On the Wasserstein distance between the empirical and the marginal distributions of weakly dependent sequences

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joint work with J. Dedecker

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- The aim is to study the behavior of $W_1(\mu_n,\mu)$ for a large class of stationary sequences, where
$$W_1(\mu_1,\mu_2) = \inf_{\pi\in M(\mu_1,\mu_2)}\int |x-y|\pi(dx,dy)\,, \qquad (1)$$
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- W_1 belongs to the general class of minimal distances, as the total variation distance. Since the cost function $c_1(x,y) = |x-y|$ is regular, W_1 can be used to compare two singular measures (not possible with the total variation distance, whose cost function is given by the discrete metric $c_0(x,y) = \mathbf{1}_{x\neq y}$).

Different representations for W_1

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$$W_1(\mu_n, \mu) = \sup_{f \in \Lambda_1} \left| \frac{1}{n} \sum_{k=1}^{n} (f(X_k) - \mu(f)) \right| , \qquad (2)$$

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- If the sequence is ergodic, since μ has a finite first moment, $W_1(\mu_n, \mu) \to 0$ a.s. and $\mathbb{E}(W_1(\mu_n, \mu)) \to 0$.

About W_r, $r \geq 1$

- For $r \geq 1$, we can define also the Wasserstein distance of order r by taking the cost function $c_r(x, y) = |x - y|^r$, so

$$W_r^r(\mu_1, \mu_2) = \inf_{\pi \in M(\mu_1, \mu_2)} \int |x - y|^r \pi(dx, dy)$$

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- In particular, if μ has an absolutely component with respect to the Lebesgue measure which does not vanishes on the support of μ, then the optimal rate $n^{-r/2}$ can be reached. But in general, the rate can be much slower!
- $W_r(\mu_n, \mu)$ is the $\mathbb{L}^r$-distance between F_n^{-1} and F^{-1} and one can say (Ebralidze (1971)) that, with $\kappa_r = 2^{r-1} r$,

$$W_r^r(\mu_n, \mu) \leq \kappa_r \int_{\mathbb{R}} |x|^{r-1} |F_n(x) - F(x)| dx$$

Natural dependency coefficients to deal with $\|W_1(\mu_n, \mu)\|_p$ $(p \geq 1)$

- Note that, for any $p \geq 1$,

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- Setting, $\mathcal{F}_0 = \sigma(X_i, i \leq 0)$ and for $n \geq 0$,

$$\alpha_{1,\mathbf{X}}(n) = \sup_{x \in \mathbb{R}} \|\mathbb{E}\left(\mathbf{1}_{X_n \leq x} | \mathcal{F}_0\right) - F(x)\|_1$$

we have

$$\left| \mathrm{Cov}(\mathbf{1}_{X_0 \leq t}, \mathbf{1}_{X_k \leq t}) \right| \leq \min(B(t), \alpha_{1,\mathbf{X}}(n))$$

- Considering the tail function $H(t) = \mathbb{P}(|X| > t)$ and setting

$$S_{\alpha,n}(t) = \sum_{k=0}^{n} \min\left\{\alpha_{1,\mathbf{X}}(k), H(t)\right\}\ ,\ t \geq 0\,,$$

we get

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- The coefficients $\alpha_{1,\mathbf{X}}(k)$ are weaker than the strong mixing coefficients of Rosenblatt ! Conditions in terms of these coefficients to get the CLT for $W_1(\mu_n,\mu)$ and bounds for $\|W_1(\mu_n,\mu)\|_p$, $p \geq 1$.

On the dependency coefficients (1)

- For a strictly stationary sequence (X_i), its strong mixing coefficients of Rosenblatt (1956) are usually defined as follows: setting $\mathcal{G}_n = \sigma(X_k, k \geq n)$,

$$\alpha(n) = \alpha(\mathcal{F}_0, \mathcal{G}_n) = \sup\{| \mathbb{P}(U \cap V) - \mathbb{P}(U)\, \mathbb{P}(V)| : U \in \mathcal{F}_0,\ V \in \mathcal{G}_n\}$$

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- Setting $\mathbf{X}_n = (X_k, k \geq n)$, we can also write

$$\alpha(n) = \frac{1}{4} \sup_{\|f\|_\infty \leq 1} \|\mathbb{E}(f(\mathbf{X}_n)|\mathcal{F}_0) - \mathbb{E}(f(\mathbf{X}_n))\|_1$$

  and if $\mathbf{X} = (X_i)_{i \in \mathbb{Z}}$ is a stationary Markov process with Kernel operator K and invariant measure ν, then

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- These coefficients have many nice properties such that a $\mathbb{L}^1$-coupling property (see the monograph by Rio'00, translated recently in english) and can be computed for M.C. that are Harris recurrent and irreducible.

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- This is a Markov chain with invariant measure λ the Lebesgue measure on $[0,1]$ and transition Markov operator given by
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- This Markov chain is not strong mixing! Indeed, $2X_{k+1} = X_k + \varepsilon_{k+1} \Rightarrow X_k$ is the fractional part of $2X_{k+1}$. Hence $\sigma(X_k) \subset \sigma(X_{k+1})$ and, by iteration, $\sigma(X_k) \subset \sigma(X_j, j \geq k+n)$ for any $n \geq 0$. Therefore
$$\frac{1}{4} \geq \alpha(n) \geq \sup_k \alpha(\sigma(X_k), \sigma(X_k)) = \frac{1}{4} .$$

On the dependency coefficients (3)

- Recall that

$$\alpha_{1,\mathbf{X}}(n) = \sup_{x \in \mathbb{R}} \left\| \mathbb{E}\left(\mathbf{1}_{X_n \le x} | \mathcal{F}_0\right) - F(x) \right\|_1$$

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- Let BV_1 be the class of bounded variation functions h such that $|h|_v \le 1$ (where $|h|_v = \|dh\|_v$ is the variation norm of the measure dh). Then

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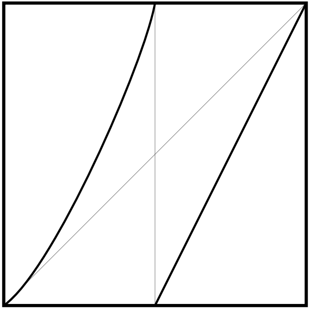
- Hence $\alpha_{1,\mathbf{X}}(n) \leq 2\alpha(n)$. For the previous $AR(1)$ example, $\alpha_{1,\mathbf{X}}(n) \leq Ce^{-\kappa n}$. These weak dependent coefficients can be computed in many situations (linear processes, random iterates,...).

Another example: intermittent Maps and their associated Markov chains

Example Let us consider a LSV map (Liverani, Saussol et Vaienti, 1999):

$$\text{for } 0 < \gamma < 1, \quad T_\gamma(x) = \begin{cases} x(1 + 2^\gamma x^\gamma) & \text{if } x \in [0, 1/2[\\ 2x - 1 & \text{if } x \in [1/2, 1] \end{cases}$$



Graph of T_γ

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- In our setting we want to analyze the concentration of the empirical measure $\tilde{\mu}_n = \frac{1}{n} \sum_{k=1}^{n} \delta_{g \circ T_\gamma^k}$.

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- We can associate a Markov chain $\mathbf{Y} = (Y_i)_{i\in Z}$ with invariant probability measure ν such that the following equality in law holds:

$$(T_\gamma, T_\gamma^2, \dots, T_\gamma^n) =^d (Y_n, Y_{n-1}, \dots, Y_1)$$

  Let $X_i = g(Y_i)$. Any information on the distribution of $W_1(\tilde{\mu}_n, \mu)$ can be derived from the distribution of $W_1(\mu_n, \mu)$.

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- The Markov operator of the chain is the Perron-Frobenius operator K (the adjoint of the composition by T in $\mathbb{L}^2(\nu)$): for any functions f and g in $\mathbb{L}^2(\nu)$,

$$\nu(f\circ T \cdot g) = \nu(f \cdot K(g))\,.$$

Some facts (2)

- For this map, Dedecker, Gouëzel, M. '10 have proved that

$$\frac{C_1}{n^{(1-\gamma)/\gamma}} \leq \alpha_{1,\mathbf{Y}}(n) = \frac{1}{2} \sup_{f \in BV_1} \nu\left(\left|K^n(f) - \nu(f)\right|\right) \leq \frac{C_2}{n^{(1-\gamma)/\gamma}}$$

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- If g is monotonic on some open interval and 0 elsewhere, and if $\mathbf{X} = (g(Y_i))_{i\in\mathbb{Z}}$, then $\alpha_{1,\mathbf{X}}(n) \le 2\alpha_{1,\mathbf{Y}}(n)$.

Application for the first and second moments of $W_1(\tilde{\mu}_n, \mu)$

- Assume that g is positive and non increasing on $(0, 1)$, with

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and

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About the CLT for $W_1(\mu_n, \mu)$

Recall that, with the notation $S_{\alpha,n}(t) = \sum_{k=0}^{n} \min\{\alpha_{1,\mathbf{X}}(k), H(t)\}$,

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$$\mathrm{Cov}\left(\int f(t)G(t)dt, \int g(t)G(t)dt\right)$$
$$= \sum_{k\in\mathbb{Z}} \mathbb{E}\left(\iint f(t)g(s)(\mathbf{1}_{X_0\leq t} - F(t))(\mathbf{1}_{X_k\leq s} - F(s))\ dtds\right)\,.$$

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- Cuny '17 proved (among many other results) that if $\mathbf{Y}$ is an ergodic sequence of martingale differences, under (5), we have both the CLT but also the FCLT.

A general FCLT in $\mathbb{L}^1(m)$

Assume ergodicity and that the random variable Y_0 is $\mathcal{F}_0$-measurable.

Theorem (Dedecker-M. '17)

Assume that, for m-almost every t, the series $U(t) = \sum_{k=1}^{\infty} \mathbb{E}_0(Y_k(t))$ *converges in probability.*

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then $\{n^{-1/2}S_{[nt]}, t \in [0,1]\}$ converges in distribution in the space $D_{\mathbb{L}_1(m)}([0,1])$ to an $\mathbb{L}_1(m)$-valued Wiener process W, whose covariance operator can be described.

Application: FCLT in $\mathbb{L}^1(m)$ for the empirical distribution

Let $Y_k(t) = \mathbf{1}_{X_k \le t} - F(t)$ where $(X_k)_k$ is an ergodic stationary sequence in $\mathbb{L}^1$ adapted to a stationary filtration $(\mathcal{F}_k)_k$. Let

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Corollary (Dedecker-M. '17)

Assume that

$$\int \sqrt{\sum_{k=0}^{\infty} \|F_{X_k|\mathcal{F}_0}(t) - F(t)\|_1} \; m(dt) < \infty . \tag{6}$$

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We have $\int \sqrt{\sum_{k=0}^{\infty} \min\{\alpha_{1,\mathbf{X}}(k), B(t)\}} \; m(dt) < \infty \Rightarrow$ (6).

Coming back to the moments of $W_1(\mu_n, \mu)$ or $W_r(\mu_n, \mu)$

Recall that, with the notation $S_{\alpha,n}(t) = \sum_{k=0}^{n} \min\{\alpha_{1,\mathbf{X}}(k), H(t)\}$,

$$\|W_1(\mu_n, \mu)\|_1 \leq 4 \int_0^{\infty} \sqrt{\min\Big\{\big(H(t)\big)^2, \frac{S_{\alpha,n}(t)}{n}\Big\}}\, dt$$

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For $p\in(1,2)$ we can get a von Bahr-Esseen type inequality.

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$$\|W_1(\mu_n, \mu)\|_1 \leq 4 \int_0^\infty \sqrt{\min\Big\{\big(H(t)\big)^2, \frac{S_{\alpha,n}(t)}{n}\Big\}}\, dt$$

and

$$\sqrt{n}\|W_1(\mu_n, \mu)\|_2 \leq 2\sqrt{2} \int_0^\infty \sqrt{S_{\alpha,n}(t)}\, dt\,.$$

For $p \in (1,2)$ we can get a von Bahr-Esseen type inequality.

Proposition (Dedecker-M. '17)

For $p \in (1,2)$ and $r \geq 1$, the following inequality holds

$$\|W_r^r(\mu_n, \mu)\|_p^p \ll \frac{1}{n^{p-1}} \int_0^1 (\alpha_{1,\mathbf{X}}^{-1}(u) \wedge n)^{p-1} Q^{pr}(u)\, du\,. \qquad (7)$$

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In the m dependent case, this becomes $\|W_r^r(\mu_n, \mu)\|_p^p \ll \frac{1}{n^{p-1}} \|X_0\|_{rp}^{rp}$.

For $u \in (0,1)$, we have

$$\alpha_{1,\mathbf{X}}^{-1}(u) = \sum_{k=0}^{\infty} \mathbf{1}_{u < \alpha_{1,\mathbf{X}}(k)}$$

so the bound writes also

$$\|W_r^r(\mu_n, \mu)\|_p^p \ll \frac{1}{n^{p-1}} \sum_{k=0}^{n} \frac{1}{(k+1)^{2-p}} \int_0^{\alpha_{1,\mathbf{X}}(k)} Q^{pr}(u)du\,.$$

or, setting $S_{\alpha,p,n}(t) = \sum_{k=0}^{n}(k+1)^{p-2} \min\{\alpha_{1,\mathbf{X}}(k), H(t)\}$

$$\|W_r^r(\mu_n, \mu)\|_p^p \ll \frac{1}{n^{p-1}} \int_0^{\infty} S_{\alpha,p,n}(t^{1/(rp)})dt\,.$$

A deviation inequality

For any $n \in \mathbb{N}$, let us introduce the following notations:

$$R_n(u) = (\min\{q \in \mathbb{N}^* : \alpha_{1,\mathbf{X}}(q) \leq u\} \wedge n)Q(u)$$

and [

$$R_n^{-1}(x) = \inf\{u \in [0,1] : R_n(u) \leq x\}.$$

The moment bound comes from

Proposition (Dedecker-M. '17)

For any positive integer n, any $x > 0$, and any $\eta \in [1,2[$, the following inequality holds:

$$\mathbb{P}\left(nW_1(\mu_n, \mu) \geq 6x\right) \leq c_1 \frac{n}{x} \int_0^{R_n^{-1}(x)} Q(u)du + c_2 \frac{n}{x^\eta} \int_{R_n^{-1}(x)}^1 R_n^{\eta-1}(u)Q(u)du,$$

Let $p \in (1,2)$ and consider the LSV map defined before with $\gamma \in (0, 1/p)$. let g be positive and non increasing on (0, 1), with

$$g(x) \le \frac{C}{x^b} \quad \text{near 0, for some } C > 0 \text{ and } b \in [0, (1-\gamma)/p).$$

Hence

$$\| W_1(\tilde{\mu}_n, \mu)) \|_p \ll \begin{cases} n^{(1-p)/p} & \text{if } b < (1-p\gamma)/p \\ (n^{(1-p)} \ln(n))^{1/p} & \text{if } b = (1-p\gamma)/p \\ n^{(pb+\gamma-1)/p\gamma} & \text{if } b > (1-p\gamma)/p. \end{cases}$$

Moreover, if $b = (1-p\gamma)/p$,

$$\mathbb{P}\left(W_1(\mu_n, \mu) \ge x\right) \ll \frac{1}{n^{p-1}x^p}$$

Note that Gouëzel '04 proved that, if $g(x) = x^{-(1-p\gamma)/p}$ then

$$\lim_{n\to\infty} \nu\left(\frac{1}{n^{1/p}} \left| \sum_{k=1}^{n} \left(g \circ T_\gamma^k - \nu(g) \right) \right| > x \right) = \mathbb{P}(|Z_p| > x)\,,$$

where Z_p is a p-stable r.v's s.t. $\lim_{x\to\infty} x^p \mathbb{P}(|Z_p| \ge x) = c \ge 0.$

- If (X_i) is a sequence of independent random variables in $\mathbb{L}^p$ with $p \geq 2$, the Rosenthal inequality says that

$$\| \sum_{i=1}^{n} X_i \|_p^p \ll \| \sum_{i=1}^{n} X_i \|_2^p + \sum_{i=1}^{n} \|X_i\|_p^p \,.$$

Moment bounds when $p > 2$: a Rosenthal-type inequality

- If (X_i) is a sequence of independent random variables in $\mathbb{L}^p$ with $p \geq 2$, the Rosenthal inequality says that

$$\| \sum_{i=1}^n X_i \|_p^p \ll \| \sum_{i=1}^n X_i \|_2^p + \sum_{i=1}^n \|X_i\|_p^p \, .$$

- Our aim is to get a moment inequality implying in the m-dependent setting that

$$\|W_r^r(\mu_n, \mu)\|_p^p \ll \frac{1}{n^{p/2}} \left(\int_0^\infty t^{r-1} \sqrt{H(t)} dt \right)^p + \frac{1}{n^{p-1}} \|X_0\|_{pr}^{pr} \, .$$

Indeed $\frac{1}{n^{1/2}} \int_0^\infty t^{r-1} \sqrt{H(t)} dt$ is a bound of $\|W_r^r(\mu_n, \mu)\|_2$.

The strategy (1)

- Our strategy will be to derive a suitable deviation bound for $W_1(\mu_n, \mu)$, i.e. for $\mathbb{P}(nW_1(\mu_n, \mu) \geq x)$ by truncating the r.v. at a level M,

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- Hence we shall make use of the following Rosenthal-type inequality for stationary m.d.s. $(D_i)_i$ adapted to a stationary filtration $(\mathcal{F}_i)_i$.

Theorem (M. & Peligrad (2013))

Let $p > 2$. Then for any $n \geq 1$,

$$\| \max_{1 \leq j \leq n} |M_j| \|_p \ll n^{1/p} \Big(\|D_1\|_p + \Big(\sum_{k=1}^{n} \frac{1}{k^{1+2\delta/p}} \|\mathbb{E}_0(M_k^2)\|_{p/2}^{\delta} \Big)^{1/(2\delta)} \Big),$$

where $\delta = \min(1, 1/(p-2))$ and $\mathbb{E}_0(D) = \mathbb{E}(D|\mathcal{F}_0)$.

The strategy (2)

- We are lead to take care of the following quantities : setting $f_x(u) = \mathbf{1}_{x \le u}$ and $Z^{(0)} = Z - \mathbb{E}(Z)$,

$$\alpha_{2,\mathbf{X}}(n) = \sup_{x,y \in \mathbb{R}} \sup_{m \ge 0} \Big\| \mathbb{E}\left(f_x^{(0)}(X_n) f_y^{(0)}(X_{n+m}) | \mathcal{F}_0 \right) - \mathbb{E}\Big(f_x^{(0)}(X_n) f_y^{(0)}(X_{n+m}) \Big) \Big\|_1$$

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- For the intermittent map, in addition to

$$\mathbf{H}_1 : \qquad \sup_{f \in BV_1} \nu\Big(\big| K^n(f) - \nu(f) \big| \Big) \leq \frac{C_1}{n^{(1-\gamma)/\gamma}}$$

we also have, for any function f in BV,

$$\mathbf{H}_2 : \qquad |K^n(f)|_v \leq C_2 |f|_v \,.$$

(See Dedecker-Gouëzel-M. '10). And then $\alpha_{2,\mathbf{Y}}(n) \ll n^{-(1-\gamma)/\gamma}$

A deviation inequality inequality

Proposition (Dedecker-M. '17)

There exists a positive universal constant c such that, for any positive integer n, any $x > 0$, any $\eta > 2$ and any $\beta \in (\eta - 2, \eta)$, the following inequality holds:

$$\mathbb{P}\left(nW_1(\mu_n, \mu) \geq x\right) \leq c\frac{n^{\eta/2}}{x^{\eta}} s_{\alpha,n}^{\eta} + \frac{n}{x^{1+\beta/2}} \int_0^{R_n^{-1}(x)} R_n^{\beta/2}(u)Q(u)du + c\frac{n}{x^{1+\eta/2}} \int_{R_n^{-1}(x)}^{1} R_n^{\eta/2}(u)Q(u)du\,,$$

where $s_{\alpha,n} = \int_0^{\infty} \sqrt{S_{\alpha,n}(t)}dt = \int_0^{\infty} \sum_{k=0}^{n} \min\left\{\alpha_{1,\mathbf{X}}(k), H(t)\right\} dt$ *and* $R_n(u) = (\alpha_{2,\mathbf{X}}^{-1}(u) \wedge n)Q(u)$.

Rosenthal-type inequality

Integrating the previous inequality, we derive

Theorem (Dedecker-M. '17)

For $p > 2$, the following inequality holds:

$$\|W_1(\mu_n, \mu)\|_p^p \ll \frac{s_{\alpha,n}^p}{n^{p/2}} + \frac{1}{n^{p-1}} \int_0^1 \left(\alpha_{2,\mathbf{X}}^{-1}(u) \wedge n\right)^{p-1} Q^p(u) du \,,$$

Application to the LSV map

- Let $p > 2$, and let g be positive and non increasing on (0, 1), with

$$g(x) \leq \frac{C}{x^b} \quad \text{near } 0, \text{ for some } C > 0 \text{ and } b \in [0, (1-\gamma)/p).$$

The following upper bounds hold.

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The following upper bounds hold.
For $\gamma \in (0, 1/2)$

$$\| W_1(\tilde{\mu}_n, \mu)) \|_p \ll \begin{cases} n^{-1/2} & \text{if } b \leq (2 - \gamma(p+2))/2p \\ n^{(pb+\gamma-1)/p\gamma} & \text{if } b > (2 - \gamma(p+2))/2p. \end{cases}$$

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For $\gamma \in [1/2, 1)$, $\| W_1(\tilde{\mu}_n, \mu)) \|_p \ll n^{(pb+\gamma-1)/p\gamma}$.

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For $\gamma \in [1/2, 1)$, $\|W_1(\tilde{\mu}_n, \mu))\|_p \ll n^{(pb+\gamma-1)/p\gamma}$.

- If $b = 0$, the bounds are optimal (see Chazottes-Gouëzel '12 and Gouëzel-Melbourne '14 where concentration inequalities have been established for intermittent maps).

On Moderate deviations

- Starting from the deviation bound and assuming that for $p > 2$,

$$\sup_{x>0} x^{p-1} \int_0^1 Q(u) \mathbb{1}_{R(u)>x} du < \infty \quad (*)$$

where $R(u) = \alpha_{2,\mathbf{Y}}^{-1}(u) Q(u)$, it follows that for any $\alpha \in]1/2, 1]$ and such that $\alpha > 1 - 1/p$,

$$\limsup_{n\to\infty} n^{\alpha p - 1} \mathbb{P}\left(n W_1(\mu_n, \mu) \geq n^{\alpha} x \right) \leq \kappa x^{-p}$$

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- In the independent setting (and more generally in the m-dependent setting),

$$(*) \iff \sup_{x>0} x^{p-1} \mathbb{E}(|X_0| \mathbf{1}_{|X_0|>x}) < \infty \iff \sup_{x>0} x^{p} \mathbb{P}(|X_0| > x) < \infty \,.$$

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- If we replace the weak dependence coefficient $\alpha_{2,\mathbf{Y}}(k)$ by the strong mixing ones, then it suffices to take $\alpha \in]1/2, 1]$. This is also true for the maximum of partial sums associated with Hölder observables of the LSV map (Dedecker-Gouëzel-M. '18).

What about the moments of $W_r^r(\mu_n, \mu)$ in higher dimensions ?

- In our proofs, the Ebralidze's inequality plays a crucial role :

$$W_r^r(\mu_n, \mu) \leq \kappa_r \int_{\mathbb{R}} |x|^{r-1} |F_n(x) - F(x)| dx$$

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- By Lemmas 5 and 6 in Fournier-Guillin '15, there exists a constant C depending only on r and d such that

$$W_r^r(\mu_n, \mu) \le C D_r(\mu_n, \mu) \, .$$

where

$$D_r(\mu_n, \mu) = \sum_{m \ge 0} 2^{rm} \sum_{\ell \ge 0} 2^{-r\ell} \sum_{F \in \mathcal{P}_\ell} |\mu_n(2^m F \cap B_m) - \mu(2^m F \cap B_m)| \, ,$$

$\mathcal{P}_\ell$ being the natural partition of $(-1, 1]^d$ into $2^{d\ell}$ translations of $(-2^{-\ell}, 2^{-\ell}]^d$
$B_0 = (-1, 1]^d$ and $B_m = (-2^m, 2^m]^d \backslash (-2^{m-1}, 2^{m-1}]^d$, for $m \ge 1$.

- Fournier-Guillin's upper bound is a modified version of the result by Dereich-Scheutzow-Schottstedt '13. With the help of this bound, they give sharp bounds for $\mathbb{E}(W_r^r(\mu_n, \mu))$ for iid random vectors with values in $\mathbb{R}^d$.

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- Starting from their upper bound, in the iid case and if $\|X\|_{rp} < \infty$ for some $p > 2$, one can for instance prove the following Rosenthal inequalities :

  If $r > d(p-1)/p$,

$$\|W_r^r(\mu_n, \mu)\|_p^p \ll \frac{1}{n^{p/2}} \left(\int_0^\infty t^{r-1} \sqrt{H(t)}\, dt \right)^p + \frac{\|X\|_{pr}^{pr}}{n^{p-1}}$$

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  If $r \in [1, d/2)$,

$$\|W_r^r(\mu_n, \mu)\|_p^p \ll \frac{\|X\|_{pr}^{pr}}{n^{pr/d}}$$

  (Work in progress with J. Dedecker)

Thank you for your attention!