Dyson Ornstein Uhlenbeck process

Cutoff phenomenon

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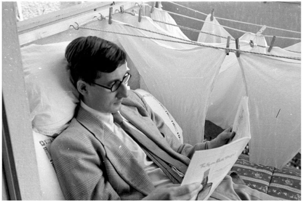
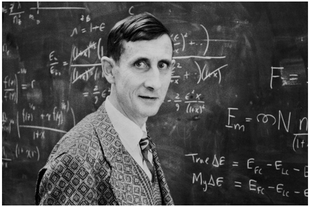
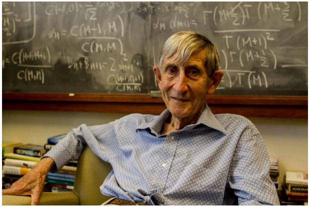
Groupe de travail « modélisation stochastique »
Jeudi 12 mai 2022
LPSM, Université Paris Cité

Freeman J. Dyson (1923 – 2020)

A Brownian Motion Model for the Eigenvalues of a Random Matrix

Journal of Mathematical Physics 3 1191–1198 (1962)

Plan

Dyson Ornstein Uhlenbeck process DOU_β

- Interacting particle system $X_t^{n,1}, \ldots, X_t^{n,n}$ on $\mathbb{R}$

$$X_0^n = x_0^n, \quad \mathrm{d}X_t^n = \sqrt{\frac{2}{n}}\mathrm{d}B_t - \frac{1}{n}\nabla H(X_t^n)\mathrm{d}t$$

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- Configuration energy with Coulomb repulsion (singular)

$$H(x) = n\sum_{i=1}^n V(x_i) + \beta \sum_{i<j} \log \frac{1}{x_i - x_j}, \quad V(x) = \frac{x^2}{2}$$

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- Convergence to equilibrium $X_t^n \xrightarrow[t\to\infty]{\mathrm{d}} P^n \propto \mathrm{e}^{-H(x)}\mathrm{d}x$

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- We take $\beta = 0$ or $\beta \geq 1$ (preserves order $x_n < \cdots < x_1$)

High dimensional random matrices

- Random matrix cases $\beta \in \{1, 2, 4\}$: matrix OU

$$\mathrm{d}M_t = \sqrt{\frac{2}{n}}\mathrm{d}B_t - M_t\mathrm{d}t$$

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$$dM_t = \sqrt{\frac{2}{n}} dB_t - M_t dt$$

  - Symmetric/Hermitian/Symplectic $n \times n$ matrices
    Real/Complex/Quaternion off-diagonal entries : $\mathbb{R}^\beta$

$$M_t \xrightarrow[t\to\infty]{d} \text{GOE/GUE/GSE} \propto e^{-n\text{Trace}(M^2)} dM$$

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  - Dyson : $(\mathrm{spectrum}(M_t))_{t\geq 0} \stackrel{\mathrm{d}}{=} \mathrm{DOU}_{\beta}$

DOU semigroup and generator

- Markov semigroup

$$\mathbb{E}(f(X_t^n) \mid X_0^n = x) = (\mathrm{e}^{t\,\mathrm{L}} f)(x)$$

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- Universality wrt β : spectrum, Poincaré, log-Sobolev

Wigner theorem and semi-circle law : scaling in n

- Empirical measure and exchangeability

$$\mu_t^n = \frac{1}{n}\sum_{i=1}^{n} \delta_{X_t^{n,i}} \quad \text{and} \quad \mathbb{E}\mu_\infty^n \sim \frac{1}{n}\sum_{i=1}^{n} P^{n,i} = P^{n,1}$$

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- Long-time behavior & mean-field limit (when $\mu_0^n \underset{n\to\infty}{\longrightarrow} \mu_0$)

$$\begin{tikzcd}
\mu_t^n \arrow[r, "t\to\infty"'] \arrow[d, "n\to\infty"'] & \mu_\infty^n \arrow[d, "n\to\infty"'] \\
\mu_t \arrow[r, "t\to\infty"'] & \mu_\infty
\end{tikzcd}$$

Mean-field limit and free probability : scaling in t

- McKean-Vlasov evolution equation

$$\partial_t \int f \mathrm{d}\mu_t = -\int x f'(x)\mu_t(\mathrm{d}x) + \frac{\beta}{2} \iint \frac{f'(x)-f'(y)}{x-y}\mu_t(\mathrm{d}x)\mu_t(\mathrm{d}y)$$

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$$s_t(z) = \int \frac{\mu_t(\mathrm{d}x)}{x-z}$$

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$$\mu_t = \mathrm{dil}_{\mathrm{e}^{-t}}\mu_0 \boxplus \mathrm{dil}_{\sqrt{1-\mathrm{e}^{-2t}}}\mu_\infty$$

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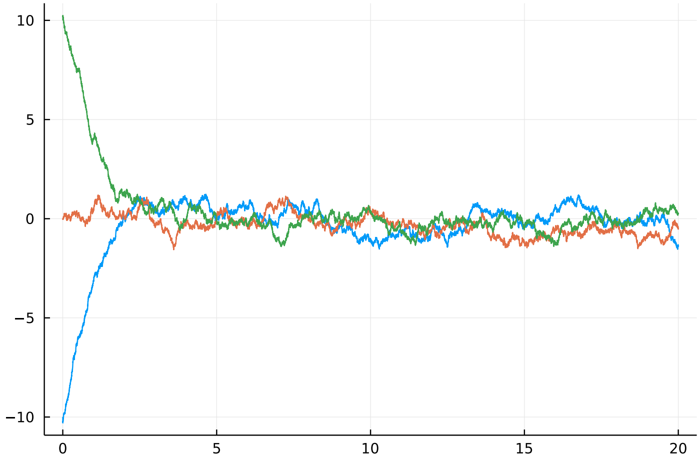
Numerical experiments



$n = 3$, $\beta = 0$: confinement and independence (OU)

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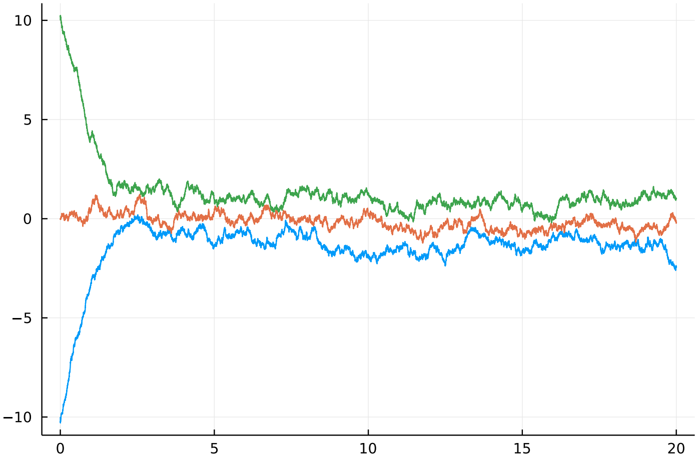
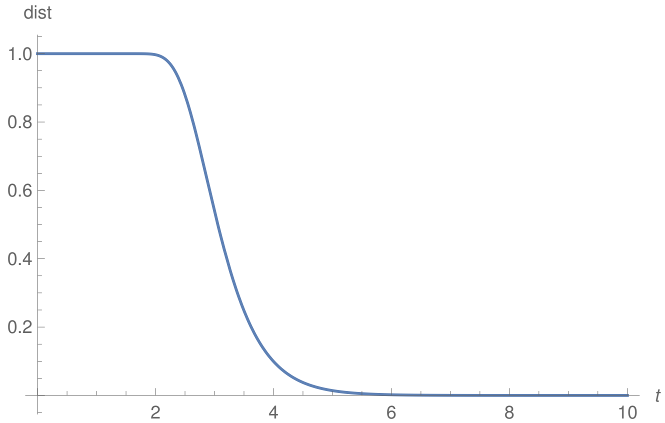


$n = 3$, $\beta = 2$: confinement and repulsion (DOU)

Cutoff for OU : Hellinger distance $\mathrm{dist}(\mathrm{Law}(X_t^n) \mid P^n)$



$$n = 50,\ \beta = 0,\ \frac{|x_0^n|^2}{n} = 1,\ \log(50) \approx 3.91$$

Expectation : cutoff phenomenon

- For all $\varepsilon \in (0,1)$

$$\lim_{n\to\infty} \sup_{x_0^n \in S_0^n} \mathrm{dist}(\mathrm{Law}(X_{t_n}^n) \mid P^n) = \begin{cases} \max & \text{if } t_n = (1-\varepsilon)c_n \\ 0 & \text{if } t_n = (1+\varepsilon)c_n \end{cases}$$

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- Universality with respect to β

Some distances or divergences

$$\chi^2(\nu \mid \mu) = \mathrm{Var}_\mu\left(\frac{d\nu}{d\mu}\right) = \left\| \frac{d\nu}{d\mu} - 1 \right\|^2_{L^2(\mu)}$$

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Some distances or divergences

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$$\text{Kullback}(\nu \mid \mu) = \int \log\frac{d\nu}{d\mu}d\nu = \int \frac{d\nu}{d\mu}\log\frac{d\nu}{d\mu}d\mu$$

$$\text{Fisher}(\nu \mid \mu) = \int\left|\nabla\log\frac{d\nu}{d\mu}\right|^2 d\nu = 4\int\left|\nabla\sqrt{\frac{d\nu}{d\mu}}\right|^2 d\mu$$

$$\text{Wasserstein}^2(\mu,\nu) = \inf_{(X_\mu,X_\nu)}\mathbb{E}(\tfrac{1}{2}|X_\mu - X_\nu|^2) = \sup_{f\mathrm{BL}}\left(\int Q_1(f)d\mu - \int f d\nu\right)$$

$$\|\mu-\nu\|_{\mathrm{TV}} = \inf_{(X_\mu,X_\nu)}\mathbb{E}(1_{X_\mu\neq X_\nu}) = \sup_{\|f\|_\infty\leq\frac{1}{2}}\left(\int f d\mu - \int f d\nu\right)$$

$$= \sup_A|\nu(A)-\mu(A)| = \tfrac{1}{2}\|\varphi_\mu - \varphi_\nu\|_{L^1(\lambda)}$$

$$\text{Hellinger}^2(\mu,\nu) = \tfrac{1}{2}\|\sqrt{\varphi_\mu} - \sqrt{\varphi_\nu}\|^2_{L^2(\lambda)}$$

Monotonicity

- With $\nu_t = \mathrm{Law}(X_t^n)$ and $\mu = P^n$

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- Fisher and Wasserstein : involve also convexity of V

Moments and cutoff

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$$m_k(t) = \mathbb{E}\left(\frac{\sum_{i=1}^{n}(X_t^{n,i})^k}{n}\right) = \mathbb{E}\int u^k \mu_t^n(\mathrm{d}u)$$

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- Eigenfunctions $\mathrm{e}^{t\mathrm{L}}\pi_k = \mathrm{e}^{-tk}\pi_k$

$$\pi_1(x) = \sum_{i=1}^n x_i \quad \text{and} \quad \pi_2(x) = \sum_{i=1}^n x_i^2 - (1 + \frac{\beta}{2}(n-1))$$

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- $\log(n)$ cutoff for $\mathbb{E}(\pi_k(X_t))$: dimension n versus e^{-kt} decay

Cutoff for DOU : processes

- **Theorem :** Assume that $\beta = 0$ or $\beta \geq 1$ and set

$$Z_t = \sum_{i=1}^{n} X_t^{n,i} \quad \text{and} \quad R_t = \sum_{i=1}^{n} (X_t^{n,i})^2 = |X_t^n|^2$$

Then $(Z_t)_{t\geq 0} \sim \mathrm{OU}$ and $(R_t)_{t\geq 0} \sim \mathrm{CIR}$

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- **Proof :** Stroock–Varadhan local martingale

$$M_t^f = f(X_t) - f(X_0) - \int_0^t \mathrm{L} f(X_s)\mathrm{d}s$$

$$\langle M\rangle_t = \int_0^t \Gamma(f)(X_s)\mathrm{d}s, \text{ take } \mathrm{L}f = -\lambda f, \text{ then } f \in \{\pi_1, \pi_2\}$$

Plan

Cutoff for OU : Mean-field case

- Theorem : if $\beta = 0$ and $\frac{|x_0^n|^2}{n} \asymp 1$ then for all $\varepsilon \in (0,1)$

$$\lim_{n\to\infty} \mathrm{dist}(\mathrm{Law}(X_{t_n}^n) \mid P^n) = \begin{cases} \max & \text{if } t_n = (1-\varepsilon)c_n \\ 0 & \text{if } t_n = (1+\varepsilon)c_n \end{cases}$$

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- Other initial conditions ?

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- Reminds behavior of second moment m_2

Cutoff for OU : Proof 1/3

- OU : $\mathrm{d}Y_t = \sqrt{2\theta}\mathrm{d}B_t - Y_t\mathrm{d}t$, $\mathbb{R}^d$, $\eta_t = \mathrm{Law}(Y_t)$

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- In particular if $\eta_0 = \delta_y$ then

$$\eta_t = \mathcal{N}\left(\mathrm{e}^{-t}y, \theta(1-\mathrm{e}^{-2t})\mathrm{I}_d\right)$$

Cutoff for OU : Proof 2/3

If $\Gamma_1 = \mathcal{N}(\mu_1, \Sigma_1)$ and $\Gamma_2 = \mathcal{N}(\mu_2, \Sigma_2)$ in $\mathbb{R}^n$ then with $m = m_1 - m_2$:

$$\chi^2(\Gamma_1 \mid \Gamma_2) = \sqrt{\frac{|\Sigma_2|}{|2\Sigma_1 - \Sigma_1^2\Sigma_2^{-1}|}}\mathrm{e}^{\frac{1}{2}\Sigma_2^{-1}(\mathrm{I}_n + 2\Sigma_1^{-1}\Sigma_2^{-1} - \Sigma_2^{-2})m\cdot m} - 1$$

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&\overset{*}{=} |\Sigma_2^{-1} m|^2 + \mathrm{Tr}(\Sigma_2^{-2}(\Sigma_2 - \Sigma_1)^2\Sigma_1^{-1})
\end{aligned}$$

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Cutoff for OU : Proof 2/3

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\mathrm{Hellinger}^2(\mu, \nu) \leq \|\mu - \nu\|_{\mathrm{TV}} &\leq \mathrm{Hellinger}(\mu, \nu)\sqrt{2 - \mathrm{Hellinger}(\mu, \nu)^2}
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Cutoff for OU : Proof 3/3

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Cutoff for OU : Proof 3/3

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$\log(n)$ cutoff for $\mathrm{dist}(\mathrm{Law}(X_t^n) \mid P^n)$: n versus e^{-t}

Plan

Cutoff for DOU: Random matrix case

- **Theorem :** Assume that $\beta \in \{1, 2, 4\}$. Let (a_n) be such that $\inf(a_n) > 0$. Then for all $\varepsilon \in (0, 1)$

$$\lim_{n\to\infty} \sup_{x_0^n \in [-a_n, a_n]} \mathrm{dist}(\mathrm{Law}(X_{t_n}^n) \mid P^n) = \begin{cases} \max & \text{if } t_n = (1-\varepsilon)c_n \\ 0 & \text{if } t_n = (1+\varepsilon)c_n \end{cases}$$

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- Proof : OU sandwich (trace Z, matrix M) + dist contract.
- Cutoff should be controlled by $|x_0^n - \rho^n|$ instead of $|x_0^n|$

Plan

Cutoff for DOU: general case

- **Theorem :** Assume that $\beta = 0$ or $\beta \geq 1$. Let (a_n) be such that $\inf(a_n) > 0$. Then for all $\varepsilon \in (0, 1)$

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- Lower bound : Contraction to OU Z

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- Lower bound : Contraction to OU Z
- Upper bound : LSI, regularization, coupling ($\sim$ exclusion)

Cutoff for DOU : Proof for general case (1/2)

- Optimal log-Sobolev inequality

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- Regularization Y^n of X^n (smoothed $Y_0^{n,i} \geq X_0^{n,i} = x_0^{n,i}$)

$$\mathrm{Entropy}(\mathrm{Law}(Y_t^n) \mid P^n) \leq C(n|x_0^n|^2 + n^2\log(n))\mathrm{e}^{-2t}$$

Cutoff for DOU : Proof for general case (2/2)

- Coalescent coupling preserving order $Y_t^{n,i} \geq X_t^{n,i}$

$$\begin{aligned}\|\mathrm{Law}(Y_t^n) - \mathrm{Law}(X_t^n)\|_{\mathrm{TV}} &\leq \mathbb{P}(Y_t^n \neq X_t^n) \\ &\leq \mathbb{P}(\inf\{s \geq 0 : A_s^n = 0\} > t)\end{aligned}$$

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- "Area" $A_t = \sum_{i=1}^n (Y_t^{n,i} - X_t^{n,i})$ is an AR sort of OU

$$\mathrm{d}A_t = -A_t \mathrm{d}t + \mathrm{d}M_t$$

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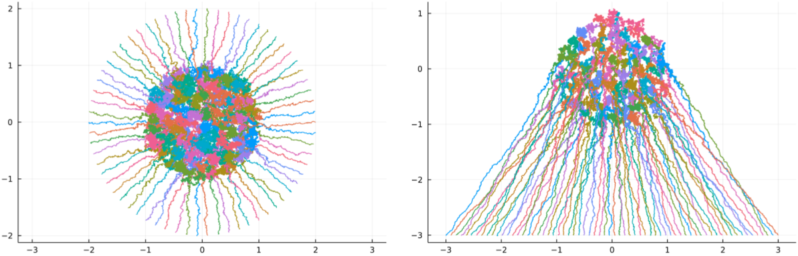
- Submartingale in $[0,1]$ $\mathrm{e}^{-\lambda A - \frac{\lambda^2}{2}\langle A \rangle}$

Thank you for your attention!



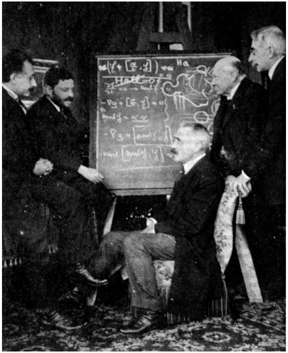
Selected bibliography and open problems

- Bibliography
  - Dyson, Anderson–Guionnet–Zeitouni, Erdős–Yau
  - Voiculescu, Rogers–Shi, Biane
  - Lassalle, Baker–Forrester
  - Lachaud, Barrera–Jara
  - Ané et al, Bakry–Gentil–Ledoux, Villani
  - Saloff-Coste, Méliot, Lacoin
  - C.–Lehec, Bolley–C.–Fontbona, Lu–Mattingly
- Problems
  - V : Exactly solvable cases (Hermite/Laguerre/Jacobi)
  - V : General strong convex case (Bakry–Émery or KLS)
  - Better initial conditions (ρ^n), other distances (Fisher, ...)
  - Non-convex interactions (such as planar DOU dynamics)

Leonard Salomon Ornstein (1880 – 1941) George Eugene Uhlenbeck (1900 – 1988)

On the theory of Brownian Motion
Physical Reviews 36 (5) 823–841 (1930)

Paul Langevin (1872 – 1946)



Sur la théorie du mouvement brownien

Comptes-rendus de l'Académie des sciences (9 mars 1908)