dots, x_n) \mapsto \frac{1}{1+\max_{i=1,\dots,n} x_i}$ is a non-increasing function, so apply Harris's Lemma :

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Step 3 : use previous bounds on $\mathbb{E}\left[\frac{1}{1+(M_n)^2}\right]$ and conclude with the concentration Lemma.

Application in the Gaussian case

Step 2 : $(x_1, \ldots, x_n) \mapsto \frac{1}{1+\max_{i=1,\ldots,n} x_i}$ is a non-increasing function, so apply Harris's Lemma :

$$\mathrm{Var}(e^{\theta M_n/2}) \leq C\frac{\theta^2}{4}\mathbb{E}\left[e^{\theta M_n}\right]\mathbb{E}\left[\frac{1}{1+(M_n)^2}\right]$$

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Notice : all we needed was a bound on the variance of M_n and the fact that the map $t' \circ t^{-1}(x)$ was dominated by a non-increasing function.

Transporting Isoperimetric inequalities

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Is it possible to transport stronger functional inequalities to obtain something relevant in the domain of attraction of the Gumbel's distribution ?

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Remark : reflects the size of $\mathrm{Var}(M_n)$ and the right tail of Gumbel's distribution (but not the left tail !).

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Sharp with respect to Extreme theory (left tail of Gumbel's distribution). Still work for log-concave measure on $\mathbb{R}_+^n$.

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Then, possibility to deal with the DGFF on $\mathbb{Z}^2$?

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- Non-product measures by Optimal Transport (Knothe-Rosenblatt ?) arguments ?

Thanks for your attention

Inverse, integrated, infinite curvature dimension criterion

Representation formula

Recall the representation formula of the variance, along the Ornstein-Uhlenbeck semigroup $(P_t)_{t\geq 0}$,

$$\mathrm{Var}_{\gamma_n}(f) = 2\int_0^\infty \int_{\mathbb{R}^n} |\nabla P_t f|^2 d\gamma_n dt. \qquad (4)$$

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Question : other functions f s.t. $I' + 2I \geq \psi_f$ for some function ψ_f ?