On Poincaré inequalities for gases

[Bolley–C.–Fontbona 2018]
[C.–Lehec 2020]
[C. 2026]

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The main result in one slide

- Complex Ginibre ensemble (CGE)
  eigenvalues of an $n \times n$ matrix with iid $\mathcal{N}_{\mathbb{C}}(0, \frac{1}{n})$ entries

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with equality iff $f = \Re(a(z_1 + \cdots + z_n)) + c$, $a \in \mathbb{C}$, $c \in \mathbb{R}$.

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- 1 of 5 proofs: Vandermonde transform, Hörmander $\bar{\partial}$, divisibility

Outline

Poincaré inequality

Bakry–Émery

Complex Ginibre

Proof

Beyond

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- $\{0\} \subset \mathrm{Spectrum}(-\mathrm{A}) \subset [0, +\infty)$

Poincaré inequality [Poincaré 1890]

- Variance

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- Poincaré inequality or exponential decay or spectral gap

$$\mathrm{Var}_\mu(f) \leq c_\mathrm{P}\, \mathcal{E}_\mu(f)$$

$$\mathrm{Var}_\mu(\mathrm{P}_t f) \leq \mathrm{e}^{-2t/c_\mathrm{P}}\, \mathrm{Var}_\mu(f)$$

$$\mathrm{Spectrum}(-\mathrm{A}) \subset \{0\} \cup [1/c_\mathrm{P}, +\infty)$$

Gaussian case: Ornstein–Uhlenbeck

Theorem ([Nash 1958, Chernoff 1981])

Let $\mu = \mathcal{N}(0, I_d)$, $E(x) = \frac{|x|^2}{2}$, *and* $\mathrm{A} = \Delta - x \cdot \nabla$.

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- Requires convexity of E: fails for planar Coulomb gases

Bakry–Émery: carré du champ and curvature

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  - Curvature $\mathrm{CD}(\rho, \infty)$: $\Gamma_2 \geq \rho\Gamma$
  - Sub-commutation: $|\nabla \mathrm{P}_t f| \leq \mathrm{e}^{-\rho t}\mathrm{P}_t|\nabla f|$

Bakry–Émery: carré du champ and curvature

- Short time expansion along the semigroup $\mathrm{P}_t = \mathrm{e}^{t\mathrm{A}}$

$$\mathrm{Var}_\mu(\mathrm{P}_t f) = \mathrm{Var}_\mu(f) - 2t \int \Gamma(f)\mathrm{d}\mu + 2t^2 \int \Gamma_2(f)\mathrm{d}\mu + O(t^3)$$

- Carré du champ: $\Gamma(f) = \frac{1}{2}\big(\mathrm{A}(f^2) - 2f\,\mathrm{A}f\big) = |\nabla f|^2$
- Its iteration, via Bochner commutation $\nabla \mathrm{A} = \mathrm{A}\nabla - \nabla^2 E \nabla$

$$\Gamma_2(f) = \tfrac{1}{2}\big(\mathrm{A}\Gamma(f) - 2\Gamma(f, \mathrm{A}f)\big) = \|\nabla^2 f\|^2_{\mathrm{HS}} + \langle \nabla f, \nabla^2 E \nabla f\rangle$$

- Equivalent, for $\rho \in \mathbb{R}$, with $c_t = \frac{1-\mathrm{e}^{-2\rho t}}{\rho}$ ($2t$ if $\rho = 0$):
  - $\nabla^2 E \succeq \rho I_d$
  - Curvature $\mathrm{CD}(\rho, \infty)$: $\Gamma_2 \geq \rho\Gamma$
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  - Local PI: $\mathrm{P}_t(f^2) - (\mathrm{P}_t f)^2 \leq c_t \mathrm{P}_t(\Gamma(f))$

Bakry–Émery: proof of the Poincaré inequality

- Interpolation along the semigroup

$$\mathrm{Var}_\mu(f) = -\int_0^\infty \partial_t \,\mathrm{Var}_\mu(\mathrm{P}_t f)\mathrm{d}t = 2\int_0^\infty \int |\nabla \mathrm{P}_t f|^2 \mathrm{d}\mu\,\mathrm{d}t$$

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- On a manifold: $\Gamma_2(f) = \|\nabla^2 f\|^2_{\mathrm{HS}} + (\mathrm{Ric} + \nabla^2 E)(\nabla f, \nabla f)$

Bakry–Émery for GβE

- Beta Hermite on $D = \{x_1 > \cdots > x_n\}$, GUE if $\beta = 2$

$$\mu_n \propto \mathrm{e}^{-E} 1_D, \quad E(x) = \frac{n}{2}|x|^2 + \beta \sum_{i<j} \log \frac{1}{x_i - x_j}$$

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- Dyson–OU, eigenvalues of Hermitian OU if $\beta = 2$ [Dyson 1962]

$$\mathrm{d}X_t^i = \sqrt{\tfrac{2}{n}}\mathrm{d}B_t^i - X_t^i \mathrm{d}t + \frac{\beta}{n}\sum_{j\neq i}\frac{\mathrm{d}t}{X_t^i - X_t^j}$$

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- Bakry program on Markov orthogonal polynomials
  [Bakry 1996, Bakry–Orevkov–Zani 2013, Bakry 2014]

Log-concave measures and the KLS conjecture

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- Link with gases: for GUE and CGE:
  linear functions of the center of mass S are *exactly* extremal

Outline

Poincaré inequality

Bakry–Émery

Complex Ginibre

Proof

Beyond

Complex Ginibre log-gas (CGE) [Ginibre 1965]

- Eigenvalues of G_n with iid $\mathcal{N}_{\mathbb{C}}(0, \frac{1}{n})$ entries

$$\mathrm{d}\mu_n = \frac{\mathrm{e}^{-n\sum_{k=1}^n |z_k|^2}}{c_n} \prod_{j<k} |z_j - z_k|^2 \mathrm{d}z = \frac{|V_n(z)|^2}{c'_n} \mathrm{d}\gamma_n(z)$$

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Langevin dynamics [Bolley–C.–Fontbona 2018]

- Planar Coulomb interacting Brownian motions, $1 \leq j \leq n$,

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- Unlike GUE, *not* the eigenvalues of the matrix OU (those are not Markov: eigenvector overlaps enter) [Grela–Warchoł 2018, Bourgade–Dubach 2020]

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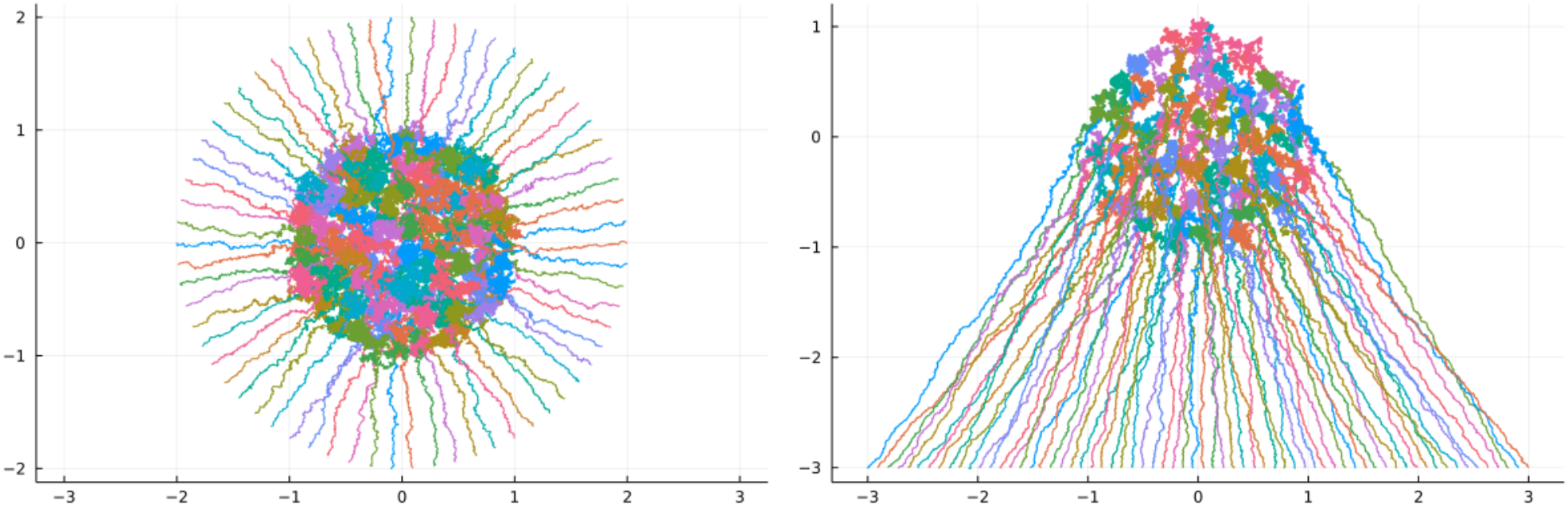
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- Dimension-free spectral gap? [BCF 2018, Lu–Mattingly 2020, Rosenzweig–Serfaty 2025]

Numerical experiment



Dynamics of $n = 66$ interacting planar BM with logarithmic repulsion and quadratic confinement (complex Ginibre log-gas [Ginibre 1965])

Starting from a circle of radius 2 (left) and from a segment (right)

Overdamped Langevin relaxing to the circular law on the unit disk

Optimal Poincaré inequality for CGE

Theorem ([C. 2026], [Suzuki 2026], [Rosenzweig 2026])

For all $n \geq 1$ and all symmetric $f \in H^1_{\mathbb{R}}(\mu_n)$,

$$\mathrm{Var}_{\mu_n}(f) \leq \frac{1}{2}\mathcal{E}_{\mu_n}(f) = \frac{1}{2n}\int_{\mathbb{C}^n} |\nabla f|^2 \mathrm{d}\mu_n.$$

Equality iff $f = \Re(a(z_1 + \cdots + z_n)) + c,\ a \in \mathbb{C},\ c \in \mathbb{R}.$

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- Same normalization as GUE, $\mathrm{Var} \leq c_{\mathrm{P}} \int |\nabla f|^2$:
  CGE $c_{\mathrm{P}} = \frac{1}{2n}$ = that of $\mathcal{N}_{\mathbb{C}}(0, \frac{1}{n} I_n)$, despite curvature $-\infty$
  GUE $c_{\mathrm{P}} = \frac{1}{n}$ = that of $\mathcal{N}(0, \frac{1}{n} I_n)$ [C.–Lehec 2020]

Consequences 1/2: linear statistics

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- Exponential concentration, universal c, C [Gromov–Milman 1983]

$$\mathbb{P}\big(|L_n(g) - \mathbb{E}L_n(g)| \geq r\big) \leq C\mathrm{e}^{-cr/\|g\|_{\mathrm{Lip}}}$$

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- Cutoff? χ^2 lower bound from eigenfunctions $\Re S$ and $\sum_k |z_k|^2$, χ^2 upper bound from PI, but they do not match
  Cutoff known for Dyson OU [Boursier–C.–Labbé 2023], open here

Outline

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Proof

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Proof 1/3: Vandermonde transform

- Ground state transform $U : L^2(\mu_n) \to L^2(\gamma_n)$ unitary

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- V_n holomorphic: $\bar\partial_{z_k}(V_n f) = V_n \bar\partial_{z_k} f$
- Hence the singular interaction disappears

$$\int |\nabla f|^2 \mathrm{d}\mu_n = \frac{4}{c_n'} \sum_{k=1}^{n} \| \bar\partial_{z_k}(V_n f) \|^2_{L^2(\gamma_n)}$$

Proof 2/3: Gaussian $\bar{\partial}$ spectral gap

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For all $g \in \mathcal{C}_c^\infty(\mathbb{C}^n, \mathbb{C})$,

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- Real analogue: Brascamp–Lieb inequality [Brascamp–Lieb 1976, Helffer–Sjöstrand 1994, Cordero-Erausquin 2005]

Proof 3/3: divisibility and projection geometry

- Permutations preserve γ_n and $\mathcal{F}_n$, so $F = \mathrm{proj}_{\mathcal{F}_n}(V_n f)$ is antisymmetric holomorphic, vanishes on $\{z_j = z_k\}$ hence $F = V_n h$ with h symmetric entire (Weierstrass)

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- Conclusion

$$\int |\nabla f|^2 \mathrm{d}\mu_n \geq \frac{4n}{c_n'} \mathrm{dist}(V_n f, \mathcal{F}_n)^2 = 4n\|f - h\|^2 \geq 2n\|f\|^2$$

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Theorem $\Leftrightarrow$ integrated Bakry–Émery: for all symmetric f,

$$\int (A_n f)^2 \mathrm{d}\mu_n \geq 2\mathcal{E}_{\mu_n}(f),$$

although $\Gamma_2 \geq 2\Gamma$ fails pointwise

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Further results [C. 2026]

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$$\mathrm{Var}_{\mu_n}(f) \leq \frac{2}{n} \mathbb{E}[a_f^* O a_f]$$

- Normal matrices: $O = I$ and $\frac{2}{n} a_f^* a_f = \frac{1}{2n} |\nabla f|^2$ for real f

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Planar Coulomb repulsion is invisible to the spectral gap: holomorphy beats convexity/curvature.

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Backup slides

- Gaussian factorization of the CGE log-gas
- Alternative proofs 1/2: spectral and Hermite–Slater
- Alternative proofs 2/2: sums of squares

Backup: Gaussian factorization

- Decomposition along $L = \mathbb{C}(1, \dots, 1)$ and $L^\perp$

$$z = \frac{S}{n}(1, \dots, 1) + W, \quad S = z_1 + \cdots + z_n, \quad V_n(z) = V_n(W)$$

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- Not complete: $\sum_j (z_j - \frac{S}{n})^2 \perp \mathbb{C}[S, \overline{S}, R]$

Alternative proofs 1/2: spectral and Hermite–Slater

- Spectral proof: two-sided Vandermonde factorization

$$-\frac{n}{2}A_n f = \frac{1}{V_n}N_{\bar{z}}(V_n f)+\frac{1}{\overline{V_n}}N_z(\overline{V_n}f), \qquad N_{\bar{z}} = \sum_{j=1}^{n}(-\partial_{z_j}+n\bar{z}_j)\bar{\partial}_{z_j}$$

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$$\Psi_\Lambda = \frac{1}{\sqrt{n!}}\det\left[h_{a_r,b_r}(z_s)\right]_{1\leq r,s\leq n}, \qquad N_{\bar{z}}\Psi_\Lambda = nq(\Lambda)\Psi_\Lambda$$

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- $q(\Lambda) = 0$: $\Psi_\Lambda \propto \det[z_s^{a_r}] \in V_n\,\mathbb{C}[z]^{\mathrm{sym}}$ (alternants)

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- Similar sums of squares for the Γ_2 deficit $\|A_n f\|^2 - 2\mathcal{E}_{\mu_n}(f)$