\begin{equation*}
\begin{aligned}
(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})\text{ is a filtered probability space, with complete and right continuous filtration.}

B = (B_t)_{t\geq 0} \text{ is a } d\text{-dimensional Brownian motion issued from the origin, } d \geq 1.
\end{aligned}
\end{equation*}

**Exercise 1** (Representation of a process). Take \(d = 1\) and \(x \in \mathbb{R}\).

1. Recall the computations and reasoning showing that the process \((Z_t)_{t\geq 0}\) defined by
   \[ Z_t = xe^{-t} + e^{-t} M_t \quad \text{where} \quad M_t = \sqrt{2} \int_0^t e^s dB_s \]
   is the unique solution of the stochastic differential equation \(Z_0 = x, \, dZ_t = \sqrt{2} dB_t - Z_t dt\).
2. Show that for all \(t \geq 0\), \(Z_t \overset{\text{law}}{=} xe^{-t} + e^{-t} B_{2t-1}\).
3. Can we have, for all \(t \geq 0\), \(Z_t = xe^{-t} + e^{-t} B_{2t-1}\)?
4. Show that the process \((M_t)_{t\geq 0}\) is a continuous local martingale with, for all \(t \geq 0\), \((M)_t = e^{2t} - 1\).
5. Deduce that there exists a Brownian motion \((W_t)_{t\geq 0}\) such that for all \(t \geq 0\), \(Z_t = xe^{-t} + e^{-t} W_{2t-1}\).

**Exercise 2** (Study of a special process). Let \(d = 1\), \(\alpha \geq 0\), \(x \geq 0\). Let \(X\) be a continuous semi-martingale taking values in \(\mathbb{R}_+\) and solving the stochastic differential equation:
\[ X_t = x + 2 \int_0^t \sqrt{X_s} dB_s + \alpha t, \quad t \geq 0. \]

Let \(f : [0, +\infty) \to [0, +\infty)\) be continuous and \(\varphi : [0, +\infty) \to (0, +\infty)\) be positive and \(\mathcal{C}^2\), solving the ordinary differential equation \(\varphi'' = 2f\varphi\) with boundary conditions \(\varphi(0) = 1\) and \(\varphi'(1) = 0\). Note that \(\varphi > 0\).

1. Could you give an explicit example of process \(X\) for special values of \(\alpha\)?
2. Show that \(\varphi\) decreases on the interval \([0, 1]\)
3. Show that \(u = \varphi'/(2\varphi)\) solves the differential equation \(u' + 2u^2 = f\)
4. Show that for all \(t \geq 0\),
   \[ u(t)X_t - \int_0^t f(s)X_s ds = u(0)x + \int_0^t u(s) dX_s - 2 \int_0^t u(s)^2 X_s ds. \]
5. For all \(t \geq 0\), let us define \(Y_t = u(t)X_t - \int_0^t f(s)X_s ds\). Show that
   \[ \varphi(t)^{-\frac{\alpha}{2}} e^{Y_t} = e^{N_t - \frac{1}{2}(N)_t} \quad \text{where} \quad N_t = u(0)x + 2 \int_0^t u(s) \sqrt{X_s} dB_s \]
6. Show that
   \[ \mathbb{E}\exp\left(-\int_0^1 f(s)X_s ds\right) = \varphi(1)^{\frac{\alpha}{2}} e^{\frac{\alpha}{2}\varphi'(0)} \]
7. From now on, let \(\lambda > 0\). Prove that
   \[ \mathbb{E}\exp\left(-\lambda \int_0^1 X_s ds\right) = (\cosh(\sqrt{2\lambda}))^{-\frac{\alpha}{2}} e^{-\frac{\alpha}{2} \sqrt{2\lambda} \tanh \sqrt{2\lambda}} \]
8. Prove that for all \(\lambda > 0\) and \(y \in \mathbb{R}\),
   \[ \mathbb{E}\exp\left(-\lambda \int_0^1 (y + B_s)^2 ds\right) = (\cosh(\sqrt{2\lambda}))^{-\frac{1}{2}} e^{-\frac{\alpha}{2} \sqrt{2\lambda} \tanh \sqrt{2\lambda}} \]
**Exercise 3** (Strict local martingales). We take $d = 3$, $X = x + B$, $0 < r < |x| < R < \infty$, and, for all $a \geq 0$,

$$T_a = \inf\{t \geq 0 : |X_t| = a\}.$$

1. Show that if $M = (M_t)_{t \geq 0}$ is a continuous local martingale with for all $t \geq 0$, $|M_t| \leq U$ where $U \in L^1$, then $M$ is a martingale. Does it remain true if the domination condition is replaced by “$M$ is u.i.”?

2. Show that if $Z = (Z_t)_{t \geq 0}$ is $d$-dimensional, adapted, taking values in an open set $D \subset \mathbb{R}^d$, such that its components are continuous local martingales, and for all $1 \leq j, k \leq d$, $\langle Z^j, Z^k \rangle = V_{j=k}$ for a finite variation process $V$, then, for all harmonic $u : D \rightarrow \mathbb{R}$, the process $u(Z)$ is a local martingale.

3. Show that $|\cdot|^{-1}$ is harmonic on $\mathbb{R}^3 \setminus \{0\}$.

4. Show that $T_R < \infty$ almost surely and

$$P(T_R < T_R) = \frac{R^{-1} - |x|^{-1}}{R^{-1} - r^{-1}}.$$

5. Deduce from the previous formula that a.s. for all $t \geq 0$, $X_t \neq 0$.

6. Show that a.s. $\lim_{t \to \infty} |B_t| = +\infty$. Hint: show that $|X|^{-1}$ is a non-negative super-martingale.

7. Show that $|X|^{-1}$ is bounded in $L^2$. Hint: density of $B_t$ in spherical coordinates.

8. Show that $|X|^{-1}$ is a continuous local martingale, but is not a martingale.

**Exercise 4** (Strict local martingales and stochastic integrals).

1. Give an example of an Itô stochastic integral which is a local martingale but not a martingale, without using the previous exercise.

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