Exam 2020/2021

October 28, 2020, from 13:45 to 16:45
Documents allowed, Internet not allowed
Do what you can, and do not worry

$(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\geq 0},\mathbb{P})$ is a filtered probability space, with complete and right continuous filtration. $B = (B_t)_{t\geq 0}$ is a $d$-dimensional Brownian motion issued from the origin, $d \geq 1$.

Exercise 1 (Representation of a process). Take $d = 1$ and $x \in \mathbb{R}$.
1. Recall the computations and reasoning showing that the process $(Z_t)_{t\geq 0}$ defined by

$$Z_t = xe^{-t} + e^{-t}M_t \quad \text{where} \quad M_t = \sqrt{2} \int_0^t e^s dB_s$$

is the unique solution of the stochastic differential equation $Z_0 = x, \, dZ_t = \sqrt{2} dB_t - Z_t \, dt$.
2. Show that for all $t \geq 0$, $Z_t \overset{\text{law}}{=} xe^{-t} + e^{-t} B_{\omega_t - 1}$.
3. Can we have, for all $t \geq 0$, $Z_t = xe^{-t} + e^{-t} B_{\omega_t - 1}$?
4. Show that the process $(M_t)_{t\geq 0}$ is a continuous local martingale with, for all $t \geq 0$, $(M)_t = e^{\omega_t} - 1$.
5. Deduce that there exists a Brownian motion $(W_t)_{t\geq 0}$ such that for all $t \geq 0$, $Z_t = xe^{-t} + e^{-t} W_{\omega_t - 1}$.

Exercise 2 (Study of a special process). Let $d = 1$, $\alpha \geq 0$, $x \geq 0$. Let $X$ be a continuous semi-martingale taking values in $\mathbb{R}_+$ and solving the stochastic differential equation:

$$X_t = x + 2 \int_0^t \sqrt{X_s} \, dB_s + \alpha t, \quad t \geq 0.$$

Let $f : [0, +\infty) \to [0, +\infty)$ be continuous and $\varphi : [0, +\infty) \to (0, +\infty)$ be positive and $\mathcal{C}^2$, solving the ordinary differential equation $\varphi'' = 2 f \varphi$ with boundary conditions $\varphi(0) = 1$ and $\varphi'(1) = 0$. Note that $\varphi > 0$.

1. Could you give an explicit example of process $X$ for special values of $\alpha$?
2. Show that $\varphi$ decreases on the interval $[0, 1]$.
3. Show that $u = \varphi'/2\varphi$ solves the differential equation $u' + 2u^2 = f$.
4. Show that for all $t \geq 0$,

$$u(t) X_t - \int_0^t f(s) X_s \, ds = u(0) x + \int_0^t u(s) \, dX_s - 2 \int_0^t [u(s)]^2 \, X_s \, ds.$$

5. For all $t \geq 0$, let us define $Y_t = u(t) X_t - \int_0^t f(s) X_s \, ds$. Show that

$$\varphi(t)^{-\frac{\alpha}{2}} e^{Y_t} = e^{N_t - \frac{1}{2} \langle N \rangle_t} \quad \text{where} \quad N_t = u(0) x + 2 \int_0^t u(s) \sqrt{X_s} \, dB_s$$

6. Show that

$$\mathbb{E} \exp \left( - \int_0^1 f(s) X_s \, ds \right) = \varphi(1)^{\frac{\alpha}{2}} e^{\frac{\alpha}{2} \varphi'(0)}$$

7. From now on, let $\lambda > 0$. Prove that

$$\mathbb{E} \exp \left( - \lambda \int_0^1 X_s \, ds \right) = (\cosh(\sqrt{2\lambda}))^{-\frac{\alpha}{2}} e^{-\frac{\alpha}{2} \sqrt{2\lambda} \tanh \sqrt{2\lambda}}$$

8. Prove that for all $\lambda > 0$ and $y \in \mathbb{R}$,

$$\mathbb{E} \exp \left( - \lambda \int_0^1 (y + R_s)^2 \, ds \right) = (\cosh(\sqrt{2\lambda}))^{-\frac{1}{2}} e^{-\frac{1}{2} \sqrt{2\lambda} \tanh \sqrt{2\lambda}}$$

1/2
Exercise 3 (Strict local martingales). We take $d = 3$, $X = x + B$, $0 < r < |x| < R < \infty$, and, for all $a \geq 0$,

$$T_a = \inf\{t \geq 0 : |X_t| = a\}.$$

1. Show that if $M = (M_t)_{t \geq 0}$ is a continuous local martingale with for all $t \geq 0$, $|M_t| \leq U$ where $U \in L^1$, then $M$ is a martingale. Does it remain true if the domination condition is replaced by “$M$ is u.i.”?

2. Show that if $Z = (Z_t)_{t \geq 0}$ is $d$-dimensional, adapted, taking values in an open set $D \subset \mathbb{R}^d$, such that its components are continuous local martingales, and for all $1 \leq j, k \leq d$, $\langle Z^j, Z^k \rangle = V_{1_{j=k}}$ for a finite variation process $V$, then, for all harmonic $u : D \to \mathbb{R}$, the process $u(Z)$ is a local martingale.

3. Show that $|\cdot|^{-1}$ is harmonic on $\mathbb{R}^3 \setminus \{0\}$.

4. Show that $T_R < \infty$ almost surely and

$$\mathbb{P}(T_r < T_R) = \frac{R^{-1} - |x|^{-1}}{R^{-1} - r^{-1}}.$$

5. Deduce from the previous formula that a.s. for all $t \geq 0$, $X_t \neq 0$.

6. Show that a.s. $\lim_{t \to \infty} |B_t| = +\infty$. Hint: show that $|X|^{-1}$ is a non-negative super-martingale.

7. Show that $|X|^{-1}$ is bounded in $L^2$. Hint: density of $B_t$ in spherical coordinates.

8. Show that $|X|^{-1}$ is a continuous local martingale, but is not a martingale.

Exercise 4 (Strict local martingales and stochastic integrals).

1. Give an example of an Itô stochastic integral which is a local martingale but not a martingale, without using the previous exercise.

– oOo –