Aspects of the Spherical Ensemble

arXiv:2606.00330

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Outline

Spherical Ensemble

Singular CLT

Spherical Ensemble

[Krishnapur 2006, 2009, Forrester–Krishnapur 2009, Haagerup–Schultz 2007]

- A, B : independent Ginibre ($n \times n$ iid $\mathcal{N}_{\mathbb{C}}(0,1)$ entries)

$$AB^{-1}$$

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- Characteristic polynomial and eigenvalues

$$P_n(z) = \det(zI_n - AB^{-1}) = \prod_{k=1}^{n} (z - \lambda_{n,k})$$

Spherical Ensemble : goals

- High-dimensional fluctuation of spectral radius

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- CLT for linear statistics

$$L_n(f) = \sum_{k=1}^{n} f(\lambda_{n,k}) \quad \text{where} \quad L_n = \sum_{k=1}^{n} \delta_{\lambda_{n,k}}$$

Spherical Ensemble : goals

- High-dimensional fluctuation of spectral radius

$$\rho_n = \max_{1 \le k \le n} |\lambda_{n,k}| \xrightarrow[n\to\infty]{\mathrm{d}} \mathrm{Law}\left(\max_{k\ge 1} \frac{1}{\sqrt{\Gamma_k}}\right)$$

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- Exact solvability and universality (arXiv:2510.18669)

Spherical Ensemble : density

- Joint density $\varphi_{n,n}(z_1, \dots, z_n)$ of eigenvalues $(\lambda_{n,k})_{1 \le k \le n}$

$$(z_1, \dots, z_n) \in \mathbb{C}^n \mapsto \frac{\prod_{k=1}^{n} (1 + |z_k|^2)^{-(n+1)}}{Z_n} \prod_{i<j} |z_i - z_j|^2$$

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- Reverse stereographic projection $T^{-1}(z) = \frac{(2z, |z|^2 - 1)}{|z|^2 + 1}$

$$(x_1, \ldots, x_n) \in (\mathbb{S}^2)^n \mapsto \frac{1}{Z_n} \prod_{i<j} \|x_i - x_j\|^2_{\mathbb{R}^3}$$

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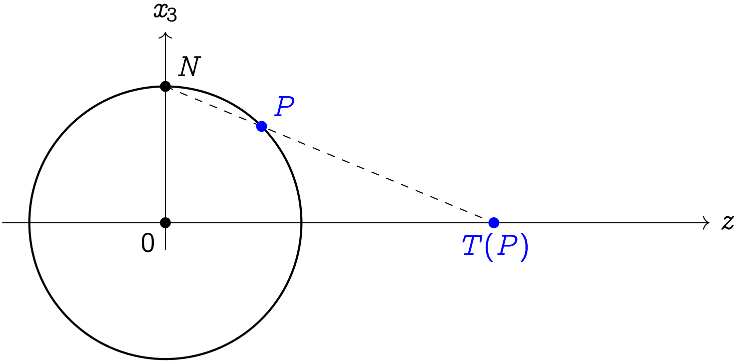
- $\rightsquigarrow$ 2D analogue of Circular Unitary Ensemble (Haar on $\mathbb{U}_n$)
- $\rightsquigarrow$ 1D complex compact manifold (Riemann surface)

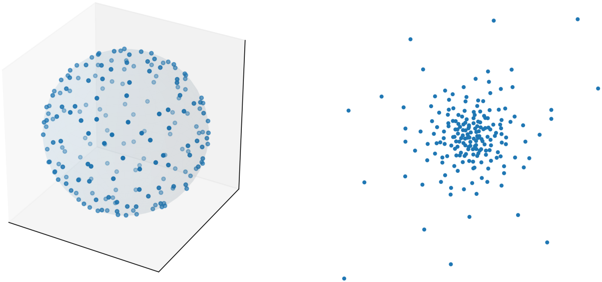
Stereographic projection



$$T(x_1, x_2, x_3) = \frac{x_1 + \mathrm{i}x_2}{1 - x_3} \text{ if } x_3 \neq 1 \text{ and } T(e_3) = \infty$$

$$\mu = \mathrm{ComplexCauchy}(\mathbb{C}) = \mathrm{Uniform}(\mathbb{S}^2) \circ T^{-1}$$

Spherical Ensemble : two views



No edge ↔ Full support and heavy-tail

Spherical symmetries ↔ Planar Möbius symmetries

Spherical Ensemble : Coulomb gas structure

- Coulomb gas with potential $Q(z) = \frac{1}{2}\log(1+|z|^2)$

$$\varphi_{n,n}(z_1,\dots,z_n) \propto \exp\left(-2(n+1)\sum_{k=1}^{n} Q(z_k) - \sum_{i\neq j}\log\frac{1}{|z_i - z_j|}\right)$$

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- $\rightsquigarrow$ Laplace/LDP via T [Hardy 2012] : speed n^2 and rate

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- $\rightsquigarrow$ LLN : $L_n \xrightarrow[n\to\infty]{\text{weak}} \mu$ a.s. (and guess for GFF CLT!)

Spherical Ensemble : determinantal structure

- k-point density with respect to $\mu^{\otimes k}$

$$\varphi_{n,k}(z_1, \ldots, z_k) = \frac{(n-k)!}{n!} \det \left[K_n(z_i, z_j) \right]_{1 \leq i,j \leq k}$$

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- Kernel (finite dimensional orthogonal projection)

$$K_n(z,w) = \sum_{k=0}^{n-1} P_{n,k}(z)\overline{P_{n,k}(w)}, \quad P_{n,k}(z) = c_{n,k}\frac{z^k}{(1+|z|^2)^{\frac{n+1}{2}}}$$

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- CLT via cumulants [Rider–Virág 2007]

$$L_n(f) - \mathbb{E}L_n(f) \xrightarrow[n\to\infty]{\mathrm{d}} \mathcal{N}\left(0, \tfrac{1}{4\pi}\|f\|^2_{H^1}\right) \quad \text{(GFF)}$$

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Characteristic polynomial and logarithmic potential

- Special singular linear statistics

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- Wishart/Laguerre Hermitization + Choleski/Bartlett facto.

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- $\mathrm{Var}(L_n(f_0)) = \frac{H_n}{2} + \frac{n}{2}\left(\frac{\pi^2}{6} - H_n^{(2)}\right) = \frac{H_n+1}{2} + o_{n\to\infty}(1)$

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- Lindeberg CLT, or Fréchet–Shohat cumulants CLT

Logarithmic singularities : definition and examples

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Definition (Logarithmic singularities)

$f : \overline{\mathbb{C}} \setminus \{p_1, \dots, p_k\} \to \mathbb{R}$, weights $c_1, \dots, c_k \in \mathbb{R} \setminus \{0\}$
$p_i \neq \infty : z \mapsto f(z) - c_i \log|z - p_i|$ has smooth extension at p_i
$p_i = \infty : z \mapsto f(1/z) - c_i \log|z|$ has smooth extension at 0

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- Reformulation

$$f = h_1 + \sum_{i=1}^{k} c_i \log d(\cdot, p_i)$$

CLT for logarithmic singularities

Theorem (CLT for logarithmic singularities)

If $f : \overline{\mathbb{C}} \setminus \{p_1, \dots, p_k\} \to \mathbb{R}$ *has logarithmic singularities at points* $p_1, \dots, p_k$ *with weights* $c_1, \dots, c_k \in \mathbb{R} \setminus \{0\}$ *then*

$$\frac{L_n(f) - \mathbb{E}L_n(f)}{\sqrt{\frac{1}{4}\log(n)}} \xrightarrow[n\to\infty]{\mathrm{d}} \mathcal{N}(0, c_1^2 + \cdots + c_k^2).$$

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- Localization at the singularities + smooth part
- Decorrelation at log-scale via kernel off-diagonal decay

$$|K_n(z,w)| = n(1 - d(z,w)^2)^{\frac{n-1}{2}} \leq ne^{-\frac{n-1}{2}d(z,w)^2}$$

CLT for logarithmic singularities

Theorem (CLT for logarithmic singularities)

If $f : \overline{\mathbb{C}} \setminus \{p_1, \ldots, p_k\} \to \mathbb{R}$ has logarithmic singularities at points $p_1, \ldots, p_k$ with weights $c_1, \ldots, c_k \in \mathbb{R} \setminus \{0\}$ then

$$\frac{L_n(f) - \mathbb{E}L_n(f)}{\sqrt{\frac{1}{4}\log(n)}} \xrightarrow[n\to\infty]{\mathrm{d}} \mathcal{N}(0, c_1^2 + \cdots + c_k^2).$$

- A sort of residue CLT
- Contains the CLT for $L_n(f_0)$
- Localization at the singularities + smooth part
- Decorrelation at log-scale via kernel off-diagonal decay

$$|K_n(z, w)| = n(1 - d(z, w)^2)^{\frac{n-1}{2}} \leq ne^{-\frac{n-1}{2}d(z,w)^2}$$

- Smooth part via $\mathrm{Var}(L_n(h)) \leq \frac{1}{2}\|h\|^2_{\mathrm{Lip.}(d)}$ or GFF CLT

Chordal distance and Green function

- (Half) chordal distance for $x, y \in \mathbb{S}^2$

$$d(x,y) = \tfrac{1}{2}\|x-y\|_{\mathbb{R}^3}, \quad d(x,y)^2 = \frac{1 - x\cdot y}{2} = \frac{1-\cos(\theta)}{2}$$

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- Complex stereographic coordinates : for all $z, w \in \mathbb{C}$,

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- Distance relation or factorization

$$|z-w| = \frac{d(z,w)}{d(z,\infty)d(w,\infty)}$$

$$f_z(w) = g_z(w) - g_\infty(w) + c_z$$

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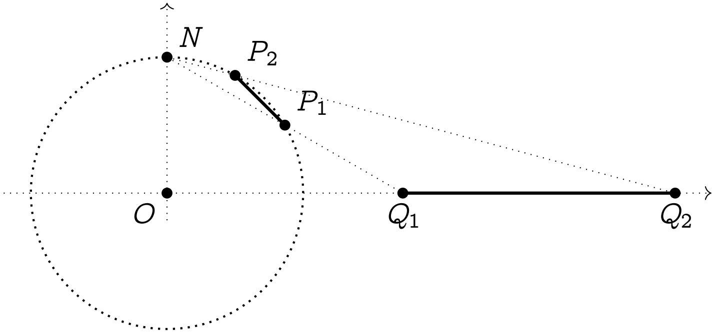
- Green function

$$g_p(q) = \log d(p,q) = -\frac{1}{2} - G(p,q), \quad \frac{\Delta G(p,\cdot)}{2\pi} = -\delta_p + \Lambda$$

Half-chordal and planar-Euclidean distances



CLT for linear statistics of Green function

Corollary (CLT for linear statistics of Green function)

$$\left(\frac{L_n(g_p)+\frac{n}{2}}{\sqrt{\frac{1}{4}\log(n)}}\right)_{p\in\overline{\mathbb{C}}} \xrightarrow[n\to\infty]{\mathrm{d}} (\zeta_p)_{p\in\overline{\mathbb{C}}}$$

where $(\zeta_p)_{p\in\overline{\mathbb{C}}}$ *is a centered Gaussian white noise*

$$\mathbb{E}(\zeta_p\zeta_q)=1_{p=q}$$

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- Multiple points via $c_1 g_{p_1} + \cdots + c_k g_{p_k}$
- Single point also via Kostlan and Fréchet–Shohat cumulants CLT : $\kappa_1(L_n(g_p)) = -\frac{n}{2}$, $\kappa_2(L_n(g_p)) = \frac{H_n}{4}$,

$$\lim_{n\to\infty} \kappa_m(L_n(g_p)) = 2^{-m}(-1)^m(m-1)!\zeta(m-1), \quad m \geq 3.$$

CLT for logarithmic potential or log|charpoly|

Corollary (CLT for the logarithmic potential)

$$\left(\frac{L_n(f_z) - \frac{n}{2}\log(1+|z|^2)}{\sqrt{\frac{1}{2}\log(n)}}\right)_{z\in\mathbb{C}} \xrightarrow[n\to\infty]{\mathrm{d}} (\xi_z)_{z\in\mathbb{C}}$$

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- ∞ in all f_z produces non-white noise

Correlation of $(g_p)_{p\in\overline{\mathbb{C}}}$ and of $(f_z)_{z\in\mathbb{C}}$

Theorem (Log-correlated random fields)

For all $p \neq q \in \overline{\mathbb{C}}$, and all $z \neq w \in \mathbb{C}$,

$$\mathrm{Var}(L_n(g_p)) = \frac{H_n}{4}$$

$$\mathrm{Cov}(L_n(g_p), L_n(g_q)) = -\frac{1}{4} - \frac{1}{2}\log d(p,q) + o_{n\to\infty}(1)$$

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- Spectrum of Berezin transform, spherical harmonics for K_n & g_p

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- From g_p to f_z via $f_z = g_z - g_\infty + c_z$

CLT universality

Theorem (CLT universality)

If AB^{-1} is in the universal spherical ensemble, then the previous singular CLT and covariance estimates remain.

- Reduction to f_z or h and Hermitization, replacement principle
- Interpolation, cumulants expansion, moment matching
- Related : [Cipolloni–Erds–Schröder 2023], [Cipolloni–Landon 2026]

Other singular test functions

Lemma (Powers and indicators)

$$\frac{L_n(|\cdot|^{-s}) - n\frac{\pi\frac{s}{2}}{\sin(\pi\frac{s}{2})}}{n^{\frac{s}{2}}} \xrightarrow[n\to\infty]{\mathrm{d}} \sum_{k=1}^{\infty}\left(\Gamma_k^{-\frac{s}{2}} - \frac{\Gamma(k-\frac{s}{2})}{\Gamma(k)}\right), \quad 0 < s < 2$$

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- Follows from Kostlan observation (determinantal structure)

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- Terra icongnita : correlation and universality

Other aspects

- FH asymptotics and GMC
  Related: [Webb–Wong 2019], [Bourgade–Dubach–Hartung–Keles 2025], [Cipolloni–Landon 2026], [Chatterjee 2025, 2026], [Byun–Forrester|Kuijlaars–Lahiry 2025], [Byun–Yang–Yoo 2026]

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- Geometric universality
  Heuristics for compact Riemann surfaces
  Key is kernel off-diagonal decay and convergence
  Related: [Berndtsson 2003], [Berman 2024], [Bourgoin 2024]

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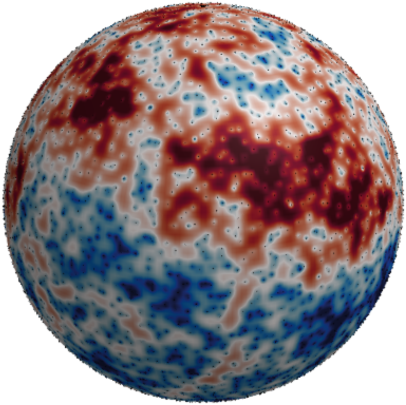
- Beta universality
  Beyond determinantal structure, plausible !
  Related : [Leblé–Serfaty 2018], [Bauerschmidt–Bourgade–Nikula–Yau 2019], [Forrester 2023]

Green field on the sphere $(L_n(g_p))_{p\in\mathbb{S}^2}$ with n=5K



Thank you for your attention!