About non-Hermitian random matrices

Djalil Chafaï

Université Paris-Dauphine

Adamczak, Bordenave, Caputo, Gozlan, Péché, Wolff, Zitt

High Dimensional Probability VII
Cargèse, Corsica
May 27, 2014

Outline

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Elementary matrix model

- Random variable X taking values in $\mathcal{M}_n(\mathbb{C})$

$$\begin{pmatrix} X_{11} & \cdots & X_{1n} \\ \vdots & \ddots & \vdots \\ X_{n1} & \cdots & X_{nn} \end{pmatrix}$$

- Independent and equally distributed entries X_{ij}
- Behavior of the spectrum of X ?

Algebraic and geometric spectra of $A \in \mathcal{M}_n(\mathbb{C})$

- Algebraic spectrum: eigenvalues (complex)
  - roots in $\mathbb{C}$ of characteristic polynomial $P_A(z) := \det(A - zI)$
  - $A = UTU^*$ and $\mathrm{diag}(T) = \lambda_1(A), \ldots, \lambda_n(A)$
  - $|\lambda_1(A)| \geqslant \cdots \geqslant |\lambda_n(A)|$
  - Spectral radius: $|\lambda_1(A)|$

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- Geometric spectrum: singular values (real $\geqslant 0$)
  - half lengths of principal axes of ellipsoid $\{Ax : \|x\|_2 = 1\}$
  - $A = UDV^*$ and $D = \mathrm{diag}(s_1(A), \ldots, s_n(A))$
  - $s_1(A) \geqslant \cdots \geqslant s_n(A)$
  - Operator norm: $s_1(A) = \max_{\|x\|_2=1} \|Ax\|_2$
  - $s_k(A) = \lambda_k(\sqrt{AA^*})$

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  - $s_k(A) = \lambda_k(\sqrt{AA^*})$
- $AA^* = A^*A$ (normal matrix) iff $\forall k,\ s_k(A) = |\lambda_k(A)|$

Weyl inequalities and determinental rigidity

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- Counting measures:

$$\mu_A = \frac{\delta_{\lambda_1(A)} + \cdots + \delta_{\lambda_n(A)}}{n} \quad \text{et} \quad \nu_A = \frac{\delta_{s_1(A)} + \cdots + \delta_{s_n(A)}}{n}$$

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- Determinental rigidity:

$$\begin{aligned} |\lambda_1(A)\cdots\lambda_n(A)| &= s_1(A)\cdots s_n(A) = |\det(A)| \\ \int \log(|\lambda|)\, d\mu_A(\lambda) &= \int \log(s)\, d\mu_{\sqrt{AA^*}}(s) = \frac{1}{n}\log|\det(A)| \end{aligned}$$

Sensitivity to perturbations

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \\ 0 & \cdots & & \ddots & 1 \\ 0 & \cdots & & & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \\ 0 & \cdots & & \ddots & 1 \\ \varepsilon_n & \cdots & & & 0 \end{pmatrix}$$

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$$AA^* = \mathrm{diag}(1,\ldots,1,0) \qquad BB^* = \mathrm{diag}(1,\ldots,1,\varepsilon_n)$$

$$A^n = 0, \lambda_k(A) = 0 \qquad B^n = \varepsilon_n I_n, \lambda_k(B) = \varepsilon_n^{1/n} e^{i2\pi k/n}$$

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$$\begin{cases} \nu_A & \to \delta_1 \\ \mu_A & = \delta_0 \end{cases} \qquad \begin{cases} \nu_B & \to \delta_1 \\ \mu_B & \to \mathrm{Uniform}(C(0,1)) \end{cases}$$

Random matrix model

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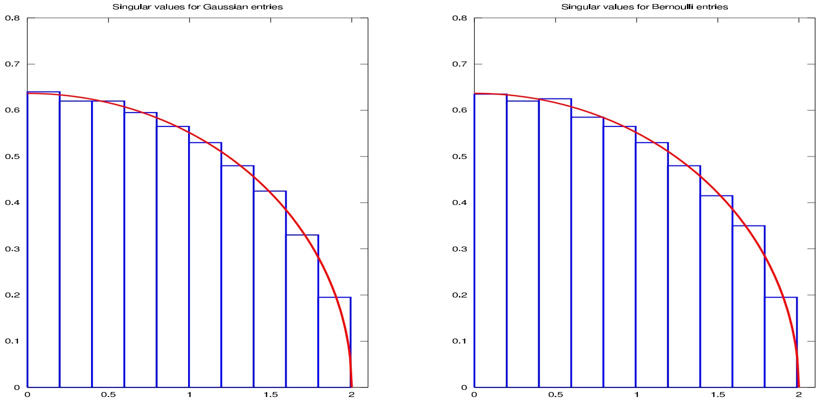
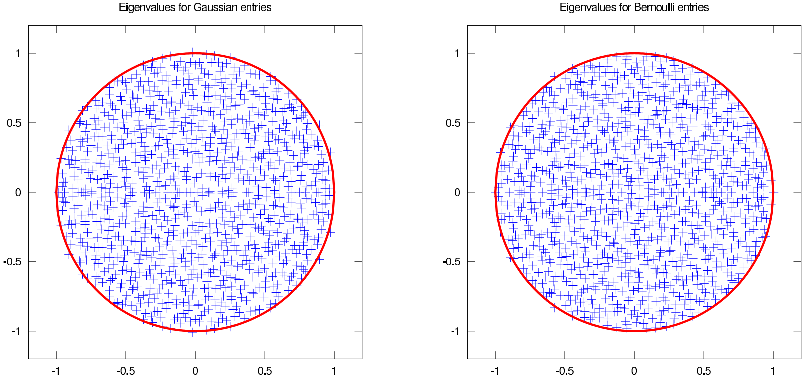
- Independent and equally distributed entries X_{ij}
- $\boxed{\text{Behavior of } \mu_X \text{ and } \nu_X \text{ when } n \to \infty \text{ ?}}$

Quarter circular law (Universality)



Singular values of $\frac{1}{\sqrt{n}}X$

Circular law (Universality)



Eigenvalues $\frac{1}{\sqrt{n}}X$

Theorem (Quarter circular law – Marchenko-Pastur)

If $\mathrm{Var}(X_{11}) = 1$ *then*

$$\nu_{\frac{1}{\sqrt{n}}X} \underset{n\to\infty}{\longrightarrow} \frac{\sqrt{4-x^2}\,\mathbf{1}_{[0,2]}}{\pi}\,dx$$

Theorem (Circular law – Girko, Bai, G.-T, Pan-Zou, Tao-Vu)

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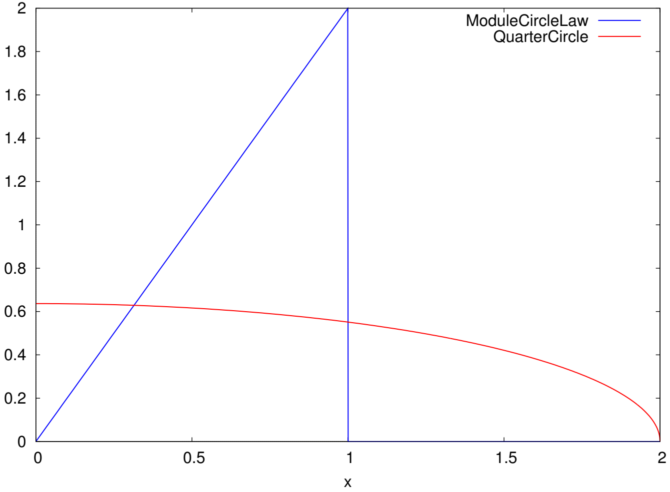
ModuleCircleLaw
QuarterCircle
2
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
0
0.5
1
1.5
2
x

Support convergence and edge behavior

If $\mathrm{Var}(X_{11}) = 1$ then quatercircular and circular laws give a.s.

$$\varliminf_{n\to\infty} s_1\left(\frac{1}{\sqrt{n}}X\right) \geqslant 2 \quad \text{and} \quad \varliminf_{n\to\infty} \left|\lambda_1\left(\frac{1}{\sqrt{n}}X\right)\right| \geqslant 1$$

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If $\mathbb{E}(X_{11}) = 0$ and $\mathbb{E}(|X_{11}|^4) < \infty$ then a.s.

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Idea: Gelfand spectral radius formula: for any matrix norm $\|\cdot\|$,

$$|\lambda_1(A)| = \lim_{k\to\infty} \left\|A^k\right\|^{1/k}$$

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Second moment stabilization:

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Law of Large Numbers!

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- H Hermitian $n \times n$ and $\eta_H := \frac{1}{n} \sum_{k=1}^{n} \delta_{\lambda_k(H)}$

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- $\boxed{\text{Not enough on } \mathbb{C} \text{ for the circular law!}}$

Tightness for free

From the strong law of large numbers (SLLN):

$$\int s^2 \, d\nu_{\frac{1}{\sqrt{n}}X}(s) = \frac{1}{n^2}\sum_{k=1}^{n} s_k(X)^2 = \frac{1}{n^2}\sum_{i,j=1}^{n} |X_{ij}|^2 \xrightarrow[n\to\infty]{a.s.} \mathbb{E}(|X_{11}|^2).$$

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Conclusion: a.s. $(\mu_{\frac{1}{\sqrt{n}}X})_{n\geqslant 1}$ is tight

Analysis of a Gaussian case (1/3)

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- $\mathrm{Tr}(GG^*) = \mathrm{Tr}(TT^*) = \mathrm{Tr}(DD^*) + \mathrm{Tr}(NN^*)$
- $(\lambda_1(G), \ldots, \lambda_n(G))$ has density

$$\varphi_n(z_1, \ldots, z_n) = c_n \exp\left(-\sum_{k=1}^n |z_k|^2\right) \prod_{1\leqslant i<j\leqslant n} |z_i - z_j|^2.$$

Analysis of a Gaussian case (2/3)

- The 1-point correlation is the density of $\mathbb{E}\mu_G$:

$$\varphi_n^{(1)}(z) = \frac{e^{-|z|^2}}{n\pi} \sum_{\ell=0}^{n-1} \frac{|z|^{2\ell}}{\ell!}$$

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- Following Mehta, this gives the mean circular law:

$$\lim_{n\to\infty} n\varphi_n^{(1)}(\sqrt{n}z) = \frac{1}{\pi}\mathbf{1}_{[0,1]}(|z|).$$

Analysis of a Gaussian case (3/3)

- Kostlan's observation:

$$(|\lambda_1(G)|,\dots,|\lambda_n(G)|) \stackrel{\text{law}}{=} (Z_{(1)},\dots,Z_{(n)})$$

where $Z_1,\dots,Z_n$ are independent with $Z_k^2 \sim \mathrm{Gamma}(k,1)$

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- Moreover if $\gamma_n := \log(n/2\pi) - 2\log(\log(n))$ then

$$\sqrt{4n\gamma_n}\left(|\lambda_1(\tfrac{1}{\sqrt{n}}G)| - 1 - \sqrt{\frac{\gamma_n}{4n}}\right) \xrightarrow[n\to\infty]{\text{law}} \mathrm{Gumbel}.$$

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- Universality: C.-Péché (2014)

Large deviations (1/2)

- Setting $V(z) = |z|^2$, the density of $\lambda_1(\frac{1}{\sqrt{n}}G), \dots, \lambda_n(\frac{1}{\sqrt{n}}G)$ is

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- Rewriting in terms of $\mu_n := \frac{1}{n}\sum_{k=1}^n \delta_{z_k}$:

$$c_n \exp\left(-n^2\left(\frac{1}{n}\sum_{k=1}^n V(z_k) - \frac{2}{n^2}\sum_{i<j} \log|z_i - z_j|\right)\right)$$

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- Approximation as $n \gg 1$:

$$\approx c_n \exp\left(-n^2 \mathcal{I}(\mu_n)\right)$$

where $\mathcal{I}$ is the logarithmic energy with external field:

$$\mathcal{I}(\mu) := \int V(z)\,d\mu + \iint \log\frac{1}{|z-w|}\,d\mu(z)d\mu(w).$$

Large deviations (2/2)

- Hiai-Petz and BenArous-Zeitouni: for every $S \subset \mathcal{M}_1(\mathbb{C})$

$$\mathbb{P}(\mu_{\frac{1}{\sqrt{n}}G} \in S) \approx \exp\left(-n^2 \inf_S \mathcal{I}\right).$$

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- Logarithmic energy = - Voiculescu free entropy

Large deviations (2/2)

- Hiai-Petz and BenArous-Zeitouni: for every $S \subset \mathcal{M}_1(\mathbb{C})$

$$\mathbb{P}(\mu_{\frac{1}{\sqrt{n}}G} \in S) \approx \exp\left(-n^2 \inf_S \mathcal{I}\right).$$

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- Interacting particles in dim > 2: Gozlan-C.-Zitt (2014)

Proof of circular law (1/4) – Logarithmic potential

- Logarithmic potential of a probability measure μ on $\mathbb{C}$

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- Hermitization and Brown spectral measure

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-U_{\mu_A}(z) &= \int_{\mathbb{C}} \log|z-\lambda|\, d\mu_A(\lambda) \\
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Proof of circular law (3/4) – Hermitization

If $\lim_{n\to\infty} \nu_{A_n - zI} = \nu_z$ weakly then do we have

$$\begin{aligned}\lim_{n\to\infty} \mu_{A_n} &= \lim_{n\to\infty} \frac{1}{2\pi}\Delta \int_0^\infty \log(s)\, d\nu_{A_n - zI}(s) \\ &\stackrel{?}{=} \frac{1}{2\pi}\Delta \int_0^\infty \log(s)\, d\nu_z(s)\end{aligned}$$

Problem: singularity of the logarithm near 0 and ∞

Proof of circular law (4/4) – Hermitization

Lemma (Hermitization (Girko, Tao-Vu, Bordenave-C.))

If A_n is a random variable on $\mathcal{M}_n(\mathbb{C})$ and if for all $z \in \mathbb{C}$

- $\nu_{A_n - zI} \underset{n\to\infty}{\longrightarrow} \nu_z$ *(deterministic)*
- $\log$ *is uniformly integrable for* $\nu_{A_n - zI}$

Then

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- Boltzmann and Shlyakhtenko, Brown and Śniady

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- Weyl random polynomials (Kabluchko-Zaporozhets 2012)

$$P_n(z) = \sum_{k=0}^{n} \frac{\xi_k}{\sqrt{k!}} z^k$$

Beyond the circular law: infinite variance (1/2)

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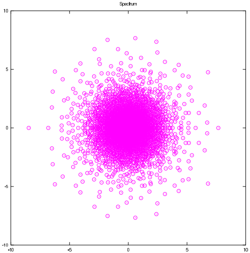

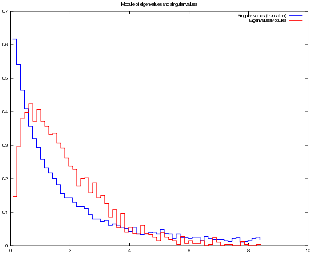

Beyond the circular law: random generators (2/2)

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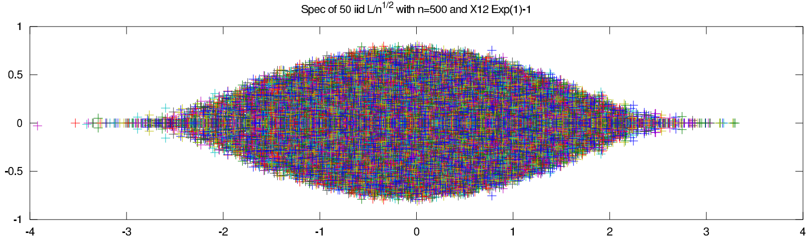

Beyond the circular law: two conjectures (1/2)

Conjecture (Bernoulli invertibility conjecture)

If $(A_{ij})_{1\leqslant i,j\leqslant n}$ are i.i.d. Bernoulli ± 1 then

$$\mathbb{P}(s_n(A)=0)=\mathbb{P}(\det(A)=0)=\left(\frac{1}{2}+o_{n\to\infty}(1)\right)^n.$$

Progresses: $1/\sqrt{2}$ (Bourgain-Vu-Wood 2010)

Beyond the circular law: two conjectures (2/2)

Conjecture (Kesten–McKay oriented d-regular graphs)

If $P_1,\ldots,P_d$ are i.i.d. $n\times n$ Haar permutation matrices then

$$\mu_{P_1+\cdots+P_d} \underset{n\to\infty}{\longrightarrow} \frac{d^2(d-1)}{\pi(d^2-|z|^2)^2}\mathbf{1}_{\{|z|<\sqrt{d}\}}dxdy$$

Brown measure of $U_1 \boxplus \cdots \boxplus U_d$ (Haagerup-Larsen)

Progresses: Rudelson-Vershynin (2012), Basak-Dembo (2012), Ben Arous & Dang (2014)

Thank you!

Bonus for curiosity!

Lemma (Rows and operator norm of the inverse (RV))

Let $A \in \mathcal{M}_n(\mathbb{C})$ with rows $R_1, \ldots, R_n$ then

$$n^{-1/2} \min_{1 \leqslant i \leqslant n} \operatorname{dist}(R_i, R_{-i}) \leqslant s_n(A) \leqslant \min_{1 \leqslant i \leqslant n} \operatorname{dist}(R_i, R_{-i}).$$

Lemma (Rows and trace norm of the inverse (TV))

Let $1 \leqslant m \leqslant n$. If $A \in \mathcal{M}_{m,n}(\mathbb{C})$ has full rank then

$$\sum_{i=1}^{m} s_i(A)^{-2} = \sum_{i=1}^{m} \operatorname{dist}(R_i, R_{-i})^{-2}.$$