Higher Order Concentration of Measure

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joint work with:

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Concentration of measure and its applications

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Topics

Higher Order Concentration Bounds in High Dimensions:

- Euclidean Spaces: Logarithmic Sobolev and Poincaré inequalities
- Spheres
- Functions of independent random variables
- Functions of weakly dependent random variables

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Recall: Logarithmic Sobolev inequality

μ sat. LSI(σ^2) if $\forall$ f bounded, locally Lipschitz

$$\mathrm{Ent}_\mu(f^2) := \int f^2 \log\left(\frac{f^2}{\int f^2 d\mu}\right) d\mu \leq 2\sigma^2 \int |\nabla f|^2 d\mu$$

$\Rightarrow \quad \mu(|f - \int f d\mu| \geq t) \leq 2e^{-c\sigma^2 t^2/\|\nabla f\|_\infty^2}$

Example: $\mu := \otimes_{i=1}^n \mu_i$, μ_i unif. distr. on $[-1, 1]$ $\quad$ μ sat. LSI($\Theta(1)$)

$f(x) := \sum_{i<j} x_i x_j \quad \rightsquigarrow \sup |\nabla f(x)| = \Theta(n^{3/2})$

$\Rightarrow \mu(|f|/n^{3/2} \geq t) \leq 2e^{-ct^2}$ $\quad$ wrong order!

Note that $|f''(x)|_{\mathrm{HS}} \leq n$ $\quad \rightsquigarrow$ Ideas:

- Use higher order derivatives
- study fluctuations of $f - \mathbb{E}f - f_1 - \ldots - f_{d-1}$ w.r.t. suitable decomp.

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Higher Order Concentration on Euclidean Spaces

Notation: $G \subset \mathbb{R}^n$ open set, $d \in \mathbb{N}$, $f \in \mathcal{C}^d(G)$.

hypermatrix of d-fold partial derivatives $f^{(d)}$

$$f^{(d)}_{i_1 \ldots i_d}(x) = \partial_{i_1 \ldots i_d} f(x).$$

Use operator and Hilbert–Schmidt type norms

$$|f^{(d)}|_{\mathrm{Op}} := \sup \left\{ f^{(d)}(x)[v_1, \ldots, v_d] \colon |v_1| = \ldots |v_d| = 1 \right\},$$

$$|f^{(d)}|_{\mathrm{HS}} := \Big(\sum_{i_1, \ldots, i_d} (\partial_{i_1 \ldots i_d} f(x))^2 \Big)^{1/2}.$$

For a prob. measure μ on G,

$$\|f^{(d)}\|_{\mathrm{Op},p} = \| |f^{(d)}|_{\mathrm{Op}} \|_p = \left(\int_G |f^{(d)}|^p_{\mathrm{Op}} d\mu \right)^{1/p},$$

any $p \in (0, \infty]$. Similarly: $\|f^{(d)}\|_{\mathrm{HS},p}$.

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Theorem (Bobkov-Götze-S (2017))

Let μ *with* $LSI(\sigma^2)$, $d \in \mathbb{N}$. $f\colon G \to \mathbb{R}$, $f \in \mathcal{C}^d(G)$, $\int f d\mu = 0$. *Assume*

$$\|f^{(k)}\|_{\mathrm{Op},2} \leq \min(1, \sigma^{d-k}) \qquad \forall k = 1, \ldots, d-1, \tag{0.1}$$

$$\|f^{(d)}\|_{\mathrm{Op},\infty} \leq 1.$$

Then, for some $c > 0$

$$\int_G \exp\left(\frac{c}{\sigma^2} |f|^{2/d}\right) d\mu \leq 2.$$

By Chebychev's inequality: $\mu(|f| \geq t) \leq 2e^{-c\sigma^2 t^{2/d}}$

may somewhat sharpen this bound by analyzing the proof

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Simplify conditions: Assume

$$\int_G \partial_{i_1 \dots i_k} f \, d\mu = 0 \quad \forall k = 1, \dots, d-1, \quad \forall 1 \le i_1, \dots, i_k \le n \tag{0.2}$$

$\rightsquigarrow$ (0.1) can be replaced by $\|f^{(d)}\|_{\mathrm{HS},2} \le 1$.

Idea: remove "lower order terms" by suitable "projections", e. g. ($d = 2$)

$$\tilde{f}(x) := f(x) - \Big(\mu(f) + \sum_{i=1}^n \mu(\partial_i f)(x_i - \mu(x_i))\Big).$$

Gaussian case: (0.2) $\Leftrightarrow$ orthogonality to all polynomials of degree $\le d-1$

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Step 1: For any $g\colon G \to \mathbb{R}$ locally Lipschitz s. th. $\|g\|_p < \infty$

$$\|g\|_p^2 - \|g\|_2^2 \le \sigma^2(p-2)\|\nabla g\|_p^2, \qquad p \ge 2.$$

Proof: (Aida-Stroock (1994)) Assume g bounded. Then,

$$\frac{d}{dp}\|g\|_p^2 = \frac{2}{p^2}\|g\|_p^{2-p}\mathrm{Ent}_\mu(|g|^p), \qquad p > 2.$$

Set $u = |g|^{p/2}$ with $|\nabla u|^2 \le \frac{p^2}{4}|g|^{p-2}|\nabla g|^2$. Hence, by Hölder's inequality,

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$$|\nabla|f^{(k-1)}(x)|_{\mathrm{Op}}| \le |f^{(k)}(x)|_{\mathrm{Op}}, \qquad \text{all } x \in G.$$

Follows by Taylor expansion.

Step 3: Apply Step 1 to $g := |f^{(k-1)}|_{\mathrm{Op}}$ and combine with Step 2:

$$\|f^{(k-1)}\|^2_{\mathrm{Op},p} - \|f^{(d-1)}\|^2_{\mathrm{Op},2} \le \sigma^2(p-2)\|\nabla|f^{(d-1)}|_{\mathrm{Op}}\|^2_p \le \sigma^2(p-2)\|f^{(d)}\|^2_{\mathrm{Op},p}.$$

By iteration and the Poincaré inequality,

$$\begin{aligned}\|f\|_p^2 \le\ & \|f\|_2^2 + \sum_{k=1}^{d-1}(\sigma^2(p-2))^k \|f^{(k)}\|^2_{\mathrm{Op},2} + (\sigma^2(p-2))^d \|f^{(d)}\|^2_{\mathrm{Op},p} \\ \le\ & \sum_{k=1}^{d-1}(\sigma^2 p)^k \|f^{(k)}\|^2_{\mathrm{Op},2} + (\sigma^2 p)^d \|f^{(d)}\|^2_{\mathrm{Op},p}.\end{aligned}$$

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By iteration and the Poincaré inequality,

$$\|f\|_p^2 \le \|f\|_2^2 + \sum_{k=1}^{d-1}(\sigma^2(p-2))^k \|f^{(k)}\|_{\mathrm{Op},2}^2 + (\sigma^2(p-2))^d \|f^{(d)}\|_{\mathrm{Op},p}^2$$

$$\le \sum_{k=1}^{d-1}(\sigma^2 p)^k \|f^{(k)}\|_{\mathrm{Op},2}^2 + (\sigma^2 p)^d \|f^{(d)}\|_{\mathrm{Op},p}^2.$$

Sketch of Proof II

Step 2: For $f\colon G \to \mathbb{R}$ C^d-function

$$|\nabla |f^{(k-1)}(x)|_{\mathrm{Op}}| \leq |f^{(k)}(x)|_{\mathrm{Op}}, \qquad \text{all } x \in G.$$

Follows by Taylor expansion.

Step 3: Apply Step 1 to $g := |f^{(k-1)}|_{\mathrm{Op}}$ and combine with Step 2:

$$\|f^{(k-1)}\|^2_{\mathrm{Op},p} - \|f^{(d-1)}\|^2_{\mathrm{Op},2} \leq \sigma^2(p-2)\|\nabla|f^{(d-1)}|_{\mathrm{Op}}\|_p^2 \leq \sigma^2(p-2)\|f^{(d)}\|^2_{\mathrm{Op},p}.$$

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$$\begin{aligned}\|f\|_p^2 \leq & \ \|f\|_2^2 + \sum_{k=1}^{d-1}(\sigma^2(p-2))^k \, \|f^{(k)}\|^2_{\mathrm{Op},2} + (\sigma^2(p-2))^d \, \|f^{(d)}\|^2_{\mathrm{Op},p} \\ \leq & \ \sum_{k=1}^{d-1}(\sigma^2 p)^k \, \|f^{(k)}\|^2_{\mathrm{Op},2} + (\sigma^2 p)^d \, \|f^{(d)}\|^2_{\mathrm{Op},p}.\end{aligned}$$

Sketch of Proof III

Use $\|f^{(k)}\|^2_{\mathrm{Op},2} \leq \min(1, \sigma^{2(d-k)})$ for all $k = 1, \ldots, d-1$ and $\|f^{(d)}\|_{\mathrm{Op},\infty} \leq 1$ to get

$$\|f\|_p^2 \leq \sigma^{2d} \sum_{k=1}^{d} p^k \leq \frac{1}{1-p^{-1}} (\sigma^2 p)^d \leq 2 (\sigma^2 p)^d,$$

i. e. $\|f\|_p \leq (2\sigma^2 p)^{d/2}$ for all $p \geq 2$. If $p < 2$, by Hölder's inequality

$$\|f\|_p \leq \|f\|_2 \leq (4\sigma^2)^{d/2}.$$

Consider $\quad p = 2k/d, \quad k = 1, 2, \ldots, \quad$ and take $2/d$-th root:

$$\||f|^{2/d}\|_k \leq \gamma k, \qquad \gamma = 4\sigma^2.$$

It follows

$$\int \exp\Big(\frac{c}{\sigma^2}|f|^{2/d}\Big)\, d\mu \leq 2,$$

$c > 0$ constant (possible choice: $c = 1/(8e)$).

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Applications: Random Matrices

- Symmetric random matrices $\Xi = (\xi_{jk}/\sqrt{N})_{1 \le j,k \le N}$, where ξ_{jk}, $j \le k$, independent with LSI(σ^2)
  $\rightsquigarrow$ μ = distr. of eigenvalues $\lambda = (\lambda_1 \le \ldots \le \lambda_N)$ sat. LSI($2\sigma^2/N$)
- β-ensembles: density on $\{\lambda \in \mathbb{R}^N \colon \lambda_1 < \ldots < \lambda_N\}$

$$\mu(d\lambda) := \frac{1}{Z_N} e^{-\beta N \mathcal{H}(\lambda)} d\lambda, \; \mathcal{H}(\lambda) := \frac{1}{2} \sum_{k=1}^{N} V(\lambda_k) - \frac{1}{N} \sum_{1 \le k < l \le N} \log |\lambda_l - \lambda_k|.$$

Here, $V \colon \mathbb{R} \to \mathbb{R}$ is a smooth strictly convex potential. Then

$$\mathcal{H}''(\lambda) \ge c \, \mathrm{Id}$$

uniformly in λ and μ sat. LSI($1/(\beta c N)$) (Bakry–Emery).

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$f : \mathbb{R} \to \mathbb{R}$, $\mathcal{C}^1$-smooth. Self-normalizing sums

$$S_N := \sum_{j=1}^{N} (f(\lambda_j) - \mathbb{E} f(\lambda_j)) \Rightarrow \mathcal{N}(0, \sigma_f^2)$$

"linear eigenvalue statistics" (cf. Johansson (1998), Pastur et al. (1996), Guionnet–Zeitouni (2000))

$f : \mathbb{R} \to \mathbb{R}$ $\mathcal{C}^2$-smooth, $f'(\lambda_j) \in L^1$, $\quad ||f''||_\infty \leq \gamma$. If

$$\tilde{S}_N := S_N - \sum_{j=1}^{N} (\lambda_j - \mathbb{E}(\lambda_j)) \mathbb{E} f'(\lambda_j), \quad \text{then}$$

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Idea: Study quadratic eigenvalue statistics, i. e. $T_N := \sum_{j \neq k} g(\lambda_j, \lambda_k)$ for $g\colon \mathbb{R}^2 \to \mathbb{R}$ smooth. “Recenter” according to (0.2) $\rightsquigarrow$

$$Q_N := \sum_{j \neq k} g(\lambda_j, \lambda_k) - \sum_{j \neq k} \mathbb{E}(g(\lambda_j, \lambda_k))$$

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Higher Order Concentration on Spheres

$$S^{n-1} = \{x \in \mathbb{R}^n \colon |x| = 1\}, \quad n \geq 2, \quad \sigma_{n-1} \text{ uniform measure}$$

- σ_{n-1} satisfies LSI with constant $1/(n-1)$
- Any $\mathcal{C}^d(S^{n-1})$-smooth fct. can be extended to $\mathcal{C}^d$-smooth fct. on $\mathbb{R}^n \setminus \{0\}$

Theorem (Bobkov-Götze-S (2017))

f $\mathcal{C}^d$-smooth in open nbhd. of S^{n-1}, $\int_{S^{n-1}} f d\sigma_{n-1} = 0$. If

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$$Q_{d,a}(\theta) := \sum_{i=1}^{n} a_i \theta_i^d, \qquad \theta \in S^{n-1},$$

$d \geq 3$, $a \in \mathbb{R}^n$ s. th. $n^{-1} \sum_{i=1}^{n} a_i^2 = 1$.

$\exists$ const. $c_d > 0$:

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In part., $Q_{d,a} - \bar{Q}_{d,a} = \mathcal{O}_{\sigma_{n-1}}(n^{-(d-1)/2})$.

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Recall: Poincaré inequality

μ sat. PI(σ^2) if $\forall$ f bounded, locally Lipschitz

$$\mathrm{Var}_\mu(f) := \int f^2 d\mu - \Big(\int f d\mu\Big)^2 \leq \sigma^2 \int |\nabla f|^2 d\mu$$

LSI(σ^2) $\Rightarrow$ PI(σ^2), PI(σ^2) $\Rightarrow \mu(|f - \int f d\mu| \geq t) \leq 2e^{-c\sigma t/\|\nabla f\|_\infty}$

Theorem (Götze-S (2018))

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Recall: Poincaré inequality

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$$\mathrm{Var}_\mu(f) := \int f^2 d\mu - \Big(\int f d\mu\Big)^2 \leq \sigma^2 \int |\nabla f|^2 d\mu$$

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Lemma: For any $g\colon G \to \mathbb{R}$ locally Lipschitz s.th. $\|g\|_p < \infty$

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Use difference operators instead of derivatives!

$$\mathfrak{h}_i f(X) = \frac{1}{2} \|f(X) - T_i f(X)\|_{i,\infty}, \qquad \mathfrak{h} f = (\mathfrak{h}_1 f, \ldots, \mathfrak{h}_n f),$$

$T_i f(X) = f(X_1, \ldots, X_{i-1}, \bar{X}_i, X_{i+1}, \ldots, X_n)$, $\bar{X}_1, \ldots, \bar{X}_n$ indep. copies

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Remarks:

- If f is multilinear polynomial $f(X) = \sum_{|I|=d} \alpha_I X_I + \dots$ with $X_1, \dots, X_n$ standardized bounded, then (0.4) $\Rightarrow$ (0.3).
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Sketch of Proof

Step 1: Introduce difference operators

$$\mathfrak{h}_i^{\pm} f(X) = \frac{1}{2} \|(f(X) - T_i f(X))_{\pm}\|_{i,\infty}$$

Boucheron / Bousquet / Lugosi / Massart (2005): For any real $p \geq 2$,

$$\|(f - \mathbb{E}f)_+\|_p \leq \sqrt{8\kappa p}\|\mathfrak{h}^+ f\|_p, \qquad \|(f - \mathbb{E}f)_-\|_p \leq \sqrt{8\kappa p}\|\mathfrak{h}^- f\|_p$$

with $\kappa < 1.271$ absolute constant.

Consequence: By Hölder's inequality and triangular ineq.,

$$\|f\|_p \leq \|f\|_2 + \sqrt{32\kappa p}\|\mathfrak{h} f\|_p, \qquad p \geq 2.$$

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Application: *U*-statistics. Combine elements of the proof of Theorem [B-G-S (2017)] for independent Rademacher variables and results on randomized *U*-statistics by de la Peña / Giné:

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$$S_f(X) := \sum_{i=1}^{n} f\Big(\sum_{j=1}^{i} X_j\Big).$$

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Ising model

Example: Ising model q^n p.m. on $\{\pm 1\}^n$ def. by normalizing $\pi(\sigma) = \exp(\frac{1}{2}\sum_{i,j} J_{ij}\sigma_i\sigma_j)$, where $J = (J_{ij})$ s.th. $J_{ii} = 0$ and

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To establish LSI: approximate tensorization result (Marton)

$$\mathrm{Ent}_{q^n}(f) \leq \frac{2C}{\beta} \sum_{i=1}^{n} \int \mathrm{Ent}_{q_i(\cdot|\bar{y}_i)}(f(\bar{y}_i, \cdot))d\bar{q}_i(\bar{y}_i)$$

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$\rightsquigarrow$ LSI with const. dep. on α only.

Application: $f = \sum_{|I|=d} a_I \sigma_I$ d-homog. polynomial, $|a_I| \leq 1$

$$\rightsquigarrow q^n(|f - \mathbb{E}f| \geq t) \leq 2\exp\Big(-\frac{t^{2/d}}{cn}\Big)$$

improves upon older results by removing logarithmic dependencies

Similar results in presence of external field, i. e. normalize

$$\pi(\sigma) = \exp\Big(\frac{1}{2}\sum_{i,j} J_{ij}\sigma_i\sigma_j + \sum_{i=1}^n h_i\sigma_i\Big)$$

with h s. th. $\|h\|_\infty \leq \tilde{\alpha}$

Ising model

Have $\|A\|_{\mathrm{Op}} \leq \|J\|_{1\to 1} \leq 1-\alpha$ and $q_i(\cdot|\bar{\sigma}_i) \in (c_\alpha, 1-C_\alpha)$

$\rightsquigarrow$ LSI with const. dep. on α only.

Application: $f = \sum_{|I|=d} a_I \sigma_I$ d-homog. polynomial, $|a_I| \leq 1$

$$\rightsquigarrow q^n(|f - \mathbb{E}f| \geq t) \leq 2\exp\Big(-\frac{t^{2/d}}{cn}\Big)$$

improves upon older results by removing logarithmic dependencies

Similar results in presence of external field, i. e. normalize

$$\pi(\sigma) = \exp\Big(\frac{1}{2}\sum_{i,j} J_{ij}\sigma_i\sigma_j + \sum_{i=1}^{n} h_i\sigma_i\Big)$$

with h s. th. $\|h\|_\infty \leq \tilde{\alpha}$

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Thank you!