

ARCHIMEDES, BARENBLATT, AND SATO-TATE

SPHERICAL PROJECTIONS AND ASSOCIATED CLASSICAL PROBABILITY DISTRIBUTIONS

DJALIL CHAFAÏ

ABSTRACT. Notes from a talk at the informal probability working group <https://probas.math.ens.psl.eu/>, through history, with elements of geometry, probability, analysis, algebra, and number theory.

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1. ARCHIMEDES' THEOREM ON THE SPHERE AND CYLINDER

“Finally although in the year 1500 Europe knew less than Archimedes who died in the year 212 BC, yet in the year 1700, Newton's *Principia* had been written and the world was well started on the modern epoch.” Alfred North Whitehead, in *Science and the Modern World* (1925), quoted notably by Arthur Koestler in *The Sleepwalkers* (1959).

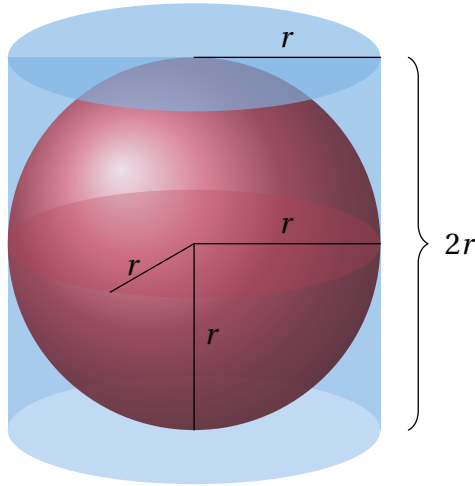


FIGURE 1. A sphere and its circumscribed cylinder, that is, the smallest cylinder containing it.

For $n = 3$ and $k = 1$, Theorem 1.1 states that the uniform distribution on the sphere, projected onto its vertical diameter, yields the uniform distribution on that diameter. This coincides with the discovery of Archimedes of Syracuse (287–212 BC): considering the section cut by two parallel planes perpendicular to the cylinder's axis, the area of the portion of the sphere between the two planes equals the area of the portion of the cylinder between them. In particular, the sphere and the lateral surface of the cylinder have the same area, which amounts to saying that the area of the sphere of radius r is $2\pi r \times 2r = 4\pi r^2$. Archimedes used the method of exhaustion, anticipating integral calculus by more than 1,800 years.

Theorem 1.1 is the generalization to all dimensions, expressed in the contemporary language of probability theory. Legend has it that Archimedes considered this his best result and asked for a drawing of a sphere and its circumscribed cylinder to be engraved on his tombstone. Cicero (106–43 BC) recounts in the *Tusculan Disputations* (V, 64–66) that he found this tomb in Syracuse in 75 BC thanks to this motif, 137 years after the death of Archimedes, who was killed by a Roman soldier during the siege of the city.

Theorem 1.1: On the sphere and cylinder (Archimedes)

Let $X = (X_1, \dots, X_n)$ be uniformly distributed on $\mathbb{S}^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$, $n \geq 2$. Then for all $1 \leq k < n$, the vector $Y = \text{proj}_{\mathbb{R}^k}(X) = (X_1, \dots, X_k)$, taking values in the ball $\mathbb{B}^k = \{x \in \mathbb{R}^k : |x| \leq 1\}$, has density

$$y \mapsto c(1 - |y|^2)_+^{\frac{n-k-2}{2}} \quad \text{where} \quad c = \frac{\Gamma(n/2)}{\pi^{k/2} \Gamma((n-k)/2)}.$$

In particular, if $n \geq 3$, then $\text{proj}_{\mathbb{R}^{n-2}}(X) = (X_1, \dots, X_{n-2})$ is uniformly distributed on the ball $\mathbb{B}^{n-2}$.

In densities, we take t_+^a to mean t^a if $t > 0$ and 0 if $t \leq 0$, including when $a \leq 0$.

The case $k = 1$ corresponds to the Funk–Hecke formula in spherical harmonic analysis.

The distribution of Y is a multivariate symmetric beta distribution on $\mathbb{B}^k$, uniform if $k = n - 2$. If $k < n - 2$, its exponent is strictly positive and, up to scaling, it has a compactly supported Barenblatt profile, considered below. The case $k = n - 1$ has exponent $-1/2$ and a density unbounded at the boundary, a kind of radial arcsine distribution, which turns out to be a natural equilibrium measure in potential theory, solving a problem of Marcel Riesz, cf. [CSW].

In particular, for every $n \geq 2$ and $k = 1$, rotational invariance of the distribution of X implies that, for every $\theta \in \mathbb{S}^{n-1}$, $\langle X, \theta \rangle \stackrel{d}{=} X_1 \sim \text{Beta}_{[-1,1]}(\frac{n-1}{2}, \frac{n-1}{2})$, with density proportional to $x \mapsto (1-x)_+^p (1+x)_+^p = (1-x^2)_+^p$ where $p = (n-3)/2$. This is the uniform distribution if $n = 3$, the semicircle distribution if $n = 4$, and the arcsine distribution if $n = 2$.

Proof. Orthogonal invariance of the distribution of X ensures that of Y . It therefore suffices to determine the distribution of $|Y|^2$. Now X has the same distribution as $Z/|Z|$, where $Z = (Z_1, \dots, Z_n) \sim \mathcal{N}(0, I_n)$. Thus Y has the same distribution as $(Z_1, \dots, Z_k)/|Z|$, so that $|Y|^2$ has the same distribution as $A/(A+B)$ with independent $A \sim \chi^2(k)$ and $B \sim \chi^2(n-k)$. It follows¹ that² $|Y|^2 \sim \text{Beta}_{[0,1]}(k/2, (n-k)/2)$. Thus $|Y|$ has density $r \mapsto c' r_+^{k-1} (1-r^2)_+^{(n-k-2)/2}$ where $c' = 2/\text{beta}(k/2, (n-k)/2)$. In polar coordinates $y = r\theta$, the volume element is $dy = r^{k-1} dr d\sigma(\theta)$, where σ is the surface measure on $\mathbb{S}^{k-1}$ (with $|\mathbb{S}^0| = 2$). By orthogonal invariance, the density of Y is therefore $c(1 - |y|^2)_+^{(n-k-2)/2}$, with

$$c = \frac{c'}{|\mathbb{S}^{k-1}|} = \frac{2}{|\mathbb{S}^{k-1}| \text{beta}(k/2, (n-k)/2)} = \frac{\Gamma(n/2)}{\pi^{k/2} \Gamma((n-k)/2)}.$$

□

Corollary 1.2: Representation of the uniform distribution on the ball, or Archimedes' theorem reversed

For all $n \geq 1$, if $Z_1, \dots, Z_n$ are i.i.d. with distribution $\mathcal{N}(0, 1)$ and E is exponential with mean 1, independent of $Z_1, \dots, Z_n$, then the random vector $(Z_1, \dots, Z_n)/\sqrt{Z_1^2 + \dots + Z_n^2 + 2E}$ is uniformly distributed on $\mathbb{B}^n$.

Proof. If $Z_1, \dots, Z_n, Z_{n+1}, Z_{n+2}$ are i.i.d. $\mathcal{N}(0, 1)$, then $(Z_1, \dots, Z_n)/\sqrt{Z_1^2 + \dots + Z_{n+2}^2}$ is uniformly distributed on $\mathbb{B}^n$ by Theorem 1.1. On the other hand, $Z_{n+1}^2 + Z_{n+2}^2 \sim \chi^2(2) = \text{Gamma}(2/2, 1/2) = \text{Exp}(1/2) \stackrel{d}{=} 2E$. □

This representation theorem for the uniform distribution extends to ℓ^p balls; see [B+].

Combined with the law of large numbers, this theorem shows that if X is uniformly distributed on $\mathbb{B}^n(\sqrt{n})$, then for every $\theta \in \mathbb{S}^{n-1}$, $\langle X, \theta \rangle \stackrel{d}{=} X_1 \stackrel{d}{\rightarrow} \mathcal{N}(0, 1)$ as $n \rightarrow \infty$. This is a special case of the central limit phenomenon for convex bodies, which was the subject of a conjecture proved in [K]. At the level of the density,

$$\left(1 - \frac{|y|^2}{n}\right)_+^{\frac{n-1}{2}} \xrightarrow{n \rightarrow \infty} e^{-\frac{|y|^2}{2}}, \quad y \in \mathbb{R}^n.$$

Finally, this theorem shows that in high dimension $n \rightarrow \infty$, most of the mass of $\mathbb{B}^n(\sqrt{n})$ lies near $\mathbb{S}^{n-1}(\sqrt{n})$. This is a special case of the thin-shell phenomenon for convex bodies, which was the subject of a conjecture

¹Recall that $\text{Gamma}(a, \lambda)$, $a, \lambda > 0$, has density $u \mapsto \Gamma(a)^{-1} \lambda^a u^{a-1} e^{-\lambda u} \mathbb{1}_{u \geq 0}$, where $\Gamma(a) = \int_0^\infty u^{a-1} e^{-u} du$. If $A_1, \dots, A_n$ are independent with $A_k \sim \text{Gamma}(a_k, \lambda)$, then $(A_1, \dots, A_n)/(A_1 + \dots + A_n) \sim \text{Dirichlet}_n(a_1, \dots, a_n)$, which does not depend on λ . This is a distribution on the simplex of discrete probability distributions $\{(p_1, \dots, p_n) \in [0, 1]^n : p_1 + \dots + p_n = 1\}$. If $(B_1, \dots, B_n) \sim \text{Dirichlet}_n(a_1, \dots, a_n)$, then for any $I_1, \dots, I_r$ forming a partition of $\{1, \dots, n\}$ into $r \geq 2$ nonempty blocks, $(\sum_{i \in I_1} B_i, \dots, \sum_{i \in I_r} B_i) \sim \text{Dirichlet}_r(\sum_{i \in I_1} a_i, \dots, \sum_{i \in I_r} a_i)$, and for every nonempty proper subset $I \subset \{1, \dots, n\}$, $\sum_{i \in I} B_i \sim \text{Beta}_{[0,1]}(\sum_{i \in I} a_i, \sum_{i \notin I} a_i)$. If $U_1, \dots, U_n$ are i.i.d. uniform on $[0, 1]$, and if $B_1, \dots, B_{n+1}$ are the lengths of the blocks in the associated partition of $[0, 1]$, the successive spacings in the order statistics, that is, the increasing rearrangement, then $(B_1, \dots, B_{n+1}) \sim \text{Dirichlet}_{n+1}(1, \dots, 1)$. Dirichlet distributions play a fundamental role in simulation, nonparametric mathematical statistics, and probability theory. Poisson–Dirichlet distributions are distributions on random mass partitions into infinitely many components, related to Dirichlet distributions through limiting and reordering procedures.

² $\text{Beta}_{[0,1]}(a, b)$, $a, b > 0$, has density $u \mapsto \text{beta}(a, b)^{-1} u^{a-1} (1-u)^{b-1}$ where $\text{beta}(a, b) = \int_0^1 u^{a-1} (1-u)^{b-1} du = \Gamma(a)\Gamma(b)/\Gamma(a+b)$.

recently proved in [KL]. An alternative proof giving an optimal constant was obtained in [CK]: for an isotropic log-concave random vector W in $\mathbb{R}^n$, $\text{Var}(|W|^2) \leq 8n$.

Theorem 1.3: Stereographic projection of the uniform distribution on the sphere

Let X be a random vector uniformly distributed on the unit sphere $\mathbb{S}^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$, $n \geq 2$. Let $Y = (Y_1, \dots, Y_{n-1}) = T(X)$ where $T : \mathbb{S}^{n-1} \rightarrow \mathbb{R}^{n-1} \cup \{\infty\}$ is the stereographic projection from the north pole e_n defined by $T(x) = (x_1, \dots, x_{n-1})/(1 - x_n)$ if $x = (x_1, \dots, x_n) \neq e_n$, and $T(e_n) = \infty$. Then for every $1 \leq k \leq n-1$, the random vector $\text{proj}_{\mathbb{R}^k}(Y) = (Y_1, \dots, Y_k)$, taking values in $\mathbb{R}^k$, has density

$$y \mapsto c(1 + |y|^2)^{-\frac{n+k-1}{2}} \quad \text{where} \quad c = \frac{\Gamma(\frac{n+k-1}{2})}{\pi^{k/2} \Gamma(\frac{n-1}{2})}.$$

For $n \geq 3$, this distribution has, up to scaling, a Barenblatt profile with full support. The case $n = 2$ would correspond to $m = 0$, excluded from the $m > 0$ of the next section. It is also a multivariate Student distribution of dimension $d = n - 1$, with location $\mu = 0$, scale matrix $\Sigma = d^{-1}I_d$, and $r = n - 1$ degrees of freedom, with density

$$y \mapsto c(1 + r^{-1}(y - \mu) \cdot \Sigma^{-1}(y - \mu))^{-(r+d)/2} \quad \text{where} \quad c = \frac{\Gamma((r+d)/2)}{\Gamma(r/2)(r\pi)^{d/2} \sqrt{\det \Sigma}}.$$

This is the distribution of $\mu + A/\sqrt{B/r}$, where $A \sim \mathcal{N}(0, \Sigma)$ and $B \sim \chi^2(r)$ are independent. For $n = 2$, this is the Cauchy distribution on $\mathbb{R}$, the distribution of the ratio of two independent standard real Gaussian random variables. For $n = 3$, this is the complex Cauchy distribution on $\mathbb{R}^2 \equiv \mathbb{C}$, the distribution of the ratio of two independent standard complex Gaussian random variables.

The case $k = 1$ shows that if X is uniform on $\mathbb{S}^{n-1}$ and $Y = T(X)$ then $\sqrt{n} Y_1 \xrightarrow{d} \mathcal{N}(0, 1)$ as $n \rightarrow \infty$, since

$$\left(1 + \frac{y^2}{n}\right)^{-\frac{n}{2}} \xrightarrow{n \rightarrow \infty} e^{-\frac{|y|^2}{2}}, \quad y \in \mathbb{R}.$$

Proof. Rotational invariance of the distribution of X implies that of $(Y_1, \dots, Y_k)$. Thus, for every $\theta \in \mathbb{R}^k$, $\langle \theta, (Y_1, \dots, Y_k) \rangle \stackrel{d}{=} |\theta| Y_1$. Now since $Y_1 = X_1/(1 - X_n)$, for every $t \in \mathbb{R}$ we obtain

$$\mathbb{P}(Y_1 \leq t) = \mathbb{P}(X_1 + t X_n \leq t) = \mathbb{P}\left(X_1 \leq \frac{t}{\sqrt{1+t^2}}\right)$$

where the second equality follows from rotational invariance of the law of X . Since $t \mapsto t/\sqrt{1+t^2}$ has inverse $u \mapsto u/\sqrt{1-u^2}$, it follows that $Y_1 \stackrel{d}{=} X_1/\sqrt{1-X_n^2}$. Since $X \stackrel{d}{=} Z/|Z|$ where $Z = (Z_1, \dots, Z_n) \sim \mathcal{N}(0, I_n)$, we obtain

$$Y_1 \stackrel{d}{=} \frac{Z_1}{\sqrt{Z_2^2 + \dots + Z_n^2}} = \frac{A}{\sqrt{B}}$$

where $A \sim \mathcal{N}(0, 1)$ and $B \sim \chi^2(n-1)$ are independent. Thus $\sqrt{n-1} Y_1$ has the Student distribution $t(n-1)$. If $G \sim \mathcal{N}(0, I_k)$ is independent of B , then for every $\theta \in \mathbb{R}^k$, $\langle \theta, G/\sqrt{B} \rangle \stackrel{d}{=} |\theta| A/\sqrt{B} \stackrel{d}{=} \langle \theta, (Y_1, \dots, Y_k) \rangle$. Thus $(Y_1, \dots, Y_k) \stackrel{d}{=} G/\sqrt{B}$ by characteristic functions (the Cramér–Wold theorem). This gives the stated result.

After this geometric-statistical proof, let us give a purely analytic proof using the change of variables formula. The north pole e_n has zero surface measure, so $T(X)$ almost surely takes values in $\mathbb{R}^{n-1}$. The inverse of T on $\mathbb{S}^{n-1} \setminus \{e_n\}$ is $F(y) = (2y/(1 + |y|^2), (|y|^2 - 1)/(1 + |y|^2))$. Set $q = 1 + |y|^2$. For $1 \leq i \leq n-1$, $\partial_i F(y) = (2e_i/q - 4y_i y/q^2, 4y_i/q^2)$ and $\langle \partial_i F(y), \partial_j F(y) \rangle = (4/q^2)\delta_{ij}$. The Jacobian is therefore

$$J_F(y) = \sqrt{\det((DF(y))^T DF(y))} = \left(\frac{2}{1 + |y|^2}\right)^{n-1}.$$

If $|\mathbb{S}^{n-1}| = 2\pi^{n/2}/\Gamma(n/2)$ denotes the surface measure of $\mathbb{S}^{n-1}$, the change of variables formula gives

$$\mathbb{E}[h(T(X))] = \frac{1}{|\mathbb{S}^{n-1}|} \int_{\mathbb{R}^{n-1}} h(y) J_F(y) dy$$

for every bounded Borel function $h : \mathbb{R}^{n-1} \rightarrow \mathbb{R}$. □

Remark 1.4: Spheres, projections, beta and Student distributions, Barenblatt profiles, and Gaussians

Theorem 1.1 shows that the linear projection of $\mathbb{S}^{n-1}$ onto $\mathbb{R}^k$ produces compactly supported profiles $c(1 - |y|^2)_+^a$ with $a = (n - k - 2)/2 \geq -1/2$, whereas Theorem 1.3 shows that the nonlinear stereographic projection of $\mathbb{S}^{n-1}$ onto $\mathbb{R}^{n-1}$ produces a heavy-tailed profile $c(1 + |y|^2)^{-b}$ with $b > 0$. The parameters a and b are integers or half-integers, and this quantization comes from the geometric realization involving

dimensions, which are integers. All of this can also be viewed as the mechanics of Eulerian special functions: Gamma and multivariate beta functions. Beta and Student distributions can be defined geometrically as continuous-parameter generalizations of the distributions of spherical projections. As explained below, they are connected by the family of Barenblatt profiles or distributions in the parameter ranges specified above. This family also contains the Gaussian distribution as a common critical case.

This part is inspired by [LPM01] and [L3GD].

2. BARENBLATT PROFILES: FAST DIFFUSION AND POROUS MEDIA

Consider the following nonlinear partial differential equation for a density u on $\mathbb{R}_+ \times \mathbb{R}^n$, $n \geq 1$:

$$\partial_t u = \Delta(u^m) = \operatorname{div}(mu^{m-1}\nabla u), \quad t > 0.$$

The parameter m is a positive real number. The case $m = 1$ is the linear heat equation. The case $0 < m < 1$ is called the fast diffusion equation. The case $m > 1$ is called the porous medium equation. Assume that

$$m > \max\left(0, 1 - \frac{2}{n}\right) \quad \text{and} \quad \beta = \frac{1}{2 + n(m-1)} > 0,$$

which will ensure that finite-mass Barenblatt profiles yield solutions.

We seek mass-preserving self-similar solutions of the form

$$u(t, x) = t^{-n\beta} B(xt^{-\beta})$$

where B is a profile to be determined. Matching powers yields $1 = 2\beta + n\beta(m-1)$, while for the profile,

$$\Delta B^m + \beta \operatorname{div}(xB) = 0 \quad \text{that is,} \quad \operatorname{div}(\nabla B^m + \beta xB) = 0.$$

Seeking a radial B with zero radial flux gives $\nabla B^m = -\beta xB$. For $m \neq 1$, on $\{B > 0\}$, this gives

$$\nabla B^{m-1} = -\frac{\beta(m-1)}{m}x.$$

Integration yields two profiles, with a constant $c > 0$ chosen so that $\int_{\mathbb{R}^n} B(x)dx = 1$:

$$B(x) = \begin{cases} \left(c + \frac{\beta(1-m)}{2m}|x|^2\right)^{-1/(1-m)} & \text{if } 0 < m < 1 \\ \left(c - \frac{\beta(m-1)}{2m}|x|^2\right)_+^{1/(m-1)} & \text{if } m > 1 \end{cases}.$$

The profile for the fast diffusion equation (case $m < 1$) has a heavy polynomial tail, and the integrability condition requires $2/(1-m) > n$, that is, $m > 1 - 2/n$. The profile for the porous medium equation (case $m > 1$) has compact support, the centered ball whose radius grows with time (slow diffusion):

$$R(t) = \sqrt{\frac{2mc}{\beta(m-1)}} t^\beta.$$

For $m = 1$, the centered self-similar solution of unit mass is the (Gaussian) heat kernel

$$u(t, x) = (4\pi t)^{-n/2} \exp(-|x|^2/(4t)).$$

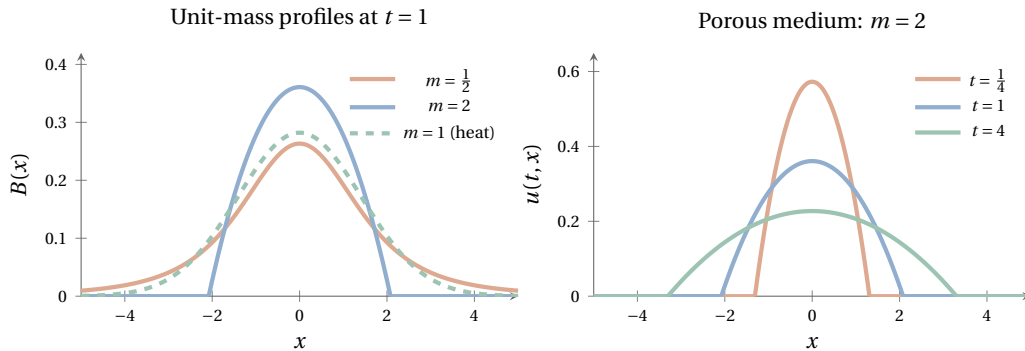


FIGURE 2. Exact unit-mass solutions in dimension one.

If $m > 1$, the extension by zero satisfies the equation in the weak sense, including at the boundary of the support. In all three cases, the measures $u(t, x)dx$ converge weakly to the Dirac mass δ_0 as $t \searrow 0$.

The Barenblatt profile B interpolates between the multivariate Student distribution (when $\max(0, 1 - 2/n) < m < 1$) and the multivariate beta distribution (when $m > 1$), passing through the Gaussian for $m = 1$. The number of degrees of freedom in the Student case is $r = 2/(1 - m) - n > 0$. With unit-mass normalization, the profiles converge to $(4\pi)^{-n/2} \exp(-|x|^2/4)$, the Gaussian profile with covariance $2I_n$, as $m \rightarrow 1$ from either side.

This part is inspired by [LPMO2]. For an in-depth study of the profiles and their role in nonlinear diffusion, see [V]. For self-similarity and intermediate asymptotics, see [B].



Grigory Isaakovich Barenblatt (1927 – 2018)

3. THE SATO–TATE DISTRIBUTION IN NUMBER THEORY VIA $\mathbb{S}^3$ AND $SU(2)$

The group $SU(2)$ and the sphere $\mathbb{S}^3$. The elements of $SU(2)$ are uniquely parametrized by

$$U = \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix} \quad \text{where } a, b \in \mathbb{C}, \quad |a|^2 + |b|^2 = 1.$$

Writing $a = x_1 + ix_2$ and $b = x_3 + ix_4$, the group $SU(2)$ is identified with the sphere $\mathbb{S}^3 \subset \mathbb{R}^4$, which is also the unit sphere of quaternions. Left and right group multiplication act orthogonally on these four coordinates, so Haar measure on $SU(2)$ corresponds to the uniform distribution on $\mathbb{S}^3$.

Archimedes gives the distribution of the trace. If $X = (X_1, X_2, X_3, X_4)$ is uniformly distributed on $\mathbb{S}^3$, then its projection (X_1, X_2) onto $\mathbb{R}^2$ is uniform on the unit disk, by Archimedes' Theorem 1.1. It follows immediately that its projection X_1 onto the line has the semicircle distribution supported on $[-1, 1]$, with density

$$\frac{2\sqrt{(1-s^2)_+}}{\pi}.$$

Consequently, if U follows Haar measure (the uniform distribution) on $SU(2)$, then

$$\text{Tr}(U) = a + \bar{a} \stackrel{d}{=} 2X_1$$

has the semicircle distribution supported on $[-2, 2]$, with density

$$\frac{\sqrt{(4-s^2)_+}}{2\pi}.$$

It has zero mean and unit variance. The eigenvalues of U are $e^{\pm i\theta}$, with $\theta \in [0, \pi]$, and $\text{Tr}(U) = 2 \cos(\theta)$. A change of variable leads to the angular Sato–Tate distribution, with density

$$\theta \in [0, \pi] \mapsto \frac{2}{\pi} \sin^2(\theta).$$

The arithmetic theorem. Let $E/\mathbb{Q}$ be an elliptic curve without complex multiplication. For every prime p of good reduction, counting all points, including the point at infinity, define

$$a_p = p + 1 - \#E(\mathbb{F}_p), \quad \text{and} \quad s_p = \frac{a_p}{\sqrt{p}} = 2 \cos \theta_p \in [-2, 2], \quad \theta_p \in [0, \pi].$$

The bound $|a_p| \leq 2\sqrt{p}$ is Hasse's theorem. The Sato–Tate equidistribution theorem, formerly a conjecture, states that the angles θ_p are equidistributed according to the angular measure above. Equivalently, if $\mathcal{P}(x)$ is the set of primes less than or equal to x with good reduction for the chosen elliptic curve E , then for every continuous real-valued function h on $[-2, 2]$,

$$\lim_{x \rightarrow \infty} \frac{1}{|\mathcal{P}(x)|} \sum_{p \in \mathcal{P}(x)} h(s_p) = \int_{-2}^2 h(s) \frac{\sqrt{4-s^2}}{2\pi} ds.$$

The Frobenius eigenvalues have modulus $\sqrt{p}$ and product p . Normalization gives a conjugacy class in $SU(2)$ with trace s_p . Archimedes' theorem identifies the distribution of the trace under Haar measure; equidistribution of the Frobenius classes is the deep arithmetic content of the Sato–Tate theorem.

This section is inspired by [LPMO3]. An introduction to Sato–Tate distributions and their connection with compact groups can be found in [S19]. An account of the first steps in the proof of the Sato–Tate theorem, for curves with potentially multiplicative reduction at at least one prime, can be found in [C]. It is related to Fermat's Last Theorem and the Taniyama–Shimura–Weil conjecture!



FIGURE 3. It turns out that the Fields Medal explicitly evokes Archimedes' Theorem 1.1 in its ancient version $n = 3$ and $k = 1$. More precisely, on the left, the portrait of Archimedes on the obverse of the Fields Medal, his name in Greek, a date, and a Latin quotation from the poet Marcus Manilius: "*Transire suum pectus mundoque potiri (to rise above oneself and conquer the world)*". On the right, on the reverse of the medal, a sentence in Latin: "*Congregati ex toto orbe mathematici ob scripta insignia tribuere (mathematicians gathered from all over the world have awarded this for outstanding contributions)*" and, in the background, a sphere inscribed in a cylinder, placed behind a branch and recalling the motif on Archimedes' tomb and his theorem "On the Sphere and Cylinder". The edge of the medal, not shown here, bears the recipient's name. Archimedes' theorem on the sphere and cylinder was considered one of the most impressive of Greek antiquity, and taught as such in all Western schools before the advent of modern mathematics, which explains its presence on the medal, designed around 1933. But who knows this nowadays? The description of the medal is that of the International Mathematical Union [IMU].



FIGURE 4. The death of Archimedes, an illustration after a painting by Édouard Vimont, reproduced in John Lord's *Beacon Lights of History*, evoking the killing of Archimedes of Syracuse during the siege of this Greek city in Sicily by Roman soldiers in 212 BC. The death of Archimedes is a classical theme in painting. This illustration has the merit of showing the sphere and cylinder very explicitly, on the wall behind the scholar's seat [Vi].

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