Dyson Ornstein Uhlenbeck process

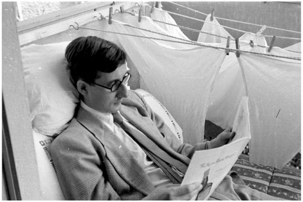
Cutoff phenomenon

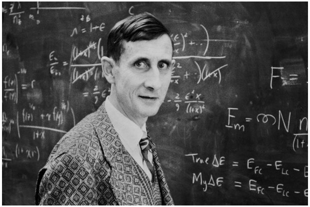
Jeanne Boursier, Djalil Chafaï*, Cyril Labbé

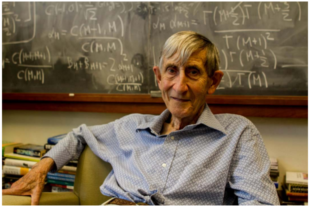
DMA, École normale supérieure – PSL
CEREMADE, Université Paris-Dauphine – PSL

Séminaire d'analyse fonctionnelle de l'IMJ
Sorbonne Université
Jeudi 17 novembre 2022

Freeman J. Dyson (1923 – 2020)







A Brownian Motion Model for the Eigenvalues of a Random Matrix

Journal of Mathematical Physics 3 1191–1198 (1962)

Plan

Dyson Ornstein Uhlenbeck process DOU_β

- Interacting particle system $X_t^{n,1},\ldots,X_t^{n,n}$ on $\mathbb{R}$

$$X_0^n = x_0^n, \quad \mathrm{d}X_t^n = \sqrt{\frac{2}{n}}\mathrm{d}B_t - \frac{1}{n}\nabla H(X_t^n)\mathrm{d}t$$

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- Configuration energy with Coulomb repulsion (singular)

$$H(x) = n\sum_{i=1}^{n} V(x_i) + \beta\sum_{i<j}\log\frac{1}{x_i - x_j}, \quad V(x) = \frac{x^2}{2}$$

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- Convergence to equilibrium $X_t^n \xrightarrow[t\to\infty]{\mathrm{d}} P^n \propto \mathrm{e}^{-H(x)}\mathrm{d}x$

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- We take $\beta = 0$ or $\beta \geq 1$ (preserves order $x_n < \cdots < x_1$)

Gaussian unitary invariant random matrices

- Random matrix cases $\beta \in \{1, 2, 4\}$: matrix OU

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  - Dyson : $(\mathrm{spectrum}(M_t))_{t\geq 0} \overset{\mathrm{d}}{=} \mathrm{DOU}_\beta$

DOU semigroup and generator

- Markov semigroup

$$\mathbb{E}(f(X_t^n) \mid X_0^n = x) = (\mathrm{e}^{t\,\mathrm{L}} f)(x)$$

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- Universality wrt β : spectrum, Poincaré, log-Sobolev

Wigner theorem and semi-circle law : scaling in n

- Empirical measure and exchangeability

$$\mu_t^n = \frac{1}{n}\sum_{i=1}^{n} \delta_{X_t^{n,i}} \quad \text{and} \quad \mathbb{E}\mu_\infty^n \sim \frac{1}{n}\sum_{i=1}^{n} P^{n,i} = P^{n,1}$$

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- Long-time behavior & mean-field limit (when $\mu_0^n \underset{n\to\infty}{\longrightarrow} \mu_0$)

$$\begin{array}{ccc} \mu_t^n & \underset{t\to\infty}{\longrightarrow} & \mu_\infty^n \\ {\scriptstyle n\to\infty}\downarrow & & \downarrow{\scriptstyle n\to\infty} \\ \mu_t & \underset{t\to\infty}{\longrightarrow} & \mu_\infty \end{array}$$

Mean-field limit and free probability : scaling in t

- McKean-Vlasov evolution equation

$$\partial_t \int f \mathrm{d}\mu_t = -\int x f'(x) \mu_t(\mathrm{d}x) + \frac{\beta}{2} \iint \frac{f'(x) - f'(y)}{x - y} \mu_t(\mathrm{d}x) \mu_t(\mathrm{d}y)$$

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$$s_t(z) = \int \frac{\mu_t(\mathrm{d}x)}{x - z}$$

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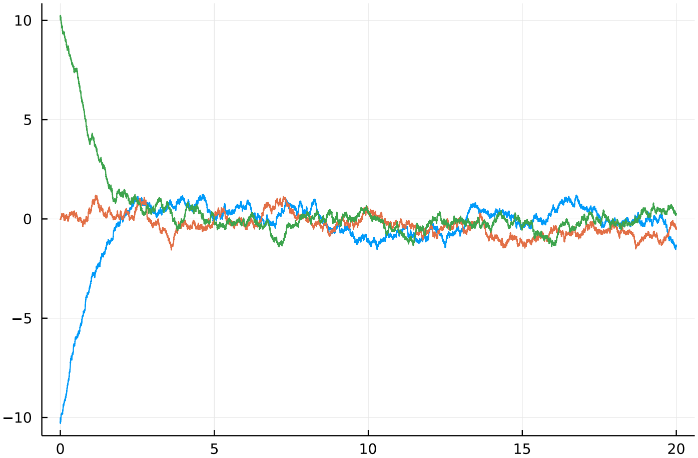
Numerical experiments



$n = 3$, $\beta = 0$: confinement and independence (OU)

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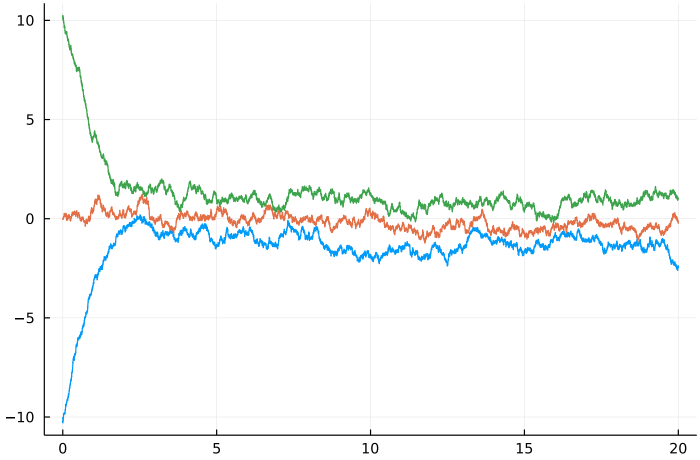
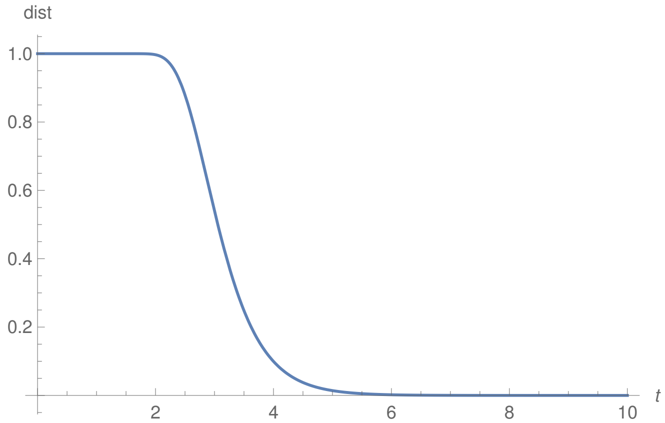


$n = 3$, $\beta = 2$: confinement and repulsion (DOU)

Cutoff for OU : Hellinger distance dist(Law(X_t^n) | P^n)



$$n = 50,\ \beta = 0,\ \frac{|x_0^n|^2}{n} = 1,\ \log(50) \approx 3.91$$

Expectation : cutoff phenomenon

- For all $\varepsilon \in (0,1)$

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- Universality with respect to β

Some distances or divergences

$$\chi^2(\nu \mid \mu) = \mathrm{Var}_\mu\left(\frac{\mathrm{d}\nu}{\mathrm{d}\mu}\right) = \left\|\frac{\mathrm{d}\nu}{\mathrm{d}\mu} - 1\right\|^2_{L^2(\mu)}$$

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Monotonicity

- With $\nu_t = \mathrm{Law}(X_t^n)$ and $\mu = P^n$

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- Fisher and Wasserstein : involve also convexity of V

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- $\log(n)$ cutoff for $\mathbb{E}(\pi_k(X_t))$: dimension n versus e^{-kt} decay

Cutoff for DOU : processes

- **Theorem :** Assume that $\beta = 0$ or $\beta \geq 1$ and set

$$Z_t = \sum_{i=1}^{n} X_t^{n,i} \quad \text{and} \quad R_t = \sum_{i=1}^{n} (X_t^{n,i})^2 = |X_t^n|^2$$

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- **Proof :** Stroock–Varadhan local martingale

$$M_t^f = f(X_t) - f(X_0) - \int_0^t \mathrm{L}f(X_s)\mathrm{d}s$$

$$\langle M \rangle_t = \int_0^t \Gamma(f)(X_s)\mathrm{d}s, \ \text{take } \mathrm{L}f = -\lambda f, \ \text{then } f \in \{\pi_1, \pi_2\}$$

Plan

The model

Non-interacting case

Random matrix case

General interacting case

Cutoff for OU : Mean-field case

- **Theorem** : if $\beta = 0$ and $\frac{|x_0^n|^2}{n} \asymp 1$ then for all $\varepsilon \in (0,1)$

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- Reminds behavior of second moment m_2

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$$2\text{Wasserstein}^2(\Gamma_1, \Gamma_2) = |m|^2 + \text{Tr}\left(\Sigma_1 + \Sigma_2 - 2\sqrt{\sqrt{\Sigma_1}\Sigma_2\sqrt{\Sigma_1}}\right)$$

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$$\text{Hellinger}^2(\Gamma_1, \Gamma_2) = 1 - \sqrt{\frac{\sqrt{\det(\Sigma_1\Sigma_2)}}{\det(\frac{\Sigma_1 + \Sigma_2}{2})}} \exp\left(-\frac{1}{4}(\Sigma_1 + \Sigma_2)^{-1} m \cdot m\right)$$

$$\|\mu - \nu\|_{\text{TV}} \leq \sqrt{2\text{Kullback}(\nu \mid \mu)}$$

Cutoff for OU : Proof 1/2

If $\Gamma_1 = \mathcal{N}(\mu_1, \Sigma_1)$ and $\Gamma_2 = \mathcal{N}(\mu_2, \Sigma_2)$ in $\mathbb{R}^n$ then with $m = m_1 - m_2$:

$$\chi^2(\Gamma_1 \mid \Gamma_2) = \sqrt{\frac{|\Sigma_2|}{|2\Sigma_1 - \Sigma_1^2\Sigma_2^{-1}|}} e^{\frac{1}{2}\Sigma_2^{-1}(I_n + 2\Sigma_1^{-1}\Sigma_2^{-1} - \Sigma_2^{-2})m\cdot m} - 1$$

$$2\mathrm{Kullback}(\Gamma_1 \mid \Gamma_2) = \Sigma_2^{-1} m \cdot m + \mathrm{Tr}(\Sigma_2^{-1}\Sigma_1 - I_n) + \log\det(\Sigma_2\Sigma_1^{-1})$$

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Cutoff for OU : Proof 2/2

If $\beta = 0$ and $X_0^n = x_0^n$ then $X_t^n \sim \mathcal{N}(x_0^n e^{-t}, \frac{1}{n}(1-e^{-t})I_n)$ and

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$$\mathrm{Kullback}(\mathrm{Law}(X_t^n) \mid P^n) = \tfrac{1}{2}\Big(n|x_0^n|^2 e^{-2t} - n e^{-2t} - n \log(1 - e^{-2t}) \Big)$$

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$$\mathrm{Hellinger}^2(\mathrm{Law}(X_t^n), P^n) = 1 - \exp\Big(-\frac{n}{4} \frac{|x_0^n|^2 e^{-2t}}{2 - e^{-2t}} + \frac{n}{4} \log\Big(4 \frac{1 - e^{-2t}}{(2 - e^{-2t})^2} \Big) \Big)$$

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$\log(n)$ cutoff for $\mathrm{dist}(\mathrm{Law}(X_t^n) \mid P^n)$: n versus e^{-t}

Plan

Cutoff for DOU: Random matrix case

- Theorem : Assume that $\beta \in \{1, 2, 4\}$. Let (a_n) be such that $\inf(a_n) > 0$. Then for all $\varepsilon \in (0, 1)$

$$\lim_{n\to\infty} \sup_{x_0^n \in [-a_n, a_n]^n} \mathrm{dist}(\mathrm{Law}(X_{t_n}^n) \mid P^n) = \begin{cases} \max & \text{if } t_n = (1-\varepsilon)c_n \\ 0 & \text{if } t_n = (1+\varepsilon)c_n \end{cases}$$

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- Proof : OU sandwich (trace Z, matrix M) + dist contract.

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- Cutoff should be controlled by $|x_0^n - \rho^n|$ instead of $|x_0^n|$

Plan

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- Theorem : Assume that $\beta = 0$ or $\beta \geq 1$. Let (a_n) be such that $\inf(a_n) > 0$. Then for all $\varepsilon \in (0,1)$

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- Lower bound : Contraction to OU Z
- Upper bound : LSI, regularization, coupling ($\sim$ exclusion)

Cutoff for DOU : Proof for general case (1/2)

- Optimal log-Sobolev inequality

$$\mathrm{Kullback}(\nu \mid P^n) \leq \frac{1}{2n}\mathrm{Fisher}(\nu \mid P^n)$$

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- Regularization Y^n of X^n (smoothed $Y_0^{n,i} \geq X_0^{n,i} = x_0^{n,i}$)

$$\mathrm{Kullback}(\mathrm{Law}(Y_t^n) \mid P^n) \leq C(n|x_0^n|^2 + n^2\log(n))\mathrm{e}^{-2t}$$

Cutoff for DOU : Proof for general case (2/2)

- Coalescent coupling preserving order $Y_t^{n,i} \geq X_t^{n,i}$

$$\begin{aligned}\|\mathrm{Law}(Y_t^n) - \mathrm{Law}(X_t^n)\|_{\mathrm{TV}} &\leq \mathbb{P}(Y_t^n \neq X_t^n) \\ &\leq \mathbb{P}(\inf\{s \geq 0 : A_s^n = 0\} > t)\end{aligned}$$

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- "Area" $A_t = \sum_{i=1}^n (Y_t^{n,i} - X_t^{n,i})$ is an AR sort of OU

$$\mathrm{d}A_t = -A_t \mathrm{d}t + \mathrm{d}M_t$$

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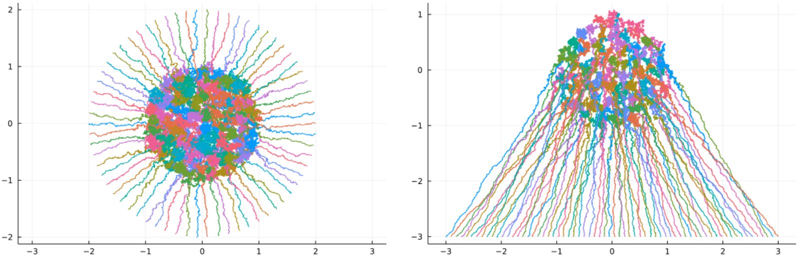
- Submartingale in $[0,1]$ $\mathrm{e}^{-\lambda A - \frac{\lambda^2}{2}\langle A \rangle}$



julia

$\mathbb{C}^n$, $-\log|\cdot|$

spectrum, eigenfunctions

Poincaré, log-Sobolev, cutoff at $\log(n)$

Open problems & Selected bibliography

- Problems

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  - C.–Lehec, Bolley–C.–Fontbona, Lu–Mattingly

Leonard Salomon Ornstein (1880 – 1941)
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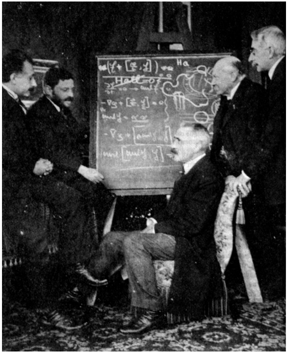
Paul Langevin (1872 – 1946)

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