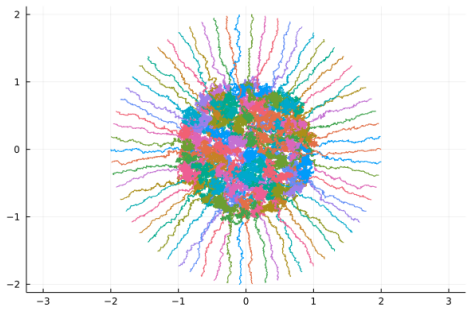
# ON POINCARÉ INEQUALITIES FOR GASES

*When holomorphy beats convexity*

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Abstract. These notes accompany a 1h30 minicourse. We prove the Poincaré inequality (PI) for the Gaussian via Hermite polynomials and via the Bakry–Émery criterion, we explain a Gaussian splitting of the center of mass, and we use it to get the optimal PI with equality cases for the Gaussian Unitary Ensemble (GUE) and for the complex Ginibre ensemble (GinUE), following mostly [15, 11] and [27].



*Overdamped Langevin dynamics associated to GinUE with $n = 66$, started from a centered discrete circle and ending with particles approximately uniformly spread over the unit disc, as predicted by the high-dimensional phenomenon discovered by Jean Ginibre [19].*

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## 1. Poincaré inequality for Langevin dynamics

*0:00–0:10. Define PI and state the spectral viewpoint. Skip the comments unless needed.*

Following [24], a probability measure $\mu$ on $\mathbb{R}^d$ satisfies a Poincaré inequality when there exists a finite constant $c > 0$ such that for all $f \in H^1(\mu)$,

$$\mathrm{Var}_\mu(f) = \int f^2 \mathrm{d}\mu - \Big(\int f \mathrm{d}\mu\Big)^2 \leq c \int |\nabla f|^2 \mathrm{d}\mu. \tag{1.1}$$

Both sides measure non constantness of $f$. The best constant, namely the smallest, is the Poincaré constant of $\mu$, denoted $c_{\mathrm{P}}$. Unlike continuity of the embedding $H^1(\mu) \subset L^2(\mu)$, (1.1) controls the centered $L^2$ norm by the gradient seminorm alone. We specialize to Boltzmann–Gibbs measures

$$\mu = Z^{-1} \mathrm{e}^{-E} \mathrm{d}x \tag{1.2}$$

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where $0 < Z = \int \mathrm{e}^{-E}\mathrm{d}x < \infty$. Sections 1–3 assume $E \in \mathcal{C}^\infty(\mathbb{R}^d, \mathbb{R})$, the singular supports below use their conservative Dirichlet forms. It is the invariant law of the overdamped Langevin diffusion

$$\mathrm{d}X_t = \sqrt{2}\mathrm{d}B_t - \nabla E(X_t)\mathrm{d}t, \tag{1.3}$$

with generator $\mathrm{L} = \Delta - \nabla E \cdot \nabla$ and semigroup $\mathrm{P}_t = \mathrm{e}^{t\,\mathrm{L}}$.

In particular $\mathrm{P}_t(f)(x) = \mathbb{E}^x(f(X_t))$, in other words $\mathrm{P}_t(\cdot)(x) = \mathrm{Law}(X_t \mid X_0 = x)$. The integration by parts states that for all $f, g \in \mathcal{C}_c^\infty$,

$$\mathcal{E}(f,g) := \int f(-\mathrm{L})g\mathrm{d}\mu = \int \nabla f \cdot \nabla g \mathrm{d}\mu. \tag{1.4}$$

We use the nonpositive self-adjoint generator L associated with the closure of $\mathcal{E}$ on $\mathcal{C}_c^\infty$. Its form domain is $H^1(\mu)$, generally larger than its operator domain $D(\mathrm{L})$. This is the Markov triple $(\mathbb{R}^d, \mathrm{L}, \mu)$ or $(\mathbb{R}^d, \mathcal{E}, \mu)$, see [3].

**Theorem 1.1** (Convex monotonicity, Poincaré inequality, exponential decay, spectral gap)**.** *For $f \in L^2(\mu)$ and $t > 0$, the variance has convex monotonicity:*

$$\partial_t \mathrm{Var}_\mu(\mathrm{P}_t f) = -2\mathcal{E}(\mathrm{P}_t f, \mathrm{P}_t f) \le 0 \quad \textit{and} \quad \partial_t^2 \mathrm{Var}_\mu(\mathrm{P}_t f) = 4\int (\mathrm{L}\,\mathrm{P}_t f)^2 \mathrm{d}\mu \ge 0. \tag{1.5}$$

*Moreover, for all $c > 0$, the following three properties are equivalent:*

1. *Poincaré inequality:* $\mathrm{Var}_\mu(f) \le c\,\mathcal{E}(f,f)$ *for all* $f \in H^1(\mu)$
2. *Exponential decay:* $\mathrm{Var}_\mu(\mathrm{P}_t f) \le \mathrm{e}^{-2t/c}\mathrm{Var}_\mu(f)$ *for all* $f \in L^2(\mu)$ *and* $t \ge 0$
3. *Spectral gap: the spectrum of* $-\mathrm{L}$ *is included in* $\{0\} \cup [1/c, +\infty)$*, and its kernel consists of the constants*

*Proof.* The monotonicity/convexity comes from $\mathrm{Var}_\mu(\mathrm{P}_t f) = \int (\mathrm{P}_t f)^2 \mathrm{d}\mu - (\int f \mathrm{d}\mu)^2$ which is a consequence of the invariance of $\mu$, together with the evolution equation $\partial_t \mathrm{P}_t = \mathrm{L}\,\mathrm{P}_t$.

(1)⇒(2). Set $v(t) = \mathrm{Var}_\mu(\mathrm{P}_t f)$. Then (1) for $\mathrm{P}_t f$ gives $v'(t) = -2\mathcal{E}(\mathrm{P}_t f, \mathrm{P}_t f) \le -\frac{2}{c}v(t)$ hence (2) by the Grönwall lemma. Conversely, differentiating (2) at $t = 0$ gives $-2\mathcal{E}(f,f) \le -\frac{2}{c}\mathrm{Var}_\mu(f)$.

(1)⇔(3). In PI, by translation invariance (by constants) and homogeneity, we can assume that $f$ is centered and of unit $L^2$ norm, and PI is then equivalent to the Rayleigh quotient on $H^1(\mu)$:

$$\frac{1}{c_{\mathrm{P}}} = \inf_{\int f \mathrm{d}\mu = 0, \int f^2 \mathrm{d}\mu = 1} \mathcal{E}(f,f). \tag{1.6}$$

By the spectral theorem, this Rayleigh-quotient bound is exactly (3). □

(i) If $\frac{1}{4}|\nabla E(x)|^2 - \frac{1}{2}\Delta E(x) \to +\infty$ as $|x| \to \infty$, then $-\mathrm{L}$ has compact resolvent, hence an orthonormal eigenbasis with finite-multiplicity eigenvalues tending to $+\infty$, by the ground-state transform $f \mapsto \mathrm{e}^{-E/2}f$ and the compact-resolvent criterion for Schrödinger operators.

(ii) If $(-\mathrm{L})e_k = \lambda_k e_k$, where $0 = \lambda_0 < \lambda_1 \le \lambda_2 \le \cdots$ and $(e_k)_{k\ge 0}$ is an orthonormal basis of $L^2(\mu)$ with $e_0 = 1$, then $\mathrm{P}_t f = \sum_{k\ge 0} \mathrm{e}^{-\lambda_k t}\langle f, e_k\rangle e_k$,

$$\mathrm{Var}_\mu(f) = \sum_{k\ge 1} |\langle f, e_k\rangle|^2, \quad \int |\nabla f|^2 \mathrm{d}\mu = \sum_{k\ge 1} \lambda_k |\langle f, e_k\rangle|^2, \quad c_{\mathrm{P}}(\mu) = \frac{1}{\lambda_1}. \tag{1.7}$$

(iii) Testing (1.1) on a linear function $f(x) = \langle \theta, x\rangle$ gives

$$c_{\mathrm{P}}(\mu) \ge \|\mathrm{Cov}(\mu)\|_{\mathrm{op}}. \tag{1.8}$$

The Kannan–Lovász–Simonovits (KLS) conjecture says that for log-concave measures, meaning when $E$ is convex, there is equality up to a universal constant. This major conjecture of geometric functional analysis was proved recently after a long and rich exploration by several mathematicians, see [6] and references therein. It is quite natural to conjecture that the optimal ratio is 4, achieved for the product one-sided exponential law!

(iv) If $\mu$ satisfies (1.1) then its image by a $\kappa$-Lipschitz map satisfies (1.1) with constant $c\kappa^2$, e.g. $\kappa = \|\Sigma\|_{\mathrm{op}}$ for $x \mapsto \Sigma x$.

(v) When $E(x) = \sum_{i=1}^n V(x_i) + \sum_{i<j} W(x_i - x_j)$ and $\mathbb{R}^d = (\mathbb{R}^k)^n$, then $\mu$ is a two-body interaction gas of $n$ particles in $\mathbb{R}^k$. This includes GUE ($k = 1$) as well as GinUE ($k = 2$).

## 2. Gaussian case: Ornstein–Uhlenbeck via Hermite polynomials

*0:10–0:20. Treat $d = 1$ on the board. Give Hermite and mention Mehler.*

Let $\gamma_d = \mathcal{N}(0, I_d)$, $E(x) = \frac{|x|^2}{2}$, $\mathrm{L} = \Delta - x \cdot \nabla$ (Ornstein–Uhlenbeck).

**Theorem 2.1** (Nash [23], Hörmander [21], Brascamp–Lieb [9], Chernoff [16])**.** *For all $d \geq 1$ and $f \in H^1(\gamma_d)$, $\mathrm{Var}_{\gamma_d}(f) \leq \int |\nabla f|^2 \mathrm{d}\gamma_d$, with equality iff $f$ is affine. Dimension-free: $c_{\mathrm{P}}(\gamma_d) = 1$.*

By scaling, we get $c_{\mathrm{P}}(\mathcal{N}(0, \sigma^2 I_d)) = \sigma^2$, and the extremal functions are affine. In particular $\mathcal{N}(0, \frac{1}{n} I_n)$ has constant $\frac{1}{n}$, while $\mathcal{N}_{\mathbb{C}}(0, \frac{1}{n} I_n) = \mathcal{N}(0, \frac{1}{2n} I_{2n})$ on $\mathbb{C}^n = \mathbb{R}^{2n}$ has constant $\frac{1}{2n}$.

*Proof. Hermite polynomials.* Since $f = \sum_\alpha a_\alpha H_\alpha$, where $(H_\alpha)$ are the Hermite polynomials, chosen orthonormal with respect to $\mathcal{N}(0, I_d)$, we have $-\mathrm{L}\, H_\alpha = |\alpha| H_\alpha$, and

$$\mathrm{Var}_{\gamma_d}(f) = \sum_{|\alpha| \geq 1} |a_\alpha|^2 \leq \sum_{|\alpha| \geq 1} |\alpha| |a_\alpha|^2 = \int |\nabla f|^2 \mathrm{d}\gamma_d, \quad \text{with equality iff } f \text{ is affine.} \tag{2.1}$$

*Mehler kernel.* Since $\mathrm{P}_t f(x) = \int f(\mathrm{e}^{-t} x + \sqrt{1 - \mathrm{e}^{-2t}} y) \mathrm{d}\gamma_d(y)$, we get $\nabla \mathrm{P}_t f = \mathrm{e}^{-t} \mathrm{P}_t \nabla f$. By the semigroup interpolation $\mathrm{Var}_{\gamma_d}(f) = -\int_0^\infty \partial_t \big( \int \mathrm{P}_t(f)^2 \mathrm{d}\gamma_d \big) \mathrm{d}t$, Jensen and invariance of $\gamma_d$, we get

$$\mathrm{Var}_{\gamma_d}(f) = 2 \int_0^\infty \int |\nabla \mathrm{P}_t f|^2 \mathrm{d}\gamma_d \mathrm{d}t \leq 2 \int_0^\infty \mathrm{e}^{-2t} \mathrm{d}t \int |\nabla f|^2 \mathrm{d}\gamma_d = \int |\nabla f|^2 \mathrm{d}\gamma_d. \tag{2.2}$$

□

## 3. Strongly log-concave measures: Brascamp–Lieb, Bakry–Émery, Caffarelli

*0:20–0:27. State BL and BE. Sketch only the resolvent argument and skip the Caffarelli proof.*

Assume $E$ smooth with $\nabla^2 E \succ 0$. The key is the Bochner commutation formula

$$-\nabla \mathrm{L}\, u = -\mathrm{L}\, \nabla u + \nabla^2 E \nabla u = (-\mathrm{L} + \nabla^2 E) \nabla u, \tag{3.1}$$

which gives, by (1.4) applied twice, for all $u \in \mathcal{C}_c^\infty$,

$$\int (\mathrm{L}\, u)^2 \mathrm{d}\mu = -\int \nabla u \cdot \nabla \mathrm{L}\, u \mathrm{d}\mu = \int \big( \|\nabla^2 u\|_{\mathrm{HS}}^2 + \langle \nabla^2 E \nabla u, \nabla u \rangle \big) \mathrm{d}\mu. \tag{3.2}$$

**Theorem 3.1** (Hörmander [21], Brascamp–Lieb [9], Helffer–Sjöstrand [20])**.** *For all $f \in H^1(\mu)$,*

$$\mathrm{Var}_\mu(f) \leq \int \langle (\nabla^2 E)^{-1} \nabla f, \nabla f \rangle \mathrm{d}\mu. \tag{3.3}$$

*In particular, if $\nabla^2 E \succeq \rho I_d$ with $\rho > 0$ then $c_{\mathrm{P}}(\mu) \leq 1/\rho$.*

*Sketch the idea only: motivates Lemma 6.2.*

*Proof.* Take centered $f \in H^1(\mu)$ with finite right-hand side, otherwise the claim is trivial. Consider the resolvent approximation $u_\varepsilon = (\varepsilon - \mathrm{L})^{-1} f$. By the spectral theorem, $-\mathrm{L}\, u_\varepsilon \to f$ in $L^2(\mu)$ as $\varepsilon \downarrow 0$, since the kernel of L consists of the constants. Integration by parts and Cauchy–Schwarz for the scalar product induced by $\nabla^2 E$ give

$$-\int f\, \mathrm{L}\, u_\varepsilon \mathrm{d}\mu = \int \langle \nabla f, \nabla u_\varepsilon \rangle \mathrm{d}\mu \leq \Big( \int \langle (\nabla^2 E)^{-1} \nabla f, \nabla f \rangle \mathrm{d}\mu \Big)^{1/2} \Big( \int \langle \nabla^2 E \nabla u_\varepsilon, \nabla u_\varepsilon \rangle \mathrm{d}\mu \Big)^{1/2}. \tag{3.4}$$

The last factor is at most $\big( \int (\mathrm{L}\, u_\varepsilon)^2 \mathrm{d}\mu \big)^{1/2}$ by (3.2), whose lower bound extends to $D(\mathrm{L})$ by graph-core approximation and lower semicontinuity. Letting $\varepsilon \downarrow 0$ yields

$$\int f^2 \mathrm{d}\mu \leq \Big( \int \langle (\nabla^2 E)^{-1} \nabla f, \nabla f \rangle \mathrm{d}\mu \Big)^{1/2} \Big( \int f^2 \mathrm{d}\mu \Big)^{1/2}. \tag{3.5}$$

Dividing by $\|f\|_{L^2(\mu)}$, unless $f = 0$, proves the claim. □

*Remark* 3.2 (Witten Laplacian). The Bochner formula (3.2) does not give $\mathrm{L}^2 \succeq \nabla^2 E$ since $\mathrm{L}^2$ acts on functions, whereas $\nabla^2 E$ acts on vector fields. On $L^2(\mu, \mathbb{R}^d)$, the Witten Laplacian on vector fields (or 1-forms) is $A^{(1)} = -\mathrm{L} + \nabla^2 E$. Also $A^{(1)} \succeq \nabla^2 E \succeq 0$, since $-\mathrm{L} \succeq 0$ componentwise. Moreover, the Bochner formula (3.2) reads $\|\mathrm{L}\, u\|_{L^2(\mu)}^2 = \langle \nabla u, A^{(1)} \nabla u \rangle_{L^2(\mu)}$.

From $\mathrm{e}^{t\mathrm{L}} = 1 + t\mathrm{L} + \frac{t^2}{2} \mathrm{L}^2 + o_{t \to 0^+}(t^2)$, we get the asymptotic expansion

$$\mathrm{P}_t(f^2) - \mathrm{P}_t(f)^2 = 2t\, \mathrm{P}_t\, \Gamma(f) - 2t^2\, \Gamma_2(f) + o_{t \to 0^+}(t^2). \tag{3.6}$$

The functional quadratic forms $\Gamma$ (carré du champ) and $\Gamma_2$ are

$$\Gamma(f) = \frac{1}{2} (\mathrm{L}(f^2) - 2f\, \mathrm{L}\, f) = |\nabla f|^2 \tag{3.7}$$

and, with $\Gamma(f,g)=\langle\nabla f,\nabla g\rangle$ and the pointwise Bochner commutation formula,

$$\Gamma_2(f)=\frac{1}{2}(\mathrm{L}\,\Gamma(f)-2\Gamma(f,\mathrm{L}\,f))=\|\nabla^2 f\|_{\mathrm{HS}}^2+\langle\nabla^2 E\nabla f,\nabla f\rangle. \tag{3.8}$$

**Theorem 3.3** (Bakry–Émery criterion [2, 3])**.** *For all $\rho>0$ the following properties are equivalent:*

(1) *$\rho$-convexity:* $\nabla^2 E\succeq\rho I_d$
(2) *Curvature criterion:* $\Gamma_2\geq\rho\Gamma$
(3) *Gradient bound:* $|\nabla\,\mathrm{P}_t\,f|^2\leq\mathrm{e}^{-2\rho t}\,\mathrm{P}_t(|\nabla f|^2)$ *for all $t$ and $f$*
(4) *Strong gradient bound:* $|\nabla\,\mathrm{P}_t\,f|\leq\mathrm{e}^{-\rho t}\,\mathrm{P}_t(|\nabla f|)$ *for all $t$ and $f$*
(5) *Local PI:* $\mathrm{P}_t(f^2)-\mathrm{P}_t(f)^2\leq\frac{1-\mathrm{e}^{-2\rho t}}{\rho}\,\mathrm{P}_t(|\nabla f|^2)$ *for all $t$ and $f$*
(6) *Local LSI:* $\mathrm{P}_t(f^2\log f^2)-\mathrm{P}_t(f^2)\log\mathrm{P}_t(f^2)\leq 2\frac{1-\mathrm{e}^{-2\rho t}}{\rho}\,\mathrm{P}_t(|\nabla f|^2)$ *for all $t$ and $f$*

*In particular, if $\nabla^2 E\succeq\rho I_d$, $\rho>0$, then $c_{\mathrm{P}}(\mu\propto\mathrm{e}^{-E})\leq 1/\rho$, with equality in the case of O.–U.*

*Proof.* Differential calculus. The diffusion property is the chain rule: for $\psi\in\mathcal{C}^2(\mathbb{R}^m)$ and smooth $f=(f_1,\ldots,f_m)$, it reads $\mathrm{L}\,\psi(f)=\sum_{i=1}^m(\partial_i\psi)(f)\,\mathrm{L}\,f_i+\sum_{i,j=1}^m(\partial^2_{ij}\psi)(f)\,\Gamma(f_i,f_j)$, where the polarized carré du champ is $\Gamma(f,g):=\frac{1}{2}\big(\mathrm{L}(fg)-f\,\mathrm{L}\,g-g\,\mathrm{L}\,f\big)=\langle\nabla f,\nabla g\rangle$. We recover the carré du champ for $m=1$ and $\psi(u)=u^2$. The basic interpolation is $\Psi(s)=\mathrm{P}_s\,\Gamma(\mathrm{P}_{t-s}\,f)$, for which $\Psi'(s)=2\,\mathrm{P}_s\,\Gamma_2(\mathrm{P}_{t-s}\,f)$. Thus $\Gamma_2\geq\rho\Gamma$ and Grönwall give the gradient bound, while its expansion at $t=0$ gives back the curvature criterion. The strong bound follows from the derivative flow estimate $\|D_xX_t^x\|_{\mathrm{op}}\leq\mathrm{e}^{-\rho t}$. For variance, the interpolation identity is

$$\mathrm{P}_t(f^2)-\mathrm{P}_t(f)^2=2\int_0^t\mathrm{P}_s\,\Gamma(\mathrm{P}_{t-s}\,f)\,\mathrm{d}s,$$

obtained by differentiating $s\mapsto\mathrm{P}_s((\mathrm{P}_{t-s}\,f)^2)$. Interpolation gives local LSI, linearization gives local PI, whose expansion at $t=0$ recovers curvature. Integration in time gives global PI/LSI. □

(i) Further formulations use admissible local $\Phi$-Sobolev and Bobkov inequalities and pairwise Wasserstein contraction, see [3].
(ii) O.–U. is the equality case of $\rho$-convexity, or constant Bakry–Émery curvature case. On a complete Riemannian manifold, $\Gamma_2(f)=\|\nabla^2 f\|_{\mathrm{HS}}^2+\langle\nabla^2 E\nabla f,\nabla f\rangle+\mathrm{Ric}(\nabla f,\nabla f)$, leading to a Ricci–Bakry–Émery curvature. Comparison theorem à la Lichnerowicz–Berger–Gromov, the Gaussian case of O.–U. playing the role of the sphere.
(iii) The theorem extends to the situation where $\rho\in\mathbb{R}$ (beyond the scope of this section), with $\frac{1-\mathrm{e}^{-2\rho t}}{\rho}$ read as $2t$ when $\rho=0$, the equality case being Brownian motion.
(iv) By using the variational formulation of the spectral gap of L, we get that the Poincaré inequality $c_{\mathrm{P}}\leq 1/\rho$ is equivalent to the integrated curvature criterion: for all $f$,

$$\int\Gamma_2(f)\mathrm{d}\mu=\int(\mathrm{L}\,f)^2\mathrm{d}\mu\geq\rho\int|\nabla f|^2\mathrm{d}\mu=\rho\int\Gamma(f)\mathrm{d}\mu. \tag{3.9}$$

(v) The global Poincaré inequality follows in one line by semigroup interpolation:

$$\mathrm{Var}_\mu(f)=-\int_0^\infty\partial_t\int\mathrm{P}_t(f)^2\mathrm{d}\mu\mathrm{d}t=2\int_0^\infty\int|\nabla\,\mathrm{P}_t\,f|^2\mathrm{d}\mu\mathrm{d}t\leq 2\int_0^\infty\mathrm{e}^{-2\rho t}\mathrm{d}t\int|\nabla f|^2\mathrm{d}\mu. \tag{3.10}$$

**Theorem 3.4** (Caffarelli [10])**.** *If $\nabla^2 E\succeq\rho I_d$, $\rho>0$, then the $\ell^2$ optimal transport map $T$ from $\mathcal{N}(0,I_d)$ to $\mu$ satisfies $\|T\|_{\mathrm{Lip}}\leq 1/\sqrt{\rho}$. In particular $c_{\mathrm{P}}(\mathcal{N}(0,I_d))=1$ gives $c_{\mathrm{P}}(\mu)\leq 1/\rho$.*

*Skip this proof in the 90-minute version.*

*Proof sketch.* By the Brenier theorem, $T=\nabla\varphi$ with $\varphi$ convex. Assume first that $E$ and $\varphi$ are smooth and that the maxima below are attained. Since $T_\#\gamma_d=\mu$, by the Monge–Ampère formula,

$$\log\det\nabla^2\varphi(x)=E(\nabla\varphi(x))-\frac{|x|^2}{2}+C. \tag{3.11}$$

Fix a unit vector $e$. At a maximum point of $\varphi_{ee}$, set $A=\nabla^2\varphi$. The maximum principle gives $\nabla\varphi_{ee}=0$ and $A_{ee}=\partial_e^2 A=\nabla^2\varphi_{ee}\preceq 0$, so twice differentiating yields

$$0\geq\mathrm{tr}(A^{-1}A_{ee})-\mathrm{tr}(A^{-1}A_eA^{-1}A_e)=\langle\nabla^2 E(\nabla\varphi)Ae,Ae\rangle-1\geq\rho|Ae|^2-1. \tag{3.12}$$

Since $\varphi_{ee}=\langle Ae,e\rangle\leq|Ae|$ by Cauchy–Schwarz, $\varphi_{ee}\leq 1/\sqrt{\rho}$ at its maximum. This holds for every $e$, hence $\|\nabla^2\varphi\|_{\mathrm{op}}\leq 1/\sqrt{\rho}$. For approximation and localization, see [10, Theorem 11]. □

*0:27–0:40. Prove tensorization and the splitting in full.*

## 4. The splitting: Gaussian factorization of the center of mass

**Lemma 4.1** (Tensorization)**.** *For all $f \in L^2(\mu_1 \otimes \mu_2)$,*

$$\mathrm{Var}_{\mu_1\otimes\mu_2}(f) \le \int \mathrm{Var}_{\mu_1}(f(\cdot,y))\mathrm{d}\mu_2(y) + \int \mathrm{Var}_{\mu_2}(f(x,\cdot))\mathrm{d}\mu_1(x). \tag{4.1}$$

*Hence $c_{\mathrm{P}}(\mu_1 \otimes \mu_2) = \max(c_{\mathrm{P}}(\mu_1), c_{\mathrm{P}}(\mu_2))$.*

*Proof.* With $g(y) = \int f(x,y)\mathrm{d}\mu_1(x)$, we have $\mathrm{Var}_{\mu_1\otimes\mu_2}(f) = \int \mathrm{Var}_{\mu_1}(f(\cdot,y))\mathrm{d}\mu_2(y) + \mathrm{Var}_{\mu_2}(g)$, and $\mathrm{Var}_{\mu_2}(g) \le \int \mathrm{Var}_{\mu_2}(f(x,\cdot))\mathrm{d}\mu_1(x)$ by convexity of $h \mapsto \mathrm{Var}_{\mu_2}(h)$ (Jensen). The lower bound on $c_{\mathrm{P}}$ comes from functions of $x$ only or of $y$ only. □

**Lemma 4.2** (Splitting with Gaussian factor)**.** *Suppose that $E(x) = \frac{\rho}{2}|x|^2 + \Phi(x)$ where $\rho > 0$ and $\Phi : \mathbb{R}^d \to \mathbb{R}\cup\{+\infty\}$ satisfies $\Phi(x+y) = \Phi(x)$ for all $x \in \mathbb{R}^d$ and $y \in F$, where $F \subset \mathbb{R}^d$ is a nonzero linear subspace. Then $\mu$ is the image of $\mathcal{N}(0, \frac{1}{\rho}I_F) \otimes \nu$ by $(y,w) \mapsto y + w$, where $\nu \propto \mathrm{e}^{-E(w)}\mathrm{d}w$ on $F^\perp$. In other words, if $X \sim \mu$, then $\mathrm{proj}_F(X) \sim \mathcal{N}(0, \frac{1}{\rho}I_F)$ is independent of $\mathrm{proj}_{F^\perp}(X) \sim \nu$. Moreover $c_{\mathrm{P}}(\mu) = \max(\frac{1}{\rho}, c_{\mathrm{P}}(\nu))$. Furthermore, if $\Phi$ is convex then both $\mu$ and $\nu$ are log-concave.*

*Proof.* For $x = y + w$, $y \in F$, $w \in F^\perp$, we have $|x|^2 = |y|^2 + |w|^2$ and $\Phi(x) = \Phi(w)$, hence $\mathrm{e}^{-E(x)} = \mathrm{e}^{-\frac{\rho}{2}|y|^2}\mathrm{e}^{-E(w)}$, and the Lebesgue measure on $\mathbb{R}^d$ is the product of the Lebesgue measures on $F$ and $F^\perp$. The claim on Poincaré comes from the tensorization Lemma 4.1, the Gaussian Theorem 2.1 with scaling, and the isometry $|\nabla f|^2 = |\nabla_F f|^2 + |\nabla_{F^\perp} f|^2$. □

For gases with quadratic confinement, $F$ is spanned by the center of mass direction, and the interaction $\Phi$ is translation invariant along $F$. The splitting says that the center of mass is *exactly* Gaussian and independent of the rest, and reduces the PI to the measure $\nu$ on $F^\perp$.

## 5. Gaussian Unitary Ensemble and beta Hermite ensembles

*0:40–0:50. Give curvature, splitting, and equality cases. Skip the three comments.*

The PI and LSI bounds of Theorem 3.1, Theorem 3.3, and Theorem 3.4 remain valid if $E = +\infty$ outside an open convex set $D$ and $E$ is smooth on $D$ and blows up at $\partial D$, by approximation with smooth convex potentials on $\mathbb{R}^d$. This is used in [15] to get the best PI and LSI constants for GUE.

Let $\beta > 0$, $D = \{x \in \mathbb{R}^n : x_1 > \cdots > x_n\}$, and

$$\mu_n = \frac{\mathrm{e}^{-E(x)}}{Z_n}\mathbf{1}_D(x)\mathrm{d}x, \quad E(x) = \frac{n}{2}|x|^2 + \Phi(x), \quad \Phi(x) = \beta\sum_{i<j}\log\frac{1}{x_i - x_j}. \tag{5.1}$$

For $\beta = 2$ this is the law of the ordered eigenvalues of the GUE, with density proportional to $\mathrm{e}^{-\frac{n}{2}\mathrm{Tr}(M^2)}$ on Hermitian $n \times n$ matrices. Functions on $D$ identify, by permutation extension, with symmetric functions off collisions. The Dyson–Ornstein–Uhlenbeck dynamics [18]

$$\mathrm{d}X_t^i = \sqrt{2/n}\mathrm{d}B_t^i - X_t^i\mathrm{d}t + \frac{\beta}{n}\sum_{j\neq i}\frac{\mathrm{d}t}{X_t^i - X_t^j}, \tag{5.2}$$

with generator $\frac{1}{n}\mathrm{L}$, is reversible for $\mu_n$. PI with constant $\frac{1}{n}$ means spectral gap 1. For $\beta \ge 1$ there are no collisions. For $0 < \beta < 1$, collisions occur, but there are solutions [15].

**Theorem 5.1** (Poincaré for G$\beta$E [15])**.** *For all $\beta > 0$, $n \ge 1$, $f \in H^1(\mu_n)$, with $S = x_1 + \cdots + x_n$,*

$$\mathrm{Var}_{\mu_n}(f) \le \frac{1}{n}\int |\nabla f|^2\mathrm{d}\mu_n, \tag{5.3}$$

*with equality iff $f = aS + b$ for some constants $a, b \in \mathbb{R}$.*

*Proof.* Since $D$ is convex and $t \mapsto -\log t$ is convex on $(0,\infty)$, it follows that $\Phi$ is convex, hence $E$ is convex with at least the curvature of the Gaussian part: $\nabla^2 E \succeq nI_n$. The PI follows then from the Bakry–Émery Theorem 3.3 or the Brascamp–Lieb Theorem 3.1 or the Caffarelli Theorem 3.4. From the splitting lemma 4.2, $f = aS + b$ is extremal since $\mathrm{Var}(S) = 1$ and $|\nabla S|^2 = n$. For the equality case, the Brascamp–Lieb argument gives

$$\mathrm{Var}_{\mu_n}(f) \le \int \langle(\nabla^2 E)^{-1}\nabla f, \nabla f\rangle\mathrm{d}\mu_n \le \frac{1}{n}\int |\nabla f|^2\mathrm{d}\mu_n.$$

Since $\ker(\nabla^2 E - nI_n) = F$, equality forces $\nabla f \in F$ almost everywhere. The connectedness of $D \cap F^\perp$ gives $f = g(S)$, Gaussian equality then gives $f = aS + b$. An alternative approach is based on a $\Gamma_2$ sum-of-squares formula for the deficit, see [15], see also the rigidity concept in [12]. □

(i) The same method gives the optimal log-Sobolev inequality $\mathrm{Ent}_{\mu_n}(f^2) \le \frac{2}{n}\int |\nabla f|^2 \mathrm{d}\mu_n$, with equality iff $f = c\mathrm{e}^{aS}$, $a, c \in \mathbb{R}$, see [15].
(ii) The interaction and in particular $\beta$ is not felt by the spectral gap. The PI constant is identical to the one of $\mathcal{N}(0, \frac{1}{n} I_n)$ and of its ordered version, obtained by setting $\beta = 0$.
(iii) By analogy with the Hermite approach for the standard Gaussian, we could use the fact that the generator of the Dyson–O.–U. dynamics preserves symmetric polynomials, and the Hermite–Lassalle symmetric polynomials $(P_k)$, indexed by partitions $k_1 \ge \cdots \ge k_n \ge 0$, form an orthogonal basis of $L^2(\mu_n)$ with $-\frac{1}{n}\mathrm{L}\, P_k = |k| P_k$, see [22, 1]. The spectrum is $\mathbb{N}$ as for O.–U., and the eigenvalue 1 is simple, with eigenfunction $S$. Combined with the splitting, this gives $c_{\mathrm{P}}(\nu_n) = \frac{1}{2n}$ for $n \ge 2$, attained by $|x|^2 - \frac{S^2}{n}$ up to a constant.
(iv) In the cases $\beta \in \{1, 2, 4\}$, we could also get the optimal PI (and LSI) from a matrix Gaussian by projection of the matrix Gaussian lift (GOE/GUE/GSE). This projection/lift approach remains valid at the level of the Dyson–Ornstein–Uhlenbeck dynamics [8].

*0:50–1:25. Main proof in full. Start this section no later than 0:50.*

## 6. Complex Ginibre ensemble

The eigenvalues, in uniform random order, of a matrix with iid $\mathcal{N}_{\mathbb{C}}(0, \frac{1}{n})$ entries [19] have law

$$\mathrm{d}\mu_n(z) = \frac{\mathrm{e}^{-n\sum_{k=1}^n |z_k|^2}}{c_n} \prod_{j<k} |z_j - z_k|^2 \mathrm{d}z = \frac{|V_n(z)|^2}{c'_n} \mathrm{d}\gamma_n(z), \quad V_n(z) = \prod_{j<k} (z_k - z_j), \tag{6.1}$$

where now $\gamma_n = \mathcal{N}_{\mathbb{C}}(0, \frac{1}{n} I_n) \propto \mathrm{e}^{-n|z|^2}\mathrm{d}z$ on $\mathbb{C}^n = \mathbb{R}^{2n}$ and $c'_n = \int |V_n|^2 \mathrm{d}\gamma_n$. This is

$$\mu_n \propto \mathrm{e}^{-E}, \quad \text{with} \quad E(z) = n|z|^2 + \Phi(z), \quad \Phi(z) = 2\sum_{j<k} \log \frac{1}{|z_j - z_k|}. \tag{6.2}$$

(i) For $n \ge 2$, since $\log \frac{1}{|\cdot|}$ on $\mathbb{R}^2$ has Hessian eigenvalues $\pm |\cdot|^{-2}$, $E$ is not convex and the Bakry–Émery curvature is $-\infty$. There is no analogue of $D$ in $\mathbb{C}$. Moreover $\log \frac{1}{|\cdot|}$ is harmonic on $\mathbb{C} \setminus \{0\}$ and superharmonic[1] on $\mathbb{C}$ since $\Delta \log |\cdot| = 2\pi\delta_0$.
(ii) The measure $\mu_n$ is reversible for a planar Coulomb overdamped Langevin dynamics with generator $\frac{1}{n}\mathrm{L}$. For $n \ge 2$, this is not the eigenvalue process of the complex matrix O.–U. diffusion: its eigenvalue quadratic variations involve eigenvector overlaps [7].

**Theorem 6.1** (Poincaré for GinUE [11], independently Suzuki [27] and Rosenzweig [26])**.** *For all $n \ge 1$ and all symmetric $f \in H^1(\mu_n, \mathbb{R})$,*

$$\mathrm{Var}_{\mu_n}(f) \le \frac{1}{2n} \int_{\mathbb{C}^n} |\nabla f|^2 \mathrm{d}\mu_n, \tag{6.3}$$

*with equality iff $f = \Re(aS) + b$, $a \in \mathbb{C}$, $b \in \mathbb{R}$, where $S = z_1 + \cdots + z_n$.*

Here $H^1(\mu_n)$ is the closure of $\mathcal{C}^\infty_c(\mathbb{C}^n)$ for the weighted Sobolev norm, and the $L^2$ spaces below are complex. Symmetrization and a smooth symmetric correction of the mean reduce the proof to centered, real, symmetric $f \in \mathcal{C}^\infty_c$.

*Proof.* Let us first prove the Poincaré inequality, in four steps.

*Vandermonde transform*[2]. Since $V_n$ is holomorphic and, for real-valued $f$, $|\nabla f|^2 = 4\sum_k |\bar\partial_k f|^2$,

$$\int |\nabla f|^2 \mathrm{d}\mu_n = \frac{4}{c'_n} \sum_{k=1}^n \|\bar\partial_k (V_n f)\|^2_{L^2(\gamma_n)}. \tag{6.4}$$

Moreover, since $f$ is symmetric, $U : f \mapsto V_n f / \sqrt{c'_n}$ is an isometry[3] from $L^2_{\mathrm{sym}}(\mu_n)$ onto $L^2_{\mathrm{anti}}(\gamma_n)$.

*Gaussian $\bar\partial$ gap.* Let $\mathcal{F}_n$ be the Bargmann–Fock space of entire functions in $L^2(\gamma_n)$ [4].

[1] Instead of being sub-harmonic, the trace weakening of convexity.
[2] Here $z_k = x_k + \mathrm{i}y_k$ and $\partial_k, \bar\partial_k = \frac{1}{2}(\partial_{x_k} \mp \mathrm{i}\partial_{y_k})$.
[3] In the mathematical physics jargon, this is a ground state transform, with respect to a holomorphic weight.

**Lemma 6.2** (Hörmander–Berndtsson estimate [21, 5])**.** *For all $g \in \mathcal{C}_c^\infty(\mathbb{C}^n, \mathbb{C})$,*

$$\sum_{k=1}^{n} \|\bar\partial_k g\|^2_{L^2(\gamma_n)} \geq n \operatorname{dist}_{L^2(\gamma_n)}(g, \mathcal{F}_n)^2. \tag{6.5}$$

Both sides measure the lack of holomorphy of $g$.

This is the Gaussian case of the complex analogue of the Brascamp–Lieb estimate (Theorem 3.1), where plurisubharmonicity (psh) plays the role of convexity, see [21, 5] and [17].

*State the Hermite basis and polynomial density without proof.*

*Proof.* An integration by parts gives that the adjoint of $a_k = \bar\partial_k$ in $L^2(\gamma_n)$ is $a_k^* = -\partial_k + n\bar z_k$, and $[a_j, a_k^*] = n\delta_{jk}$, $[a_j, a_k] = 0$. Let $H_{p,q} = \prod_k (a_k^*)^{q_k} z_k^{p_k}$, $p, q \in \mathbb{N}^n$ (complex Hermite polynomials). Since $a_k z^p = 0$, the commutation relations give $a_k^* a_k H_{p,q} = n q_k H_{p,q}$ and $\|a_k^* H_{p,q}\|^2 = n(q_k+1)\|H_{p,q}\|^2$. The $H_{p,q}$ are products of one-variable functions, and in one variable $H_{p,q}$ is an eigenfunction of $a^*a$ with eigenvalue $nq$ and has weight $\mathrm{e}^{\mathrm{i}(p-q)\theta}$ under the rotation $z \mapsto \mathrm{e}^{\mathrm{i}\theta} z$, which preserves each factor of $\gamma_n$. Since $(p-q, q)$ determines $(p, q)$, the $H_{p,q}$ are orthogonal. Moreover $H_{p,q} = n^{|q|} z^p \bar z^q +$ lower degree, so they span $\mathbb{C}[z, \bar z]$, which is dense in $L^2(\gamma_n)$ (Section 2, after scaling). Thus $\hat H_{p,q} = H_{p,q}/\|H_{p,q}\|$ is an orthonormal basis, and $\mathcal{F}_n$ is the closed span of $\hat H_{p,0}$. With $c_{p,q} = \langle g, \hat H_{p,q}\rangle$, we get $\langle a_k g, \hat H_{p,q}\rangle = \langle g, a_k^* \hat H_{p,q}\rangle = \sqrt{n(q_k+1)}\, c_{p,q+e_k}$, and by Parseval

$$\sum_{k=1}^{n} \|a_k g\|^2 = \sum_{p,q} n|q| |c_{p,q}|^2 \geq n \sum_{p,\, q\neq 0} |c_{p,q}|^2 = n\|g - \operatorname{proj}_{\mathcal{F}_n} g\|^2. \tag{6.6}$$

□

Applying Lemma 6.2 to $g = V_n f$ in (6.4) gives

$$\int |\nabla f|^2 \mathrm{d}\mu_n \geq \frac{4n}{c_n'} \operatorname{dist}_{L^2(\gamma_n)}(V_n f, \mathcal{F}_n)^2. \tag{6.7}$$

*Divisibility.* Set $G = \operatorname{proj}_{\mathcal{F}_n}(V_n f)$. Since $f$ is symmetric, $V_n f$ is antisymmetric. Since the projection commutes with permutations, $G$ is antisymmetric. Thus $G = V_n h$ for an entire symmetric $h$, and

$$\mathcal{F}_n^{\mathrm{div}} = \{h \text{ entire symmetric} : V_n h \in \mathcal{F}_n\} = U^{-1}(\mathcal{F}_n \cap L^2_{\mathrm{anti}}(\gamma_n)) \subset L^2(\mu_n), \tag{6.8}$$

is a closed subspace containing the constants. By isometry, $h = \operatorname{proj}_{\mathcal{F}_n^{\mathrm{div}}}(f)$, and

$$\operatorname{dist}_{L^2(\gamma_n)}(V_n f, \mathcal{F}_n)^2 = \|V_n f - V_n h\|^2_{L^2(\gamma_n)} = c_n' \|f - h\|^2_{L^2(\mu_n)} = c_n' \operatorname{dist}_{L^2(\mu_n)}(f, \mathcal{F}_n^{\mathrm{div}})^2. \tag{6.9}$$

*Rotation.* Rotations $z \mapsto \mathrm{e}^{\mathrm{i}\theta} z$ preserve $\mu_n$ and $\mathcal{F}_n^{\mathrm{div}}$, hence commute with the projection. Write the Taylor series of $h$ as $h(z) = \sum_{m\geq 0} h_m(z)$, where $h_m$ is homogeneous of degree $m$. The degree-zero term is the projection of $f$ onto the constants, so it vanishes since $\int f \mathrm{d}\mu_n = 0$. Since $\langle h, \bar h\rangle = \int h^2 \,\mathrm{d}\mu_n$, rotational invariance lets us average this integral over rotations. For each fixed $z$, the angular average of $h(\mathrm{e}^{\mathrm{i}\theta} z)^2$ is zero: its Taylor series has no constant term. Fubini applies since the rotational average of $|h(\mathrm{e}^{\mathrm{i}\theta} z)|^2$ integrates to $\|h\|^2_{L^2(\mu_n)}$. Thus $h \perp \bar h$. For $r = f - h - \bar h$, the projection identity gives $f - h \perp h$ in $L^2(\mu_n)$. Since $f$ is real-valued, we get $\bar f = f$, so conjugating this identity gives $f - \bar h \perp \bar h$. Together with $h \perp \bar h$, these identities yield $r \perp h$ and $r \perp \bar h$, and therefore, denoting $\|\cdot\| = \|\cdot\|_{L^2(\mu_n)}$,

$$\operatorname{dist}(f, \mathcal{F}_n^{\mathrm{div}})^2 = \|f - h\|^2 = \|h\|^2 + \|r\|^2 = \frac{\|f\|^2 + \|r\|^2}{2} \geq \frac{1}{2}\|f\|^2. \tag{6.10}$$

Combining (6.7) and (6.9) with the preceding estimate,

$$\int |\nabla f|^2 \mathrm{d}\mu_n \geq 4n \operatorname{dist}_{L^2(\mu_n)}(f, \mathcal{F}_n^{\mathrm{div}})^2 \geq 2n\|f\|^2_{L^2(\mu_n)} = 2n \operatorname{Var}_{\mu_n}(f). \tag{6.11}$$

*Optimality.* Since $\Phi$ is invariant along $F = \mathbb{C}(1, \dots, 1)$, the splitting Lemma 4.2 with $\rho = 2n$ gives $S \sim \mathcal{N}_{\mathbb{C}}(0, 1)$, independent of $z - \frac{S}{n}(1, \dots, 1)$. Hence $\operatorname{Var}_{\mu_n}(\Re(aS)) = \frac{|a|^2}{2} = \frac{1}{2n}|\nabla \Re(aS)|^2$ proving optimality. By (6.4) and closedness of the Gaussian $\bar\partial$ form, the spectral identities extend to $H^1(\mu_n)$. Keeping track of the losses gives, for all symmetric centered real $f$,

$$\int |\nabla f|^2 \mathrm{d}\mu_n - 2n \operatorname{Var}_{\mu_n}(f) = 2n\|r\|^2 + 4n \sum_{m\geq 2} (m-1)\|g_m\|^2_{L^2(\gamma_n)}, \tag{6.12}$$

where $g_m$ is the orthogonal projection of $Uf$ on the closed span of the $H_{p,q}$ with $|q| = m$, namely the eigenspace of $\sum_k a_k^* a_k$ for the eigenvalue $nm$ (and not the part of degree $m$ in $\bar z$, since $H_{p,q}$

contains lower order terms $z^{p-j}\bar{z}^{q-j}$). In particular, equality forces $r = 0$ and $g_m = 0$ for $m \geq 2$. Then $f = h + \bar{h}$ and $V_n\bar{h} = V_n f - V_n h \perp \mathcal{F}_n$ lies in the closed span of the $H_{p,q}$ with $|q| = 1$, killed by all $a_j a_k$, so $V_n\overline{\partial_j \partial_k h} = 0$, hence $h$, affine, symmetric and centered, is $aS$ and $f = \Re(2aS)$. □

*1:25–1:30. Mention only (i), then take questions.*

(i) Symmetry is needed: for $f = \Re z_1$, $\int |\nabla f|^2 \mathrm{d}\mu_n = 1$ while $\mathrm{Var}_{\mu_n}(f) = \frac{1}{2}\mathbb{E}|z_1|^2 = \frac{n+1}{4n}$. In terms of dynamics, the spectral gap of $-\frac{1}{n}\,\mathrm{L}$ on symmetric functions is exactly 2, uniformly in $n$, while it is at most $\frac{4}{n+1}$ on all functions.

(ii) For the generator $\mathrm{L}/n$, the center of mass satisfies $\mathrm{d}S_t = \sqrt{2}\,\mathrm{d}B_t^{(2)} - 2S_t\mathrm{d}t$. If $|S_0| \asymp \sqrt{n}$, this marginal has a total variation cutoff at $\frac{1}{4}\log n$ with an $O(1)$ window, giving a mixing lower bound for the unlabelled process. A full cutoff needs a matching upper bound controlling the centered configuration, the curvature hypothesis of [12] does not hold here.

(iii) For real $g$, the linear statistic $L_n(g) = \sum_k g(z_k)$ satisfies $|\nabla L_n(g)|^2 \leq n\|g\|^2_{\mathrm{Lip}}$, hence $\mathrm{Var}(L_n(g)) \leq \frac{1}{2}\|g\|^2_{\mathrm{Lip}}$ for all $n$, consistent with the CLT without normalization [25].

(iv) This is compatible with concentration at the $n^2$ scale. For $\widehat{\mu}_n = \frac{1}{n}\sum_k \delta_{z_k}$, the preceding bound gives $\mathrm{Var}(\int g\mathrm{d}\widehat{\mu}_n) \leq \|g\|^2_{\mathrm{Lip}}/(2n^2)$, while [13] prove nonasymptotic concentration of $\widehat{\mu}_n$ around the circular law in Wasserstein distance at this same scale.

(v) See [11] for other proofs and extensions, including an LSI for radial test functions, sum-of-squares deficit formulas, polynomial eigenfunctions, etc. See [14] for a Lean formalization.

(vi) At fixed diffusive inverse temperature, [26] gives LSI uniform in the particle number. This does not give uniform GinUE LSI: GinUE corresponds to inverse temperature $2n$ there.

## References


[1] T. H. Baker and P. J. Forrester, *The Calogero–Sutherland model and generalized classical polynomials*, Comm. Math. Phys. 188 (1997), 175–216.
[2] D. Bakry and M. Émery, *Diffusions hypercontractives*, Séminaire de Probabilités XIX, Lecture Notes in Math. 1123, Springer (1985), 177–206.
[3] D. Bakry, I. Gentil, and M. Ledoux, *Analysis and geometry of Markov diffusion operators*, Springer (2014).
[4] V. Bargmann, *On a Hilbert space of analytic functions and an associated integral transform*, Comm. Pure Appl. Math. 14 (1961), 187–214.
[5] B. Berndtsson, *Curvature of vector bundles associated to holomorphic fibrations*, Ann. of Math. 169 (2009), 531–560.
[6] P. Bizeul, B. Klartag, and J. Lehec, *Presenting a proof of the KLS conjecture*, arXiv:2610.05474.
[7] F. Bolley, D. Chafaï, and J. Fontbona, *Dynamics of a planar Coulomb gas*, Ann. Appl. Probab. 28 (2018), 3152–3183, arXiv:1706.08776.
[8] J. Boursier, D. Chafaï, and C. Labbé, *Universal cutoff for Dyson Ornstein–Uhlenbeck process*, Probab. Theory Related Fields 185 (2023), 449–512, arXiv:2107.14452.
[9] H. J. Brascamp and E. H. Lieb, *On extensions of the Brunn–Minkowski and Prékopa–Leindler theorems, including inequalities for log concave functions, and with an application to the diffusion equation*, J. Funct. Anal. 22 (1976), 366–389.
[10] L. A. Caffarelli, *Monotonicity properties of optimal transportation and the FKG and related inequalities*, Comm. Math. Phys. 214 (2000), 547–563.
[11] D. Chafaï, *An optimal Poincaré inequality for the complex Ginibre log-gas*, arXiv:2608.19358.
[12] D. Chafaï and M. Fathi, *On cutoff via rigidity for high-dimensional curved diffusions*, C. R. Math. 363 (2025), 1103–1121, arXiv:2412.15969.
[13] D. Chafaï, A. Hardy, and M. Maïda, *Concentration for Coulomb gases and Coulomb transport inequalities*, J. Funct. Anal. 275 (2018), arXiv:1610.00980.
[14] D. Chafaï, *Automatic formalization of a full paper in Lean*, blog post (2026), djalil.chafai.net.
[15] D. Chafaï and J. Lehec, *On Poincaré and logarithmic Sobolev inequalities for a class of singular Gibbs measures*, Geometric Aspects of Functional Analysis, Lecture Notes in Math. 2256, Springer (2020), arXiv:1805.00708.
[16] H. Chernoff, *A note on an inequality involving the normal distribution*, Ann. Probab. 9 (1981), 533–535.
[17] D. Cordero-Erausquin, *On Berndtsson's generalization of Prékopa's theorem*, Math. Z. 249 (2005), 401–410.
[18] F. J. Dyson, *A Brownian-motion model for the eigenvalues of a random matrix*, J. Math. Phys. 3 (1962).
[19] J. Ginibre, *Statistical ensembles of complex, quaternion, and real matrices*, J. Math. Phys. 6 (1965), 440–449.
[20] B. Helffer and J. Sjöstrand, *On the correlation for Kac-like models in the convex case*, J. Statist. Phys. 74 (1994), 349–409.
[21] L. Hörmander, *$L^2$ estimates and existence theorems for the $\bar{\partial}$ operator*, Acta Math. 113 (1965), 89–152.
[22] M. Lassalle, *Polynômes de Hermite généralisés*, C. R. Acad. Sci. Paris Sér. I 313 (1991), 579–582.
[23] J. Nash, *Continuity of solutions of parabolic and elliptic equations*, Amer. J. Math. 80 (1958), 931–954.
[24] H. Poincaré, *Sur les équations aux dérivées partielles de la physique math.*, Amer. J. Math. 12 (1890), 211–294.
[25] B. Rider and B. Virág, *The noise in the circular law and the GFF*, Int. Math. Res. Not. (2007), rnm006.
[26] M. Rosenzweig, *Uniform LSI for the 2D Coulomb gas at the diffusive temperature scale*, arXiv:2608.24863.
[27] K. Suzuki, *Spectral gap for unlabelled Ginibre interacting Brownian motion*, arXiv:2608.17437.