

ARCHIMEDES, BARENBLATT, AND SATO–TATE

SPHERICAL PROJECTIONS AND ASSOCIATED CLASSICAL PROBABILITY LAWS

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ABSTRACT. These notes explore mathematical history and current developments, starting from a result of Greek antiquity, with elements of geometry, probability, analysis, algebra, and number theory. The starting point is a modern formulation of Archimedes’ theorem on the sphere and the cylinder. Through projection, it naturally gives rise to classical probability laws such as the beta, Student, and Gaussian laws. These laws are then connected to the Barenblatt profile in the analysis of nonlinear partial differential equations. Finally, the semicircle law is a special case linked to the three names in the title, notably to Sato and Tate’s equidistribution theorem for elliptic curves in number theory.

These notes offer a journey through history, with elements of geometry, probability, analysis, algebra, and number theory. The first part explores probability laws arising from projections of spheres, in particular beta, Student, and Gaussian laws, and their connections with current developments in the analysis of high-dimensional convex bodies. The second part connects these laws to the Barenblatt profiles that occur in the fundamental solution of a nonlinear diffusion partial differential equation. The third part presents a connection with a Sato–Tate equidistribution theorem for elliptic curves in number theory. In particular, the famous semicircle law $(2\pi)^{-1} \sqrt{(4-x^2)_+} dx$, which is both a symmetric beta law and a Barenblatt profile, is connected to the three names in the title:

- (1) **Archimedes:** marginal law of the uniform law on the sphere $\mathbb{S}^3$.
- (2) **Barenblatt:** profile of the fundamental solution of the 1D porous medium equation $\partial_t u = \partial_{xx}(u^3)$.
- (3) **Sato–Tate:** limiting law of the normalized Frobenius trace via Haar measure on $SU(2)$.

These are concise and eclectic expository notes intended to arouse curiosity. They continue and provide an updated elaboration of [R], [PR], [L], [Ch], among other possible references.

1. ARCHIMEDES’ THEOREM ON THE SPHERE AND THE CYLINDER

“Finally although in the year 1500 Europe knew less than Archimedes who died in the year 212 BC, yet in the year 1700, Newton’s Principia had been written and the world was well started on the modern epoch.” Alfred North Whitehead, in *Science and the Modern World* (1925), quoted in particular by Arthur Koestler in *The Sleepwalkers* (1959).

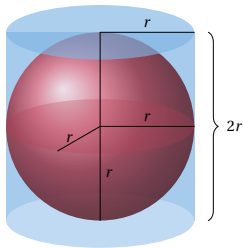


FIGURE 1. A sphere and its circumscribed cylinder, that is, the smallest possible cylinder containing it.

For $n = 3$ and $k = 1$, Theorem 1.1 states that the uniform law on the sphere, projected onto its vertical diameter, gives the uniform law on that diameter. This coincides with the discovery of Archimedes of Syracuse (287 – 212 BC): considering the section by two parallel planes perpendicular to the axis of the cylinder, the area of the portion of the sphere between the two planes equals the area of the portion of the cylinder between the two planes. In particular, the sphere and the lateral surface of the cylinder have the same area, which amounts to saying that the area of the sphere of radius r is $2\pi r \times 2r = 4\pi r^2$. Archimedes used the method of exhaustion, anticipating integral calculus by more than 1800 years. Theorem 1.1 is the generalization to all dimensions, expressed in the contemporary language of probability theory. Legend has it that Archimedes considered this his best result, and asked for a drawing of a sphere and its circumscribed

cylinder to be engraved on his tombstone. Cicero (106 – 43 BC) recounts in the *Tusculan Disputations* (V, 64–66) having rediscovered this tomb in Syracuse in 75 BC thanks to this motif, 137 years after the death of Archimedes, who was killed by a Roman soldier during the siege of the city.

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These notes were polished with the help of artificial intelligence and translated into English by machine translation.

Theorem 1.1: On the sphere and the cylinder (Archimedes)

Let $X = (X_1, \dots, X_n)$ be uniform on the sphere $\mathbb{S}^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$, $n \geq 2$. Then for every $1 \leq k < n$, its projection $Y = \text{proj}_{\mathbb{R}^k}(X) = (X_1, \dots, X_k)$, in the ball $\mathbb{B}^k = \{x \in \mathbb{R}^k : |x| \leq 1\}$, has density

$$y \mapsto c(1 - |y|^2)^{\frac{n-k-2}{2}} \quad \text{where} \quad c = \frac{\Gamma(n/2)}{\pi^{k/2} \Gamma((n-k)/2)}.$$

In particular, if $n \geq 3$, then $\text{proj}_{\mathbb{R}^{n-2}}(X) = (X_1, \dots, X_{n-2})$ has the uniform law on the ball $\mathbb{B}^{n-2}$.

The uniform projection property on $\mathbb{B}^{n-2}$ is the generalization presented in [R]. See also [L]. The historical result is reproduced in [W]. Modern extensions of Archimedes' ideas are presented in [Be], while connections with Gaussian laws and Stein's method are developed in [PR].

In densities, we adopt the convention that t_+^a equals t^a if $t > 0$ and 0 if $t \leq 0$, including when $a \leq 0$.

For $k = 1$, this gives the zonal integration formula, a special case of the Funk–Hecke formula for the constant (degree-zero) spherical harmonic. The associated orthogonal polynomials are the Gegenbauer polynomials with parameter $(n-2)/2$ for $n \geq 3$, and the first-kind Chebyshev polynomials for $n = 2$ (the limiting parameter).

The law of Y is a *multivariate symmetric beta law* on $\mathbb{B}^k$, uniform if $k = n-2$. If $k < n-2$, its exponent is strictly positive and it has, up to dilation, a *compactly supported Barenblatt profile*, cf. Section 2. The case $k = n-1$ has exponent $-1/2$ and a density unbounded at the boundary, a kind of radial arcsine.

Uniform laws on balls and spheres and their projections are natural equilibrium measures in potential theory, cf. for example [CSW] for a recent result related to the case $k = n-1$.

In particular, for every $n \geq 2$, with $k = 1$, rotation invariance of the law of X implies that for every $\theta \in \mathbb{S}^{n-1}$, $\langle X, \theta \rangle \stackrel{d}{=} X_1 \sim \text{Beta}_{[-1,1]}(\frac{n-1}{2}, \frac{n-1}{2})$, with density proportional to $x \mapsto (1-x)_+^p (1+x)_+^p = (1-x^2)_+^p$ where $p = (n-3)/2$. This gives the uniform law for $n = 3$, the *semicircle* for $n = 4$, and the *arcsine* for $n = 2$.

Proof. Orthogonal invariance of the law of X ensures that of the law of Y . It therefore suffices to determine the law of $|Y|^2$. Now X has the same law as $Z/|Z|$ where $Z = (Z_1, \dots, Z_n) \sim \mathcal{N}(0, I_n)$. Thus Y has the same law as $(Z_1, \dots, Z_k)/|Z|$, so that $|Y|^2$ has the same law as $A/(A+B)$ with independent $A \sim \chi^2(k)$ and $B \sim \chi^2(n-k)$. It follows that $|Y|^2 \sim \text{Beta}_{[0,1]}(k/2, (n-k)/2)$. Thus $|Y|$ has density $r \mapsto c' r^{k-1} (1-r^2)_+^{(n-k-2)/2}$ where $c' = 2/\text{beta}(k/2, (n-k)/2)$. In polar coordinates $y = r\theta$, the volume element is $dy = r^{k-1} dr d\sigma(\theta)$, where σ is surface measure on $\mathbb{S}^{k-1}$ (with $|\mathbb{S}^0| = 2$). By orthogonal invariance, the density of Y is therefore $c(1 - |y|^2)_+^{(n-k-2)/2}$, with

$$c = \frac{c'}{|\mathbb{S}^{k-1}|} = \frac{2}{|\mathbb{S}^{k-1}| \text{beta}(k/2, (n-k)/2)} = \frac{\Gamma(n/2)}{\pi^{k/2} \Gamma((n-k)/2)}.$$

□

Corollary 1.2: Uniform law on the ball: Archimedes' theorem in reverse

For every $n \geq 1$, if $Z_1, \dots, Z_n$ are i.i.d. with law $\mathcal{N}(0, 1)$ and E is exponential with mean 1, independent of $Z_1, \dots, Z_n$, then the random vector $(Z_1, \dots, Z_n)/\sqrt{Z_1^2 + \dots + Z_n^2 + 2E}$ has the uniform law on $\mathbb{B}^n$.

Proof. If $Z_1, \dots, Z_n, Z_{n+1}, Z_{n+2}$ are i.i.d. $\mathcal{N}(0, 1)$, then $(Z_1, \dots, Z_n)/\sqrt{Z_1^2 + \dots + Z_{n+2}^2}$ has the uniform law on $\mathbb{B}^n$ by Theorem 1.1. On the other hand, $Z_{n+1}^2 + Z_{n+2}^2 \sim \chi^2(2) = \text{Gamma}(2/2, 1/2) = \text{Exp}(1/2) \stackrel{d}{=} 2E$. □

This representation theorem for the uniform law generalizes to ℓ^p balls, see [B+].

Theorem 1.3: Stereographic projection of the uniform law on the sphere

Let X be uniform on the unit sphere $\mathbb{S}^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$, $n \geq 2$. Let $Y = (Y_1, \dots, Y_{n-1}) = S(X)$ be its image under stereographic projection S from the north pole e_n . Then for every $1 \leq k \leq n-1$, the random vector $\text{proj}_{\mathbb{R}^k}(Y) = (Y_1, \dots, Y_k)$, taking values in $\mathbb{R}^k$, has density

$$y \mapsto c(1 + |y|^2)^{-\frac{n+k-1}{2}} \quad \text{where} \quad c = \frac{\Gamma(\frac{n+k-1}{2})}{\pi^{k/2} \Gamma(\frac{n-1}{2})}.$$

For $n \geq 3$, this law has, up to dilation, a *Barenblatt profile with full support*. The case $n = 2$ would correspond to $m = 0$, excluded from the setting $m > 0$ of Section 2. The full vector Y has a *multivariate Student law* of

¹The law $\text{Beta}_{[0,1]}(a, b)$, $a, b > 0$, has density $u \mapsto \text{beta}(a, b)^{-1} u^{a-1} (1-u)^{b-1}$ where $\text{beta}(a, b) = \int_0^1 u^{a-1} (1-u)^{b-1} du = \Gamma(a)\Gamma(b)/\Gamma(a+b)$.

dimension $d = n - 1$, location $\mu = 0$, scale matrix $\Sigma = d^{-1}I_d$, with $r = n - 1$ degrees of freedom² Its k -dimensional marginals are Student laws of dimension $d = k$, with $r = n - 1$ degrees of freedom and $\Sigma = (n - 1)^{-1}I_k$. Such a Student law is that of $\mu + A/\sqrt{B/r}$ where $A \sim \mathcal{N}(0, \Sigma)$ and $B \sim \chi^2(r)$ are independent. For $n = 2$, the full vector Y has the *Cauchy law* on $\mathbb{R}$, the law of the ratio of two independent standard real Gaussian random variables. For $n = 3$, this is the *Cauchy law* on $\mathbb{R}^2 \equiv \mathbb{C}$, the law of the ratio of two independent standard complex Gaussians.

Proof. Stereographic projection $S: \mathbb{S}^{n-1} \rightarrow \mathbb{R}^{n-1} \cup \{\infty\}$ is defined by

$$S(x) = \frac{(x_1, \dots, x_{n-1})}{1 - x_n} \quad \text{if } x = (x_1, \dots, x_n) \neq e_n \quad \text{and} \quad S(e_n) = \infty.$$

The projection from the north pole e_n fixes the horizontal unit sphere $\{x \in \mathbb{R}^n : |x| = 1, x_n = 0\}$.

Rotation invariance of X implies that of $(Y_1, \dots, Y_k)$. Thus, for every $\theta \in \mathbb{R}^k$, $\langle \theta, (Y_1, \dots, Y_k) \rangle \stackrel{d}{=} |\theta| Y_1$. Now since $Y_1 = X_1 / (1 - X_n)$, it follows that, for every $t \in \mathbb{R}$,

$$\mathbb{P}(Y_1 \leq t) = \mathbb{P}(X_1 + tX_n \leq t) = \mathbb{P}\left(X_1 \leq \frac{t}{\sqrt{1+t^2}}\right)$$

where the second equality follows from rotation invariance of the law of X . Since $t \mapsto t/\sqrt{1+t^2}$ has inverse $u \mapsto u/\sqrt{1-u^2}$, it follows that $Y_1 \stackrel{d}{=} X_1/\sqrt{1-X_n^2}$. Since $X \stackrel{d}{=} Z/|Z|$ where $Z = (Z_1, \dots, Z_n) \sim \mathcal{N}(0, I_n)$, we obtain

$$Y_1 \stackrel{d}{=} \frac{Z_1}{\sqrt{Z_2^2 + \dots + Z_n^2}} = \frac{A}{\sqrt{B}}$$

where $A \sim \mathcal{N}(0, 1)$ and $B \sim \chi^2(n-1)$ are independent. Thus $\sqrt{n-1}Y_1$ has the Student law $t(n-1)$. If $G \sim \mathcal{N}(0, I_k)$ is independent of B , then for every $\theta \in \mathbb{R}^k$, $\langle \theta, G/\sqrt{B} \rangle \stackrel{d}{=} |\theta|A/\sqrt{B} \stackrel{d}{=} \langle \theta, (Y_1, \dots, Y_k) \rangle$. Hence $(Y_1, \dots, Y_k) \stackrel{d}{=} G/\sqrt{B}$ by characteristic functions (the Cramér–Wold theorem). This yields the stated result.

After this geometric and probabilistic proof, let us give a purely analytic proof using the change-of-variables formula. The north pole e_n has zero surface measure, so $S(X)$ almost surely takes values in $\mathbb{R}^{n-1}$. The inverse of S on $\mathbb{S}^{n-1} \setminus \{e_n\}$ is $F(y) = (2y/(1+|y|^2), (|y|^2-1)/(1+|y|^2))$. Set $q = 1+|y|^2$. For $1 \leq i \leq n-1$, $\partial_i F(y) = (2e_i/q - 4y_i y/q^2, 4y_i/q^2)$ and $\langle \partial_i F(y), \partial_j F(y) \rangle = (4/q^2)\delta_{ij}$. The Jacobian is therefore

$$J_F(y) = \sqrt{\det((DF(y))^T DF(y))} = \left(\frac{2}{1+|y|^2}\right)^{n-1}.$$

Applying the change-of-variables formula to any bounded Borel function $h: \mathbb{R}^{n-1} \rightarrow \mathbb{R}$ and using $|\mathbb{S}^{n-1}| = 2\pi^{n/2}/\Gamma(n/2)$ for the surface measure of the unit sphere, we obtain

$$\mathbb{E}[h(S(X))] = \frac{1}{|\mathbb{S}^{n-1}|} \int_{\mathbb{R}^{n-1}} h(y) J_F(y) dy.$$

□

Remark 1.4: Central limit phenomenon through high-dimensional projection

The standard Gaussian law appears through high-dimensional projection.

- For every fixed $k \geq 1$, if X has the uniform law on $\mathbb{S}^{n-1}(\sqrt{n})$, $n > k$, then $\text{proj}_{\mathbb{R}^k}(X) \stackrel{d}{\rightarrow} \mathcal{N}(0, I_k)$ as $n \rightarrow \infty$. Indeed, $X \stackrel{d}{=} \sqrt{n}Z/|Z|$ where $Z \sim \mathcal{N}(0, I_n)$, and $\sqrt{n}/|Z| = 1 + o_n(1)$ by the law of large numbers. Standard Gaussian laws can therefore be viewed as uniform laws on spheres of high dimension and large radius. This phenomenon, whose discovery is often attributed to Émile Borel or Henri Poincaré, appears earlier in the work of Ferdinand Gustav Mehler [M]. This can also be checked directly from the formula in Theorem 1.1:

$$\left(1 - \frac{|y|^2}{n}\right)_+^{\frac{n-k-2}{2}} \xrightarrow{n \rightarrow \infty} e^{-\frac{|y|^2}{2}}, \quad y \in \mathbb{R}^k, \quad \text{and} \quad \frac{\Gamma(n/2)}{n^{k/2} \pi^{k/2} \Gamma((n-k)/2)} \xrightarrow{n \rightarrow \infty} \frac{1}{(2\pi)^{k/2}}.$$

In the same spirit, with stereographic projection, Theorem 1.3 for $k = 1$ shows that if X has the uniform law on $\mathbb{S}^{n-1}$ and $Y = S(X)$ then $\sqrt{n}Y_1 \stackrel{d}{\rightarrow} \mathcal{N}(0, 1)$ as $n \rightarrow \infty$, since

$$\left(1 + \frac{y^2}{n}\right)^{-\frac{n}{2}} \xrightarrow{n \rightarrow \infty} e^{-\frac{y^2}{2}}, \quad y \in \mathbb{R}, \quad \text{and} \quad \frac{\Gamma(n/2)}{\sqrt{n\pi} \Gamma((n-1)/2)} \xrightarrow{n \rightarrow \infty} \frac{1}{\sqrt{2\pi}}.$$

- Corollary 1.2, combined with the law of large numbers for the denominator, shows that if X has the uniform law on $\mathbb{B}^n(\sqrt{n})$, then for every $\theta \in \mathbb{S}^{n-1}$, $\langle X, \theta \rangle \stackrel{d}{=} X_1 \stackrel{d}{\rightarrow} \mathcal{N}(0, 1)$ as $n \rightarrow \infty$. This is a special case of the central limit phenomenon for convex bodies, proved in [K]. On the other hand,

²With density $y \in \mathbb{R}^d \mapsto c(1+r^{-1}(y-\mu) \cdot \Sigma^{-1}(y-\mu))^{-(r+d)/2}$ where $c = \Gamma((r+d)/2)/(\Gamma(r/2)(r\pi)^{d/2}\sqrt{\det \Sigma})$.

Corollary 1.2 also shows that in high dimension $n \rightarrow \infty$, most of the mass of $\mathbb{B}^n(\sqrt{n})$ lies near $\mathbb{S}^{n-1}(\sqrt{n})$. This is a special case of the thin-shell phenomenon for convex bodies, recently proved with an optimal constant in [CK]: for an isotropic log-concave vector W in $\mathbb{R}^n$, $\text{Var}(|W|^2) \leq 8n$.

Remark 1.5: Spheres, projections, beta and Student laws, Barenblatt profiles, and Gaussians

Theorem 1.1 shows that the linear projection of $\mathbb{S}^{n-1}$ onto $\mathbb{R}^k$ produces compactly supported profiles $c(1 - |y|^2)_+^a$ with $a = (n - k - 2)/2 \geq -1/2$, whereas Theorem 1.3 shows that the nonlinear stereographic projection of $\mathbb{S}^{n-1}$ onto $\mathbb{R}^{n-1}$ produces a heavy-tailed profile $c(1 + |y|^2)^{-b}$, with $b > 0$. The parameters a and b are integers or half-integers, and this quantization comes from the geometric realization, which involves dimensions, which are integers. All of this can also be viewed as a calculus of Eulerian special functions, Gamma and multivariate beta. Beta and Student laws can be defined geometrically as continuous-parameter generalizations of spherical projection laws. As explained below, they are connected by the family of Barenblatt profiles or laws in the parameter ranges specified above. This family also contains the Gaussian law as a common critical case.

Remark 1.6: Dirichlet laws

The proof of Theorem 1.1 hints at the workings of Dirichlet laws. If $A_1, \dots, A_n$ are independent with $A_k \sim \text{Gamma}(a_k, \lambda)$, then $(A_1, \dots, A_n)/(A_1 + \dots + A_n) \sim \text{Dirichlet}_n(a_1, \dots, a_n)$, which does not depend on λ . This is a law on the simplex of discrete probability laws $\{(p_1, \dots, p_n) \in [0, 1]^n : p_1 + \dots + p_n = 1\}$. If $(B_1, \dots, B_n) \sim \text{Dirichlet}_n(a_1, \dots, a_n)$, then for any $I_1, \dots, I_r$ forming a partition of $\{1, \dots, n\}$ into $r \geq 2$ nonempty blocks, $(\sum_{i \in I_1} B_i, \dots, \sum_{i \in I_r} B_i) \sim \text{Dirichlet}_r(\sum_{i \in I_1} a_i, \dots, \sum_{i \in I_r} a_i)$, and for every nonempty proper subset $I \subset \{1, \dots, n\}$, $\sum_{i \in I} B_i \sim \text{Beta}_{[0,1]}(\sum_{i \in I} a_i, \sum_{i \notin I} a_i)$. If $U_1, \dots, U_n$ are i.i.d. uniform on $[0, 1]$, and if $B_1, \dots, B_{n+1}$ are the lengths of the blocks in the associated partition of $[0, 1]$, the successive spacings in the order statistics, that is, the increasing rearrangement, then $(B_1, \dots, B_{n+1}) \sim \text{Dirichlet}_{n+1}(1, \dots, 1)$. Dirichlet laws play a fundamental role in simulation, nonparametric mathematical statistics, and probability theory. Poisson–Dirichlet laws are laws on random partitions of mass into infinitely many components, related to Dirichlet laws through limiting and rearrangement procedures.

^aThe law $\text{Gamma}(a, \lambda)$, $a, \lambda > 0$, has density $u \mapsto \Gamma(a)^{-1} \lambda^a u^{a-1} e^{-\lambda u} \mathbb{1}_{u \geq 0}$, where $\Gamma(a) = \int_0^\infty u^{a-1} e^{-u} du$.

2. BARENBLATT PROFILES: FAST DIFFUSION AND POROUS MEDIA

Consider the following nonlinear partial differential equation for a density u on $\mathbb{R}_+ \times \mathbb{R}^n$, $n \geq 1$:

$$\partial_t u = \Delta(u^m) = \text{div}(m u^{m-1} \nabla u), \quad t > 0.$$

The parameter m is a positive real number. The case $m = 1$ is the linear heat equation. The case $0 < m < 1$ is called the fast diffusion equation. The case $m > 1$ is called the porous medium equation. Assume that

$$m > \max\left(0, 1 - \frac{2}{n}\right) \quad \text{and} \quad \beta = \frac{1}{2 + n(m-1)} > 0,$$

which will ensure that Barenblatt profiles of finite mass are solutions.

We seek mass-preserving self-similar solutions of the form

$$u(t, x) = t^{-n\beta} B(x t^{-\beta})$$

where B is a profile to be determined. The powers yield the condition $1 = 2\beta + n\beta(m-1)$, while for the profile,

$$\Delta B^m + \beta \text{div}(xB) = 0 \quad \text{that is,} \quad \text{div}(\nabla B^m + \beta x B) = 0.$$

Seeking a radial B with zero radial flux gives $\nabla B^m = -\beta x B$. For $m \neq 1$, on $\{B > 0\}$, this gives

$$\nabla B^{m-1} = -\frac{\beta(m-1)}{m} x.$$

Integration yields two profiles, with a constant $c > 0$ chosen so that $\int_{\mathbb{R}^n} B(x) dx = 1$:

$$B(x) = \begin{cases} \left(c + \frac{\beta(1-m)}{2m} |x|^2\right)^{-1/(1-m)} & \text{if } 0 < m < 1 \\ \left(c - \frac{\beta(m-1)}{2m} |x|^2\right)_+^{1/(m-1)} & \text{if } m > 1 \end{cases}.$$



FIGURE 2. The Fields Medal, designed around 1933, explicitly evokes Archimedes' Theorem 1.1 in its ancient version $n = 3$ and $k = 1$. More precisely, on the left, Archimedes' portrait on the obverse of the Fields Medal, his name in Greek, a date, and a Latin quotation from the poet Marcus Manilius: "*Transire suum pectus mundoque potiri (to rise above oneself and conquer the world).*" On the right, on the reverse of the medal, a Latin sentence: "*Congregati ex toto orbe mathematici ob scripta insignia tribuere (mathematicians gathered from all over the world have given an award for outstanding contributions)*" and, in the background, a sphere inscribed in a cylinder, placed behind a branch and recalling the motif on Archimedes' tomb and his theorem "On the sphere and the cylinder." The edge of the medal, not reproduced here, bears the recipient's name. The description of the medal is taken from [IMU].

The solution of the fast diffusion equation (case $m < 1$) has a heavy polynomial tail, and integrability requires $2/(1 - m) > n$, that is, $m > 1 - 2/n$. The profile for the porous medium equation (case $m > 1$) has compact support, the centered ball whose radius increases with time (slow diffusion):

$$u(t, x) = t^{-n\beta} c^{1/(m-1)} \left(1 - \frac{|x|^2}{R(t)^2}\right)_+^{1/(m-1)} \quad \text{where} \quad R(t) = \sqrt{\frac{2mc}{\beta(m-1)}} t^\beta.$$

For $m = 1$, the centered self-similar solution of unit mass is the (Gaussian) heat kernel

$$u(t, x) = \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|x|^2}{4t}\right).$$

For $n = 1$ and $m = 3$, we explicitly recover the semicircle law:

$$\beta = \frac{1}{4}, \quad c = \frac{1}{\pi\sqrt{3}}, \quad B(x) = \left(\frac{1}{\pi\sqrt{3}} - \frac{x^2}{12}\right)_+^{1/2},$$

and the solution of unit mass is written as

$$u(t, x) = \frac{2}{\pi R(t)^2} \sqrt{(R(t)^2 - x^2)_+}, \quad R(t) = \left(\frac{48t}{\pi^2}\right)^{1/4}.$$

The arcsine law formally corresponds to the case $m = -1$, outside this setting.

If $m > 1$, extension by zero satisfies the equation in the weak sense, including at the boundary of the support. In all three cases, the measures $u(t, x)dx$ converge weakly to the Dirac mass δ_0 as $t \searrow 0$.

The Barenblatt profile B interpolates between the multivariate Student law (when $\max(0, 1 - 2/n) < m < 1$) and the multivariate beta law (when $m > 1$), passing through the Gaussian for $m = 1$. The number of degrees of freedom in the Student case is $r = 2/(1 - m) - n > 0$. With unit-mass normalization, the profiles converge to $(4\pi)^{-n/2} \exp(-|x|^2/4)$, the Gaussian profile with covariance $2I_n$, as $m \rightarrow 1$ from either side.

The correspondence with spherical projections is therefore explicit: writing d for the ambient dimension of the sphere and k for that of the projection (thus $n = k$ in the diffusion equation), identifying the exponents gives $m = 1 + 2/(d - k - 2)$ if $1 \leq k < d - 2$, and $m = 1 - 2/(d + k - 1)$ if $1 \leq k \leq d - 1$ and $d + k > 3$,



Grigory Isaakovich Barenblatt (1927–2018)

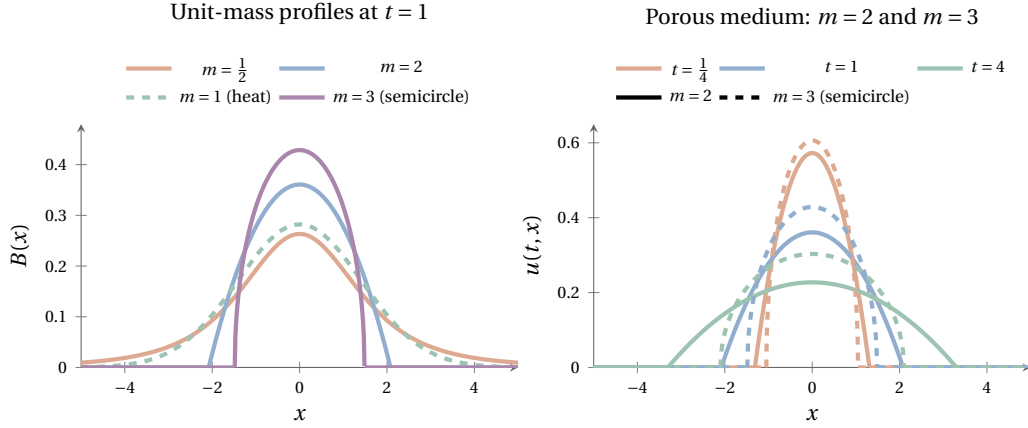


FIGURE 3. Exact unit-mass solutions in 1D.

respectively for the orthogonal projections and stereographic marginals of Theorems 1.1 and 1.3. In the second case, $\max(0, 1 - 2/k) < m < 1$. This is a correspondence between densities and profiles of unit mass, up to dilation. In particular, $d = 4$ and $k = 1$ give $m = 3$ in the first case: this is the semicircle.

We refer to [V] for an in-depth study of the profiles and their role in nonlinear diffusion, and to [B] for self-similarity and intermediate asymptotics. The semicircle profile is also preserved by a nonlinear nonlocal McKean–Vlasov evolution equation. Its Cauchy–Stieltjes transform satisfies a complex Burgers equation, and its complex logarithmic potential a Hamilton–Jacobi equation. The semicircle law plays the role of the Gaussian law in Dan-Virgil Voiculescu’s free probability theory. Finally, a famous theorem of Eugene Wigner states that the semicircle law appears as the limiting law of the empirical spectral measures of Hermitian $n \times n$ Wigner random matrices³, in the high-dimensional limit $n \rightarrow \infty$. However, Section 3 shows that its simplest spectral setting is that of 2×2 matrices following Haar measure on $SU(2)$, linked to Archimedes’ principle for S^3 .

3. THE SATO–TATE LAW IN NUMBER THEORY VIA S^3 AND $SU(2)$

The group $SU(2)$ and the sphere S^3 . The elements of $SU(2)$ are uniquely parametrized by

$$U = \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix}, \quad a, b \in \mathbb{C}, \quad |a|^2 + |b|^2 = 1.$$

Writing $a = x_1 + ix_2$ and $b = x_3 + ix_4$ identifies the group $SU(2)$ with the sphere $S^3 \subset \mathbb{R}^4$, which is also the unit sphere of the quaternions. Left and right multiplication act orthogonally on these four real coordinates, so that Haar measure on $SU(2)$ corresponds to the uniform law on S^3 .

Distribution of the trace via Archimedes. If $X = (X_1, X_2, X_3, X_4)$ has the uniform law on S^3 , then its projection (X_1, X_2) onto $\mathbb{R}^2$ is uniform on the unit disk (Theorem 1.1). It follows immediately that X_1 has the semicircle law supported on $[-1, 1]$, with density $s \mapsto 2\sqrt{(1 - s^2)_+}/\pi$. Consequently, if U follows Haar measure on $SU(2)$, then

$$\text{Tr}(U) = a + \bar{a} \stackrel{d}{=} 2X_1$$

has the semicircle law supported on $[-2, 2]$, with density (of mean zero and variance 1)

$$s \mapsto \frac{\sqrt{(4 - s^2)_+}}{2\pi}.$$

Elliptic curves and reduction modulo a prime. An *elliptic curve* over $\mathbb{Q}$ can be represented by a *Weierstrass equation* of the form (the condition $\Delta_E \neq 0$ guarantees smoothness)

$$E: y^2 = x^3 + ax + b \quad \text{where } a, b \in \mathbb{Z} \text{ satisfy } \Delta_E := -16(4a^3 + 27b^2) \neq 0.$$

The smooth projective curve E is obtained by adding its unique point at infinity $\mathcal{O}$ to the affine model above. The set $E(\mathbb{Q}) = \{(x, y) \in \mathbb{Q}^2 : y^2 = x^3 + ax + b\} \cup \{\mathcal{O}\}$ of rational points, where $\mathcal{O}$ is the point at infinity, is endowed with an abelian group law by the so-called chord-and-tangent construction, with $\mathcal{O}$ as the identity element. For a prime number p not dividing $2\Delta_E$ (a sufficient condition for *good reduction*), reduction modulo p gives a smooth curve $\tilde{E}$ over $\mathbb{F}_p$, and we define the *Frobenius coefficient*

$$a_p(E) := p + 1 - \#\tilde{E}(\mathbb{F}_p).$$

³Their entries strictly above the diagonal are i.i.d., centered, and have variance $1/n$.

The term $p + 1$ is the number of points of a smooth projective conic over $\mathbb{F}_p$, and $a_p(E)$ measures the deviation from this reference value. A theorem of Helmut Hasse (1898 – 1979) dating from 1936 states that for every prime number p of good reduction for E ,

$$|a_p(E)| \leq 2\sqrt{p}.$$

This allows us to compare the coefficients $a_p(E)$ for different primes by considering the normalized traces $s_p := a_p(E)/\sqrt{p} \in [-2, 2]$. The table below illustrates these quantities for the curve

$$E_{11}: y^2 + y = x^3 - x^2,$$

whose only prime of bad reduction is 11. The values of $\#\tilde{E}_{11}(\mathbb{F}_p)$ are obtained by direct enumeration of the solutions in $\mathbb{F}_p^2$, with the point at infinity added.

p	$\#\tilde{E}_{11}(\mathbb{F}_p)$	a_p	$s_p = a_p/\sqrt{p}$
2	5	-2	-1.414
3	5	-1	-0.577
5	5	+1	+0.447
7	10	-2	-0.756
13	10	+4	+1.109
17	20	-2	-0.485
19	20	+0	+0.000
23	25	-1	-0.209

TABLE 1. Frobenius coefficients for the elliptic curve $E_{11}: y^2 + y = x^3 - x^2$. Prime $p = 11$ excluded (bad reduction).

At this stage, the law of the normalized traces s_p is not obvious. In the 1960s, Mikio Satō (1928 – 2023) and John Tate (1925 – 2019) independently formulated a conjecture predicting the semicircle law for curves without complex multiplication. To understand the shape of this law, let us return to $SU(2)$.

From point counting to Frobenius classes. Hasse’s bound allows us to write uniquely $s_p = 2\cos\theta_p$, with $\theta_p \in [0, \pi]$. The matrix

$$\tilde{M}_p = \begin{pmatrix} e^{i\theta_p} & 0 \\ 0 & e^{-i\theta_p} \end{pmatrix} \in SU(2)$$

has trace s_p . Every matrix in $SU(2)$ is conjugate within this group to such a matrix, and its trace completely determines its conjugacy class. Each prime of good reduction is thus associated with a class, called the *normalized Frobenius class*. This name comes from the Frobenius endomorphism of the reduced curve, $(x, y) \mapsto (x^p, y^p)$. Its action on the torsion points of order a power of a prime $\ell \neq p$, described by the Tate module, has characteristic polynomial $T^2 - a_p(E)T + p$. Its two complex roots are $\sqrt{p}e^{\pm i\theta_p}$: normalizing them recovers the eigenvalues of $\tilde{M}_p$. The question therefore becomes: are Frobenius classes distributed like the conjugacy classes of a matrix drawn from Haar measure on $SU(2)$? If so, their traces follow the semicircle law given by Archimedes’ theorem.

Statement of the Sato–Tate theorem. An elliptic curve is said to be *without complex multiplication* if its only endomorphisms over $\mathbb{Q}$ are multiplication by integers. This is in particular the case for E_{11} . The following theorem was established between 2006 and 2011 by Laurent Clozel, Michael Harris, Nicholas Shepherd-Barron and Richard Taylor, then Tom Barnet-Lamb, David Geraghty, Harris and Taylor [B++].

Theorem 3.1: Sato–Tate, 2006–2011

Let $E/\mathbb{Q}$ be an elliptic curve without complex multiplication, and $P(x)$ the set of primes $p \leq x$ of good reduction for E . Then, against bounded continuous test functions (weak convergence),

$$\frac{1}{\#P(x)} \sum_{p \in P(x)} \delta_{s_p} \xrightarrow{x \rightarrow \infty} \frac{\sqrt{(4-s^2)_+}}{2\pi} ds.$$

Archimedes’ theorem identifies the candidate measure: the semicircle is the law of the trace of a Haar-distributed matrix in $SU(2)$. The deep arithmetic content of the Sato–Tate theorem is the equidistribution of Frobenius classes according to the image of this Haar measure on the space of conjugacy classes. In angular coordinates $s = 2\cos(\theta)$, the semicircle has density $\theta \in [0, \pi] \mapsto (2/\pi)\sin^2(\theta)d\theta$.

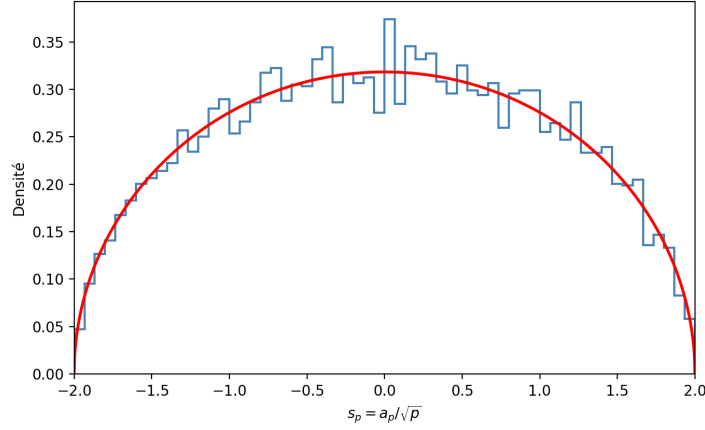


FIGURE 4. Histogram of the normalized traces $(s_p)_{p \leq 10^5}$ for the curve $E_{11} : y^2 + y = x^3 - x^2$, overlaid with the semicircle density $s \mapsto \sqrt{(4 - s^2)_+} / (2\pi)$. SageMath code:

```
E = EllipticCurve('11a3')
traces = []
for p in prime_range(100001):
    if E.has_good_reduction(p):
        traces.append(float(E.ap(p) / sqrt(RR(p))))
G = histogram(traces, bins=60, density=True, histtype='step',
              color='steelblue', linewidth=1.5)
G += plot(sqrt(4-x^2)/(2*pi), (x, -2, 2), color='red', thickness=2)
G.show()
```

Complex multiplication. The assumption of no complex multiplication is essential. A curve has *complex multiplication* (CM) when its algebra $\text{End}_{\overline{\mathbb{Q}}}(E) \otimes_{\mathbb{Z}} \mathbb{Q}$ is an imaginary quadratic field K . For example, $E : y^2 = x^3 - x$ has the endomorphism $(x, y) \mapsto (-x, iy)$ and has complex multiplication by $\mathbb{Z}[i]$, with $K = \mathbb{Q}(i)$. For such a curve, the law of the normalized traces s_p differs radically from the semicircle:

- as p ranges over the primes of good reduction that *split* in K , the s_p are equidistributed according to the *arcsine law* on $[-2, 2]$, with density $s \mapsto 1/(\pi\sqrt{4 - s^2})$ for $|s| < 2$.
- if p is *inert* in K , then $a_p(E) = 0$, hence $s_p = 0$, and the law of the s_p is the Dirac mass δ_0 .

This result, due to Erich Hecke (1887 – 1947), and the density $1/2$ of each of the two families of primes give the limiting law $\frac{1}{2}\delta_0 + \frac{1}{2}\mu_{\text{arcsinus}}$, see [S19]. In particular, the set of primes such that $s_p = 0$ has density $1/2$. For $E : y^2 = x^3 - x$, for example, $a_p = 0$ for every prime $p \equiv 3 \pmod{4}$.

The interpretation in terms of Haar measure is just as simple here. The torus

$$T = \{\text{diag}(e^{i\theta}, e^{-i\theta}) : \theta \in [0, 2\pi)\} \subset \text{SU}(2)$$

is isomorphic to the circle: its Haar measure corresponds to an angle θ uniform on $[0, 2\pi)$. The trace $2\cos\theta$ therefore follows, by a change of variables, the arcsine law on $[-2, 2]$. For a CM curve over $\mathbb{Q}$, the Sato–Tate group is the normalizer of T in $\text{SU}(2)$: it has two components of equal Haar mass, T and a second component all of whose matrices have trace zero. We thus recover the mixture $\frac{1}{2}\mu_{\text{arcsinus}} + \frac{1}{2}\delta_0$.

A glimpse of the proof. The first step can be formulated in terms of polynomials. The Chebyshev polynomials of the second kind, defined by $U_0(t) = 1$, $U_1(t) = 2t$ and $U_{n+1}(t) = 2tU_n(t) - U_{n-1}(t)$, satisfy

$$\int_{-2}^2 U_n(s/2) \frac{\sqrt{4 - s^2}}{2\pi} ds = 0, \quad n \geq 1.$$

Since they form a basis of the space of polynomials, the density of polynomials in the continuous functions on $[-2, 2]$ reduces Sato–Tate to

$$\frac{1}{\#P(x)} \sum_{p \in P(x)} U_n(s_p/2) \xrightarrow{x \rightarrow \infty} 0, \quad n \geq 1.$$

For $n = 1$ and $n = 2$, since $U_1(s/2) = s$ and $U_2(s/2) = s^2 - 1$, this amounts to recovering the zero mean and second moment equal to 1 of the semicircle:

$$\frac{1}{\#P(x)} \sum_{p \in P(x)} s_p \rightarrow 0, \quad \frac{1}{\#P(x)} \sum_{p \in P(x)} s_p^2 \rightarrow 1.$$

In $SU(2)$, $U_n(\text{Tr}(U)/2)$ is the character of the n th symmetric power of the natural representation. The condition above is Weyl's equidistribution criterion for conjugacy classes.

The arithmetic difficulty is to prove these limits for every n . Modularity relates the $a_p(E)$ to the Fourier coefficients of a modular form. The L -functions associated with its symmetric powers then encode the preceding sums; in the unitary normalization, their holomorphy and nonvanishing on the line $\text{Re}(s) = 1$ yield the desired limits. Potential automorphy theorems provide these analytic properties. For this deep part of the proof, see [Ca] for the case with multiplicative reduction at one place and [B++] for the general case.

REFERENCES

- [B] G. I. Barenblatt, *Scaling, Self-Similarity, and Intermediate Asymptotics*, CUP, 1996
- [B+] F. Barthe, O. Guédon, S. Mendelson and A. Naor, *A probabilistic approach to the geometry of the ℓ_p^n -ball*, Annals of Probab. **33** (2005)
- [B++] T. Barnet-Lamb, D. Geraghty, M. Harris and R. Taylor, *A Family of Calabi–Yau Varieties and Potential Automorphy II*, (2011)
- [Be] B. Beauzamy, *Archimedes' Modern Works*, Société de Calcul Mathématique, 2012.
- [Ca] H. Carayol, *La conjecture de Sato–Tate [d'après Clozel, Harris, Shepherd-Barron, Taylor]*, Séminaire Bourbaki, Astérisque **317** (2008)
- [Ch] D. Chafaï, *Archimedes, Barenblatt, and Sato-Tate*, *Libres pensées d'un mathématicien ordinaire* blog, October 16, 2026.
- [CK] Y. Chen and B. Klartag, *Digesting the proof of the sharp thin-shell inequality*, arXiv:2607.23307
- [CSW] D. Chafaï, E. B. Saff and R. S. Womersley, *On the solution of a Riesz equilibrium problem...*, J. of Math. Analysis and App. **515** (2022)
- [IMU] International Mathematical Union, <https://www.mathunion.org/imu-awards/fields-medal> (description of the medal)
- [K] B. Klartag, *A central limit theorem for convex sets*, Inventiones Mathematicae **168** (2007)
- [L] G. Letac, *From Archimedes to statistics: the area of the sphere*, lecture notes for the Rovinj summer school, 2004.
- [M] F. G. Mehler, *Ueber die Entwicklung einer Function von beliebig vielen Variablen nach Laplaceschen Functionen höherer Ordnung*, Journal für die reine und angewandte Mathematik **66** (1866)
- [PR] J. Pitman and N. Ross, *Archimedes, Gauss, and Stein*, Notices of the AMS **59** (2012)
- [R] W. Rudin, *A Generalization of a Theorem of Archimedes*, The American Mathematical Monthly **80** (1973)
- [S19] A. V. Sutherland, *Sato–Tate distributions*, in *Analytic Methods in Arithmetic Geometry*, Contemporary Mathematics **740** (2019)
- [V] J. L. Vázquez, *Smoothing and Decay Estimates for Nonlinear Diffusion Equations: Equations of Porous Medium Type*, OUP, 2006
- [Vi] Ministère de l'Éducation nationale, Odysseum, *Archimède. Sources iconographiques*, illustration no. 48.
- [W] Wikipedia, *On the Sphere and Cylinder*.

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