On the convex Poincaré inequality

Based on joint work with Radosław Adamczak.

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Prelude: classical setting [Bobkov–Ledoux, '97]

If μ satisfies the Poincaré inequality, i.e. $\forall\ f\colon \mathbb{R}^n \to \mathbb{R}$,

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then $\forall\ f : \mathbb{R}^n \to \mathbb{R}$ with $|\nabla f| \le c < \sqrt{2\lambda}$,

$$\mathbb{E}_\mu\, fe^f - \ln(\mathbb{E}_\mu\, e^f)\, \mathbb{E}_\mu\, e^f =: \mathrm{Ent}_\mu(e^f) \le C(c, \lambda)\, \mathbb{E}_\mu\, |\nabla f|^2 e^f.$$

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Implies two-level concentration: $f : (\mathbb{R}^n)^N \to \mathbb{R}$,

$$\left.\begin{array}{l} \displaystyle\sum_{i=1}^N |\nabla_i f|^2 \le \alpha^2 \\ \displaystyle\max_{1\le i\le N} |\nabla_i f| \le \beta \end{array}\right\} \implies \mathbb{P}_{\mu^{\otimes N}}\big(\, f - \mathbb{E}_{\mu^{\otimes N}} f \ \ge t\big) \le \ \exp\Big(-C_1 \frac{t^2}{\alpha^2} \wedge \frac{t}{\beta}\Big)$$

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Example: $n = 1,\ f(x) = f_N(x) = (x_1 + \cdots + x_N)/\sqrt{N}$.

Convex setting

Standing assumption:

μ satisfies the *convex Poincaré* inequality, i.e. $\forall$ convex $f : \mathbb{R}^n \to \mathbb{R}$,

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Goal: $\forall$ convex or concave $f : \mathbb{R}^n \to \mathbb{R}$ with $|\nabla f| \leq c$,

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- Linked to weak transportation inequalities.

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Main result

Theorem (Adamczak-St., '17+)

$\forall$ convex or concave $f\colon \mathbb{R}^n \to \mathbb{R}$ with $|\nabla f| \le c < \sqrt{2\lambda}/e$,

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$$F(1) = \mathbb{E}_\mu\, f^2 e^f \lesssim \mathbb{E}_\mu\,|\nabla f|^2 e^f.$$

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Proof of Proposition 2

f convex or concave, $\mathrm{Med}_\mu f = 0$, $|\nabla f| \leq c$.

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Proof. $X \sim \mu$; $\widetilde{\mathbb{E}} :=$ expectation wrt density $|\nabla f(X)|^2 / \mathbb{E}|\nabla f(X)|^2$.

$$\mathbb{E}|\nabla f(X)|^2 e^{-|f(X)|} = \mathbb{E}|\nabla f(X)|^2 \cdot \widetilde{\mathbb{E}} e^{-|f(X)|} \ge \mathbb{E}|\nabla f(X)|^2 \cdot e^{-\widetilde{\mathbb{E}}|f(X)|}$$

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$$\leq c\sqrt{2/\lambda}\,\mathbb{E}|\nabla f(X)|^2$$

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$\Phi(f)$ is convex with median 0, for $\Phi(x) = \begin{cases} xe^{x/2} & \text{for } x \ge -2, \\ -2/e & \text{for } x < -2. \end{cases}$

$$\mathbb{E} f(X)^2 e^{f(X)} \le \mathbb{E}\Phi(f(X))^2 =: a^2 \quad \overset{?}{\lesssim} \quad b^2 := \mathbb{E}|\nabla f(X)|^2 e^{f(X)}$$

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$$\begin{aligned} a^2 &\le \frac{2}{\lambda}\mathbb{E}|\nabla f(X)|^2 (1 + f(X)/2)^2 e^{f(X)} 1_{\{f(X) \ge -2\}} \\ &\le \ldots \le \frac{2}{\lambda}(b + ca/2)^2 . \end{aligned}$$

Difficulty for concave functions

Question

If $f : \mathbb{R}^n \to \mathbb{R}$ is convex and 1-Lipschitz, then, for $t \geq 0$,

$$\mathbb{P}(f(X) \geq \operatorname{Med} f(X) + t) \leq 2 \exp(-C(\lambda) t).$$

Do we also have

$$\mathbb{P}(f(X) \leq \operatorname{Med} f(X) - t) \overset{?}{\leq} 2 \exp(-C(\lambda) t) \text{ ?}$$

Difficulty: for f convex, e^f, f_+^p, ... are convex,
but e^{-f}, f_-^p, ... are not always convex/concave.

Lower tail – can you do better?

Lemma

M such that $\mathbb{P}(|X - \mathbb{E}X| \leq M) \geq 3/4$. If $f : \mathbb{R}^n \to \mathbb{R}$ is convex, then

$$\mathbb{P}(f(X) \leq \operatorname{Med} f(X) - t) \leq 8 \exp\Big(-\frac{\sqrt{2\lambda}}{16e\mathbb{E}|\nabla f(X)|} t\Big)$$

for $t \geq 32M\mathbb{E}|\nabla f(X)|$.

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Proof. W.l.o.g. $\operatorname{Med} f(X) = 0$.

$$\left.\begin{array}{c}\mathbb{P}(f(X)\ge 0)\ge 1/2\\ \mathbb{P}(|X-\mathbb{E}X|\le M)\ge 3/4\\ \mathbb{P}(|\nabla f(X)| < 8\mathbb{E}|\nabla f(X)|)\ge 7/8\end{array}\right\} \implies \exists x_0 \begin{cases} f(x_0)\ge 0\\ |x_0-\mathbb{E}X|\le M\\ |\nabla f(x_0)| < 8\mathbb{E}|\nabla f(X)|\end{cases}$$

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$$\widetilde{f}(x) := f(x_0) + \langle \nabla f(x_0), x - x_0 \rangle, \quad x \in \mathbb{R}^n$$

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But $\mathbb{P}(|X - \mathbb{E}X| \leq M) \geq 3/4$ and

$$\mathbb{P}(|X - \mathbb{E}X| \geq M + t) \leq 8 \exp\Big(-\frac{\sqrt{2\lambda}}{e} t\Big), \quad t \geq 0,$$

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so for $t/(16\mathbb{E}|\nabla f|) \geq 2M$,

$$\mathbb{P}(f(X) \leq -t) \leq \mathbb{P}(|X - \mathbb{E}X| \geq M + t/(16\mathbb{E}|\nabla f(X)|)) \leq \ldots.$$

Summary

Assumption: $\mu \sim X$ satisfies convex Poincaré.

Theorem

$\forall$ convex or concave $f : \mathbb{R}^n \to \mathbb{R}$ with $|\nabla f| \leq c < \sqrt{2\lambda}/e$,

$$\mathrm{Ent}_\mu(e^f) \leq C\,\mathbb{E}_\mu\,|\nabla f|^2 e^f,$$

where $C = C(\lambda, c, n)$.

Question

If $f : \mathbb{R}^n \to \mathbb{R}$ is convex and 1-Lipschitz, do we have

$$\mathbb{P}(f(X) \leq \mathrm{Med}\, f(X) - t) \overset{?}{\leq} 2\exp(-C(\lambda)t) \ ?$$