Exam 2020/2021

October 28, 2020, from 13:45 to 16:45

Documents allowed, Internet not allowed
Do what you can, and do not worry

\((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})\) is a filtered probability space, with complete and right continuous filtration. 
\(B = (B_t)_{t \geq 0}\) is a \(d\)-dimensional Brownian motion issued from the origin, \(d \geq 1\).

**Exercise 1** (Representation of a process). Take \(d = 1\) and \(x \in \mathbb{R}\).
1. Recall the computations and reasoning showing that the process \((Z_t)_{t \geq 0}\) defined by
   \[ Z_t = x e^{-t} + e^{-t} M_t \quad \text{where} \quad M_t = \sqrt{2} \int_0^t e^s dB_s \]
   is the unique solution of the stochastic differential equation \(Z_0 = x, dZ_t = \sqrt{2} dB_t - Z_t dt\).
2. Show that for all \(t \geq 0\), \(Z_t \overset{\text{law}}{=} x e^{-t} + e^{-t} \mathcal{B}_{e^t - 1}\).
3. Can we have, for all \(t \geq 0\), \(Z_t = x e^{-t} + e^{-t} \mathcal{B}_{e^t - 1}\)?
4. Show that the process \((M_t)_{t \geq 0}\) is a continuous local martingale with, for all \(t \geq 0\), \((M)_t = e^{2t} - 1\).
5. Deduce that there exists a Brownian motion \((W_t)_{t \geq 0}\) such that for all \(t \geq 0\), \(Z_t = x e^{-t} + e^{-t} W_{e^t - 1}\).

**Exercise 2** (Study of a special process). Let \(d = 1\), \(\alpha \geq 0\), \(x \geq 0\). Let \(X\) be a continuous semi-martingale taking values in \(\mathbb{R}_+\) and solving the stochastic differential equation:
\[
X_t = x + 2 \int_0^t \sqrt{X_s} dB_s + \alpha t, \quad t \geq 0.
\]
Let \(f : [0, +\infty) \to [0, +\infty)\) be continuous and \(\varphi : [0, +\infty) \to (0, +\infty)\) be positive and \(\mathcal{C}^2\), solving the ordinary differential equation \(\varphi'' = 2 f \varphi\) with boundary conditions \(\varphi(0) = 1\) and \(\varphi'(1) = 0\). Note that \(\varphi > 0\).
1. Could you give an explicit example of process \(X\) for special values of \(\alpha\)?
2. Show that \(\varphi\) decreases on the interval \([0, 1]\)
3. Show that \(u = \varphi'/(2 \varphi)\) solves the differential equation \(u' + 2 u^2 = f\)
4. Show that for all \(t \geq 0\),
   \[ u(t) X_t - \int_0^t f(s) X_s \, ds = u(0) x + \int_0^t u(s) \, ds - 2 \int_0^t u(s)^2 X_s \, ds. \]
5. For all \(t \geq 0\), let us define \(Y_t = u(t) X_t - \int_0^t f(s) X_s \, ds\). Show that
   \[ \varphi(t)^{-\frac{\alpha}{2}} \varphi(t)^{\frac{\alpha}{2}} = e^{N - \frac{1}{2} \langle N \rangle_t} \quad \text{where} \quad N_t = u(0) x + 2 \int_0^t u(s) \sqrt{X_s} \, dB_s \]
6. Show that
   \[ \mathbb{E} \exp \left( - \int_0^1 f(s) X_s \, ds \right) = \varphi(1)^{\frac{\alpha}{2}} e^{\frac{\alpha}{2} \varphi(0)} \]
7. From now on, let \(\lambda > 0\). Prove that
   \[ \mathbb{E} \exp \left( - \lambda \int_0^1 X_s \, ds \right) = (\cosh(\sqrt{2} \lambda))^{-\frac{\alpha}{2}} e^{-\frac{\alpha}{2} \sqrt{2} \lambda \tanh \sqrt{2} \lambda} \]
8. Prove that for all \(\lambda > 0\) and \(y \in \mathbb{R}\),
   \[ \mathbb{E} \exp \left( - \lambda \int_0^1 (y + B_s)^2 \, ds \right) = (\cosh(\sqrt{2} \lambda))^{-\frac{1}{2}} e^{-\frac{\lambda}{2} \sqrt{2} \lambda \tanh \sqrt{2} \lambda} \]
**Exercise 3** (Strict local martingales). We take \( d = 3, \ X = x + B, \ 0 < r < |x| < R < \infty, \) and, for all \( a \geq 0, \)

\[
T_a = \inf\{ t \geq 0 : |X_t| = a \}.
\]

1. Show that if \( M = (M_t)_{t \geq 0} \) is a continuous local martingale with for all \( t \geq 0, \ |M_t| \leq U \) where \( U \in L^1 \),
then \( M \) is a martingale. Does it remain true if the domination condition is replaced by “\( M \) is u.i.”?

2. Show that if \( Z = (Z_t)_{t \geq 0} \) is \( d \)-dimensional, adapted, taking values in an open set \( D \subset \mathbb{R}^d \), such that
its components are continuous local martingales, and for all \( 1 \leq j,k \leq d, \ \langle Z^j, Z^k \rangle = V_{1,j=k} \) for a
finite variation process \( V \), then, for all harmonic \( u : D \to \mathbb{R} \), the process \( u(Z) \) is a local martingale.

3. Show that \(|\bullet|^{-1}\) is harmonic on \( \mathbb{R}^3 \setminus \{0\} \).

4. Show that \( T_R < \infty \) almost surely and

\[
\mathbb{P}(T_R < T_r) = \frac{R^{-1} - |x|^{-1}}{R^{-1} - r^{-1}}.
\]

5. Deduce from the previous formula that a.s. for all \( t \geq 0, \ X_t \neq 0. \)

6. Show that a.s. \( \lim_{t \to \infty} |B_t| = +\infty. \) Hint: show that \( |X|^{-1} \) is a non-negative super-martingale.

7. Show that \(|X|^{-1}\) is bounded in \( L^2. \) Hint: density of \( B_t \) in spherical coordinates.

8. Show that \(|X|^{-1}\) is a continuous local martingale, but is not a martingale.

**Exercise 4** (Strict local martingales and stochastic integrals).

1. Give an example of an Itô stochastic integral which is a local martingale but not a martingale,
without using the previous exercise.