Riesz Energy Problems and Integral Identities

Unexpected phenomena for equilibrium measures

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Coulomb gases and universality
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Plan

Integral Identities and Special Functions ($\lambda \in [0,1]$)

$$\int_0^1 \left((\lambda + r)\log(\lambda + r) - (\lambda - r)\log|\lambda - r|\right)\frac{r\,\mathrm{d}r}{\sqrt{1-r^2}} = ?$$

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$$\int_0^1 ((\lambda + r)\log(\lambda + r) - (\lambda - r)\log|\lambda - r|)\frac{r\,\mathrm{d}r}{\sqrt{1-r^2}} = \frac{\pi}{2}\left(\lambda^2 + \frac{1}{2} - \log 2\right)$$

$$\int_0^1 K\left(\frac{4\lambda r}{(r+\lambda)^2}\right)\frac{(\lambda - r)r\,\mathrm{d}r}{\sqrt{1-r^2}} = \frac{\pi^2}{8}\left(\frac{3\lambda^2}{2} - 1\right)$$

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Mathematica...

$$\int_0^1 (\lambda + r)\log(\lambda + r)\frac{r\,\mathrm{d}r}{\sqrt{1-r^2}} = \frac{{}_3F_2\left(\frac{1}{2},1,1;\frac{3}{2},\frac{5}{2};\frac{1}{\lambda^2}\right)}{3\lambda} - \pi\,\frac{{}_3F_2\left(1,1,\frac{3}{2};2,3;\frac{1}{\lambda^2}\right)}{32\lambda^2} + \frac{(4\lambda+\pi)\log(\lambda)+\pi}{4}$$

Plan

Marcel Riesz

MARCEL RIESZ

(1886, Győr, Hungary – 1969, Lund, Sweden)

- Intégrales de Riemann–Liouville et potentiels
  Acta Sci. Math. Szeged 9 (1938): 1–42, 116–118

Riesz original problem (1931, 1938)

- Riesz s-kernel in $\mathbb{R}^d$, $-2 < s < d$

$$K_s = \begin{cases} \dfrac{1}{s\,|\cdot|^s} & \text{if } s \neq 0 \\ -\log|\cdot| & \text{if } s = 0 \end{cases}$$

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- Potential $U^{\mu} := K_s * \mu$
- Equilibrium Measure on ball $B_R := \{x \in \mathbb{R}^d : |x| \leq R\}$

$$\mu_{\mathrm{eq}} = \arg \min_{\substack{\mu \\ \mathrm{supp}(\mu) \subset B_R}} I_s(\mu)$$

Marcel Riesz original problem ($d \geq 2$)

- Equilibrium Measure on B_R

$$\mu_{\text{eq}} = \begin{cases} \sigma_R & \text{if } -2 < s \leq d-2 \\ \dfrac{\Gamma(1+\frac{s}{2})}{R^s \pi^{\frac{d}{2}} \Gamma(1+\frac{s-d}{2})} \dfrac{\mathbb{1}_{B_R}}{(R^2 - |x|^2)^{\frac{d-s}{2}}} \mathrm{d}x & \text{if } d-2 < s < d \end{cases}$$

(σ_R is uniform law on sphere of radius R)

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$$\int_{|y| \leq R} \frac{|x-y|^{-s}}{(R^2-|y|^2)^{\frac{d-s}{2}}} \mathrm{d}y = \frac{\pi^{\frac{d}{2}+1}}{\Gamma(\frac{d}{2}) \sin(\frac{\pi}{2}(d-s))}, \quad x \in B_R$$

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  - Dyda – Kuznetsov – Kwaśnicki (Constr. Approx. 2017)

About Mellin transform and Riesz kernels

- Reciprocal pair of integral transform of $f : (0, +\infty) \to \mathbb{R}$

$$\mathcal{M}f(z) := \int_0^\infty f(x)x^{z-1}\mathrm{d}x, \quad \alpha < \Re z < \beta,$$

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- Riesz potential of radial function : if $f(x) = \varphi(|x|^2)$ then

$$(K_s * f)(x) = \psi(|x|^2) \quad \text{where} \quad \mathcal{M}\psi(z) = \eta_{d,s}(z)\mathcal{M}(\varphi)(z + \tfrac{d-s}{2})$$

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- More on `https://djalil.chafai.net/blog/2022/08/26/mellin-transform-and-riesz-potentials/`

External Field Equilibrium Problem

- Energy with external field V on $\mathbb{R}^d$

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- Frostman or Euler–Lagrange characterization of μ_{eq}

$$K_s * \mu + V \begin{cases} = c & \text{q.e. on } \operatorname{supp}(\mu) \\ \geq c & \text{q.e. outside } \operatorname{supp}(\mu) \end{cases}$$

Coulomb case : $s = d - 2$

- Laplace fundamental solution

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- If $V = |\cdot|^\alpha$, $\alpha > 0$, then

$$\mu_{\mathrm{eq}} = \alpha(\alpha + d - 2)\, |\cdot|^{\alpha-2}\, \mathbb{1}_{B_R} \mathrm{d}x$$

(with $R = \left(\frac{1}{\alpha}\right)^{\frac{1}{\alpha+d-2}}$)

Continuous case $-2 < s < 0$ and $\alpha = 2$

Equilibrium problem :

$$\arg\min_{\mu}\left\{ -\iint |x-y|^{|s|}\mathrm{d}\mu(x)\mathrm{d}\mu(y) + 2\int |x|^2\,\mathrm{d}\mu(x)\right\}$$

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- Hertrich-Gräf–Beinert-Steidl arXiv:2211.01804
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- Explicit formulas for μ_{eq}

Plan

Motivation

Riesz Energy Problems

Findings and Open Questions

Iterated Coulomb case : $s = d - 2n,\ n = 1, 2, 3, \ldots$

Proposition (CSW 2022)

If $s = d - 2n$ then, restricted to the interior of μ_{eq},

$$\mu_{\mathrm{eq}} = \frac{\Delta^n V}{c_d C_{d,n}} \quad \textit{where} \quad C_{d,n} := (-1)^{n-1}(2n-2)!!$$

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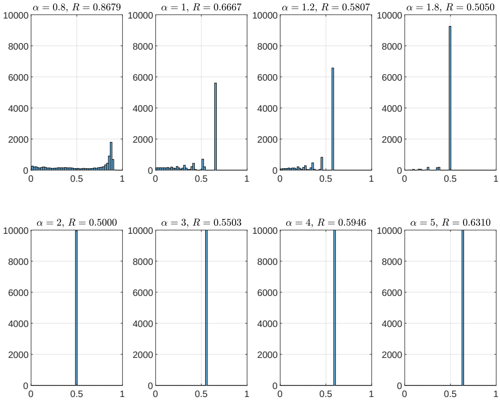
- Key : $-\Delta K_u = c_{d,u} K_{u+2},\ c_{d,u} := d - 2 - u$
- In particular : if $s = d - 4$ and $V = |\cdot|^\alpha$, $\alpha \geq 2$, then $C_{d,2} < 0$ while $\Delta V = \alpha(\alpha + d - 2)\,|\cdot|^{\alpha-2} \geq 0$ and thus μ_{eq} is necessarily singular!

Condensation phenomenon for $s = d - 4$?



Point norms histograms $|x_j|, 1 \le j \le N = 10^4$ of a discrete approximation of $\mu_{\text{eq}},\ d = 4,\ s = d - 4 = 0,\ V = |\cdot|^\alpha,\ \alpha > 0.$

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  - *If* $0 < \alpha < 2$ *then* $\mu_{\mathrm{eq}} = \beta f m_d + (1-\beta)\sigma_R$ *where*

$$\beta = \frac{2-\alpha}{s+2}, \quad f = \frac{\alpha+s}{R^{\alpha+s}|\mathbb{S}^{d-1}|}\mathbb{1}_{B_R}, \quad R = \left(\frac{2}{(\alpha+s+2)\alpha\gamma}\right)^{\frac{1}{\alpha+s}}$$

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  - *If* $\alpha = 1$ *and* $\gamma \geq 1$, *then* $\mu_{\mathrm{eq}} = \delta_0$
  - *If* $\alpha > 1$, *then* μ_{eq} *is as above*

Theorem (CSW JMAA 2022 : $V = \gamma\,|\cdot|^2$, $\gamma > 0$)

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- Initial motivation : $(d, s, \alpha) = (3, 0, 2)$ and Gumbel at edge.

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  - Non-locality of fractional Laplacian

Link : obstacle problems (thanks, Sylvia!)

- Fundamental solution (fractional Laplacian) :

$$-\Delta_s K_s = \delta_0, \quad K_s = (-\Delta_s)^{-1}, \quad -\Delta_s U = \mu, \quad U := K_s * \mu$$

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- Variational : convex duality for energies

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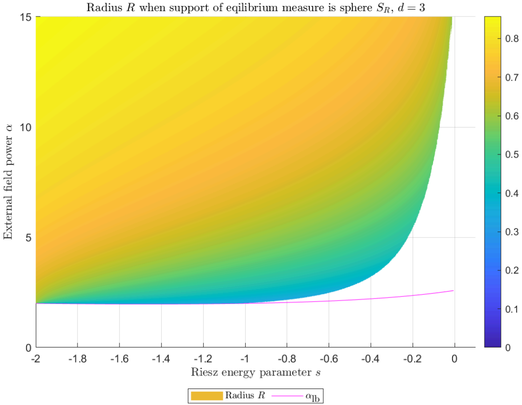
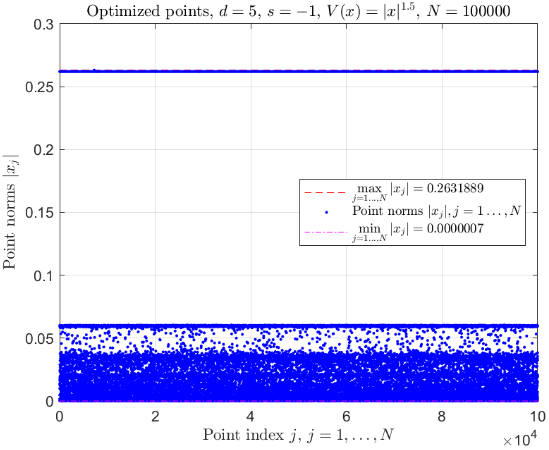
- Other norms in kernel and external field

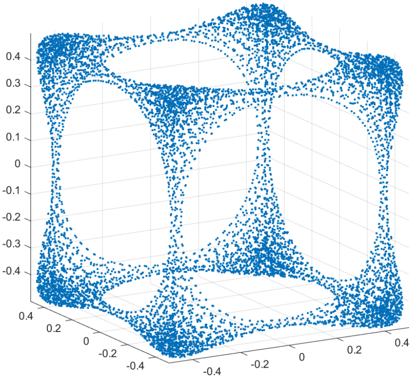


A plot of the radius R when the support of the equilibrium measure is the sphere S_R for $d = 3$ and various s and α.



Point norms $|x_j|$, $1 \le j \le N = 10^5$,
$d = 5,\ s = d - 6 = -1,\ V = |\cdot|^{3/2}$.

More than 70% on outer boundary!



Projection on $x_1 = 0$ of a numerical approximation of μ_{eq},
$N = 10^4,\ d = 4,\ s = d - 4 = 0,\ V = |\cdot|_4^4,$
$|x|_p := (|x_1|^p + \cdots + |x_d|^p)^{1/p}.$

Naoum Samoilovitch Landkof

(1915, Kharkov, Russian Empire – 2004, Israel)

- Foundations of Modern Potential Theory
  Grundlehren der mathematischen Wissenschaften 180
  Springer 1972 (translated from Russian, Moscow 1966)