(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}) is a filtered probability space, with complete and right continuous filtration. 

\( B = (B_t)_{t \geq 0} \) is a d-dimensional Brownian motion issued from the origin, \( d \geq 1 \).

**Exercise 1** (Representation of a process). Take \( d = 1 \) and \( x \in \mathbb{R} \).
1. Recall the computations and reasoning showing that the process \( (Z_t)_{t \geq 0} \) defined by
   \[
   Z_t = xe^{-t} + e^{-t}M_t \quad \text{where} \quad M_t = \sqrt{2} \int_0^t e^s dB_s
   \]
   is the unique solution of the stochastic differential equation \( Z_0 = x, \, dZ_t = \sqrt{2} dB_t - Z_t \, dt \).
2. Show that for all \( t \geq 0 \), \( Z_t \overset{\text{law}}{=} xe^{-t} + e^{-t} B_{2t} \).
3. Can we have, for all \( t \geq 0 \), \( Z_t = xe^{-t} + e^{-t} B_{2t-1} \)?
4. Show that the process \( (M_t)_{t \geq 0} \) is a continuous local martingale with, for all \( t \geq 0 \), \( \langle M \rangle_t = e^{2t} - 1 \).
5. Deduce that there exists a Brownian motion \( (W_t)_{t \geq 0} \) such that for all \( t \geq 0 \), \( Z_t = xe^{-t} + e^{-t} W_{2t-1} \).

**Exercise 2** (Study of a special process). Let \( d = 1, \, \alpha \geq 0, \, x \geq 0 \). Let \( X \) be a continuous semi-martingale taking values in \( \mathbb{R}_+ \) and solving the stochastic differential equation:
\[
X_t = x + 2 \int_0^t \sqrt{X_s} dB_s + \alpha t, \quad t \geq 0.
\]

Let \( f : [0, +\infty) \to [0, +\infty) \) be continuous and \( \varphi : [0, +\infty) \to (0, +\infty) \) be positive and \( \mathcal{C}^2 \), solving the ordinary differential equation \( \varphi'' = 2f(\varphi) \) with boundary conditions \( \varphi(0) = 1 \) and \( \varphi'(1) = 0 \). Note that \( \varphi > 0 \).

1. Could you give an explicit example of process \( X \) for special values of \( \alpha \)?
2. Show that \( \varphi \) decreases on the interval \([0, 1]\)
3. Show that \( u = \varphi'(1)/(2\varphi) \) solves the differential equation \( u' + 2u^2 = f \)
4. Show that for all \( t \geq 0 \),
   \[
u(t)X_t - \int_0^t f(s)X_s ds = u(0)x + \int_0^t u(s)X_s ds - 2 \int_0^t u(s)^2 X_s ds.
   \]
5. For all \( t \geq 0 \), let us define \( Y_t = u(t)X_t - \int_0^t f(s)X_s ds \). Show that
   \[
   \varphi(t)^{-\frac{\alpha}{2}} e^{Y_t} = e^{N_t - \frac{1}{2} \langle N \rangle_t} \quad \text{where} \quad N_t = u(0)x + 2 \int_0^t u(s) \sqrt{X_s} dB_s
   \]
6. Show that
   \[
   \mathbb{E} \exp \left( -\int_0^t f(s)X_s ds \right) = \varphi(1)^{-\frac{\alpha}{2}} e^{\frac{\alpha}{2} \varphi(0)}
   \]
7. From now on, let \( \lambda > 0 \). Prove that
   \[
   \mathbb{E} \exp \left( -\lambda \int_0^t X_s ds \right) = (\cosh(\sqrt{2\lambda}))^{-\frac{\alpha}{2}} e^{-\frac{\alpha}{2} \sqrt{2\lambda} \tanh \sqrt{2\lambda}}
   \]
8. Prove that for all \( \lambda > 0 \) and \( y \in \mathbb{R} \),
   \[
   \mathbb{E} \exp \left( -\lambda \int_0^t (y + B_s)^2 ds \right) = (\cosh(\sqrt{2\lambda}))^{-\frac{1}{2}} e^{-\frac{\alpha}{2} \sqrt{2\lambda} \tanh \sqrt{2\lambda}}
   \]
Exercise 3 (Strict local martingales). We take $d = 3$, $X = x + B$, $0 < r < |x| < R < \infty$, and, for all $a \geq 0$, 

$$T_a = \inf\{t \geq 0 : |X_t| = a\}.$$

1. Show that if $M = (M_t)_{t \geq 0}$ is a continuous local martingale with for all $t \geq 0$, $|M_t| \leq U$ where $U \in L^1$, then $M$ is a martingale. Does it remain true if the domination condition is replaced by “$M$ is u.i.”?

2. Show that if $Z = (Z_t)_{t \geq 0}$ is $d$-dimensional, adapted, taking values in an open set $D \subset \mathbb{R}^d$, such that its components are continuous local martingales, and for all $1 \leq j, k \leq d$, $\langle Z^j, Z^k \rangle = V_{1 = k}$ for a finite variation process $V$, then, for all harmonic $u : D \to \mathbb{R}$, the process $u(Z)$ is a local martingale.

3. Show that $|\cdot|^{-1}$ is harmonic on $\mathbb{R}^3 \setminus \{0\}$.

4. Show that $T_R < \infty$ almost surely and

$$\mathbb{P}(T_r < T_R) = \frac{R^{-1} - |x|^{-1}}{R^{-1} - r^{-1}}.$$  

5. Deduce from the previous formula that a.s. for all $t \geq 0$, $X_t \neq 0$.

6. Show that a.s. $\lim_{t \to \infty} |B_t| = +\infty$. Hint: show that $|X|^{-1}$ is a non-negative super-martingale.

7. Show that $|X|^{-1}$ is bounded in $L^2$. Hint: density of $B_t$ in spherical coordinates.

8. Show that $|X|^{-1}$ is a continuous local martingale, but is not a martingale.

Exercise 4 (Strict local martingales and stochastic integrals).

1. Give an example of an Itô stochastic integral which is a local martingale but not a martingale, without using the previous exercise.