\( (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}) \) is a filtered probability space, with complete and right continuous filtration. 

\( B = (B_t)_{t \geq 0} \) is a \( d \)-dimensional Brownian motion issued from the origin, \( d \geq 1 \).

**Exercise 1** (Representation of a process). Let \( d = 1 \) and \( x \in \mathbb{R} \).

1. Recall the computations and reasoning showing that the process \((Z_t)_{t \geq 0}\) defined by
   \[
   Z_t = xe^{-t} + e^{-t} M_t, \quad \text{where} \quad M_t = \sqrt{2} \int_0^t e^s \, dB_s
   \]

   is the unique solution of the stochastic differential equation \( Z_0 = x, dZ_t = \sqrt{2} dB_t - Z_t \, dt \).

2. Show that for all \( t \geq 0 \), \( Z_t \) is a continuous local martingale with respect to the filtration \((\mathcal{F}_t)_{t \geq 0}\).

3. Can we have, for all \( t \geq 0 \), \( Z_t = e^{-t} B_{e^{t-1}} \)?

4. Show that the process \((M_t)_{t \geq 0}\) is a continuous local martingale with respect to the filtration \((\mathcal{F}_t)_{t \geq 0}\) such that for all \( t \geq 0 \), \( Z_t = e^{-t} + e^{-t} W_{e^{t-1}} \).

5. Deduce that there exists a Brownian motion \((W_t)_{t \geq 0}\) such that for all \( t \geq 0 \), \( Z_t = e^{-t} + e^{-t} W_{e^{t-1}} \).

**Elements of solution for Exercise 1.** Ornstein–Uhlenbeck and Dubins–Schwarz!

1. The Itô formula \( X_t Y_t = X_0 Y_0 + \int_0^t (X_s dY_s + Y_s dX_s) \) gives, with \((X_t, Y_t) = (e^{-t}, M_t)\),
   \[
   e^{-t} M_t = \sqrt{2} \int_0^t e^{-s} \, dB_s - \int_0^t e^{-s} M_s \, ds = \sqrt{2} B_t - \int_0^t e^{-s} M_s \, ds
   \]
   which gives
   \[
   xe^{-t} + e^{-t} M_t = x + \sqrt{2} B_t - \int_0^t (xe^{-s} + e^{-s} M_s) \, ds.
   \]

   On the other hand, to establish the uniqueness, let us suppose that \((X_t)_{t \geq 0}\) is a continuous semi-martingale solution of the SDE. Then by combining the Itô formula above for the function \( f(u, v) = uv \) and the semi-martingale \((e^t, X_t)\), and the SDE satisfied by \( X \), we obtain
   \[
   e^t X_t - x = \int_0^t e^s \, dX_s + \int_0^t e^s X_s \, ds = \sqrt{2} \int_0^t e^s \, dB_s.
   \]

2. Since \( M_t \) is a Wiener integral, it is Gaussian with zero mean and variance \( 2 \int_0^t e^{2s} \, ds = e^{2t} - 1 \), which gives \( Z_t \sim \mathcal{N}(xe^{-t}, 1 - e^{-2t}) \), and it turns out that this is also the law of \( xe^{-t} + e^{-t} B_{e^{t-1}} \).

3. No because the process on the right hand side would not be adapted. Indeed the random variable \( B_{e^{t-1}} \) is \( \mathcal{F}_{e^{t-1}} \)-measurable instead of being \( \mathcal{F}_t \)-measurable.

4. The process \( M \) is a Wiener (–Itô) integral, in particular it is a Gaussian square integrable martingale, and in particular a local martingale. Moreover \( (M)_t = E(M_t^2) = 2 \int_0^t e^{2s} \, ds = e^{2t} - 1 \).

5. The process \( M \) is a continuous local martingale with respect to the filtration \((\mathcal{F}_t)_{t \geq 0}\) with \( M_0 = 0 \) and \( (M)_\infty = \infty \). Therefore the Dubins–Schwarz theorem states that \((W_t)_{t \geq 0} = (M_{T_t})_{t \geq 0}\) where \( T_t = \inf\{s \geq 0 : (M)_s > t\} \) is a Brownian motion for the filtration \((\mathcal{F}_t)_{t \geq 0}\), and \((W_{(M)_t})_{t \geq 0} = (M_t)_{t \geq 0}\).

**Exercise 2** (Study of a special process). Let \( d = 1, \alpha \geq 0, x \geq 0 \). Let \( X \) be a continuous semi-martingale taking values in \( \mathbb{R}_+ \) and solving the stochastic differential equation:

\[
X_t = x + 2 \int_0^t \sqrt{X_s} \, dB_s + \alpha t, \quad t \geq 0.
\]

Let \( f : [0, +\infty) \to [0, +\infty) \) be continuous and \( \varphi : [0, +\infty) \to (0, +\infty) \) be positive and \( C^2 \), solving the ordinary differential equation \( \varphi'' = 2f(\varphi) \) with boundary conditions \( \varphi(0) = 1 \) and \( \varphi'(1) = 0 \). Note that \( \varphi > 0 \).
1. Could you give an explicit example of process $X$ for special values of $\alpha$?
2. Show that $\varphi$ decreases on the interval $[0,1]$.
3. Show that $u = \varphi'/2\varphi$ solves the differential equation $u' + 2u^2 = f$.
4. Show that for all $t \geq 0$,
   \[ u(t)X_t - \int_0^t f(s)X_sds = u(0)x + \int_0^t u(s)dX_s - 2\int_0^t u(s)^2X_sds. \]
5. For all $t \geq 0$, let us define $Y_t = u(t)X_t - \int_0^t f(s)X_sds$. Show that
   \[ \varphi(t)^{-\frac{1}{2}}e^{Y_t} = e^{N_t - \frac{1}{2}(N)_t}, \]
   where $N_t = u(0)x + 2\int_0^t u(s)\sqrt{X_s}dB_s$.
6. Show that
   \[ \mathbb{E}\exp\left( -\int_0^1 f(s)X_sds \right) = \varphi(1)^{-\frac{1}{2}}e^{\frac{\varphi'}{2}(0)} \]
7. From now on, let $\lambda > 0$. Prove that
   \[ \mathbb{E}\exp\left( -\lambda \int_0^1 X_sds \right) = (\cosh(\sqrt{2\lambda}))^{-\frac{1}{2}}e^{-\frac{\lambda}{2}\sqrt{\varphi(t)^{-1}}}, \]
8. Prove that for all $\lambda > 0$ and $y \in \mathbb{R}$,
   \[ \mathbb{E}\exp\left( -\lambda \int_0^1 (y + B_s)^2ds \right) = (\cosh(\sqrt{2\lambda}))^{-\frac{1}{2}}e^{-\frac{\lambda}{2}\sqrt{\varphi(t)^{-1}}} \]

**Elements of solution for Exercise 2.** This is on squared Bessel processes: [1, Exercise 5.31 p. 145–146].

1. We know from the course (Itô+Lévy) that if $\alpha = n \in \{1,2,\ldots\}$ then $X$ has the law of $|x + W|^2$ where $W$ is a $n$-dimensional Brownian motion issued from the origin (squared Bessel process).
2. We have $\varphi'(1) = 0$ and $\varphi'' \geq 0$ hence $\varphi' \leq 0$ on $[0,1]$.
3. We have $u' = \frac{\varphi''\varphi - \varphi'^2}{2\varphi^2}$ thus
   \[ u' + 2u^2 = \frac{\varphi''\varphi - \varphi'^2}{2\varphi^2} + \frac{\varphi'^2}{2\varphi} = \frac{\varphi''}{2\varphi} = f. \]
4. The Itô formula for $F(x_1,x_2) = x_1x_2$ and $(u(t),X_t)$ gives
   \[ u(t)X_t - u(0)x = \int_0^t X_su'(s)ds + \int_0^t u(s)dX_s \]
   \[ = \int_0^t X_s(f(s) - 2u^2(s))ds + \int_0^t u(s)dX_s \]
   and it remains to use the result of the previous question.
5. We have, using the SDE solved by $X$ for the second step, and in the third step the definition of $Y$
   and the previous question,
   \[ N_t - \frac{1}{2}(N)_t = u(0)x + 2\int_0^t u(s)\sqrt{X_s}dB_s - 2\int_0^t u(s)^2X_sds \]
   \[ = u(0)x + \int_0^t u(s)(dX_s - \alpha ds) - 2\int_0^t u(s)^2X_sds \]
   \[ = Y_t - \alpha \int_0^t u(s)ds. \]
Since
   \[ \int_0^t u(s)ds = \int_0^t \frac{\varphi'(s)}{2\varphi(s)}ds = \frac{\log\varphi(1) - \log\varphi(0)}{2} = \frac{\log\varphi(1)}{2}, \]
we obtain
   \[ e^{N_t - \frac{1}{2}(N)_t} = e^{Y_t\varphi(1)^{-\alpha t}}. \]
6. The process $e^{N_{t-1/2}(N)}$ is a Doléans-Dade exponential, hence a continuous local martingale. In order to show that it is a martingale for $t \in [0, 1]$, it suffices to show that it is dominated by an integrable random variable. From the previous computations, for all $t \in [0, 1]$,

$$e^{N_{t-1/2}(N)} = e^{Y_t - a \int_0^t u(s) ds}.$$ 

Now, $Y_t = u(t) X_t - f_t^0 f(s) X_s ds \leq 0$ since $u \leq 0$ (recall that $\varphi' \leq 0$), while $X, f \geq 0$. Hence $(e^{N_{t-1/2}(N)})_{t \in [0, 1]}$ is a martingale.

Next, $u(1) = \varphi'(1)/(2\varphi(1)) = 0$, we get, from the previous question with $t = 1$,

$$\varphi(1)^{-\lambda} e^ {-\lambda f(t) X_t ds} = E(e^{N_{t-1/2}(N_1)})$$

$$= E(e^{N_{0-1/2}(N_0)})$$

$$= e^{\varphi(0) x}$$

$$= e^{\frac{\varphi'(0)}{2\varphi(0)}}$$

$$= e^{\frac{\varphi'(0)}{2}}.$$

7. We take $f$ constant and equal to $\lambda > 0$. The differential equation solved by $\varphi$ writes $\varphi'' = 2\lambda \varphi$ with $\varphi(0) = 1$ and $\varphi'(1) = 0$. The associated equation has two roots $\pm \sqrt{2\lambda}$ hence $\varphi(x) = a e^{\sqrt{2\lambda} x} + b e^{-\sqrt{2\lambda} x}$. The boundary conditions give $\alpha + \beta = 1$ and $ae^{2\sqrt{2\lambda}} = \beta$, hence

$$\alpha = \frac{1}{1 + e^{2\sqrt{2\lambda}}} = \frac{e^{-\sqrt{2\lambda}}}{2 \cosh(\sqrt{2\lambda})}$$

and

$$\beta = \frac{e^{2\sqrt{2\lambda}}}{1 + e^{2\sqrt{2\lambda}}} = \frac{e^{\sqrt{2\lambda}}}{2 \cosh(\sqrt{2\lambda})}.$$

This gives

$$\varphi(1) = \frac{1}{\cosh(\sqrt{2\lambda})}$$

and

$$\varphi'(0) = \sqrt{2\lambda} (\alpha - \beta) = -\sqrt{2\lambda} \tanh(\sqrt{2\lambda}).$$

8. If we take $\alpha = 1$ then $X$ has the law of the squared Bessel process $(x + B)^2$.

**Exercise 3** (Strict local martingales). We take $d = 3$, $X = x + B$, $0 < r < |x| < R < \infty$, and, for all $a \geq 0$,

$$T_a = \inf\{t \geq 0 : |X_t| = a\}.$$

1. Show that if $M = (M_t)_{t \geq 0}$ is a continuous local martingale with for all $t \geq 0, |M_t| \leq U$ where $U \in L^1$, then $M$ is a martingale. Does it remain true if the domination condition is replaced by “$M$ is u.i.”?

2. Show that if $Z = (Z_t)_{t \geq 0}$ is $d$-dimensional, adapted, taking values in an open set $D \subset \mathbb{R}^d$, such that its components are continuous local martingales, and for all $1 \leq j, k \leq d$, $\langle Z^j, Z^k \rangle = \mathbb{V}_{j=k}$ for a finite variation process $V$, then, for all harmonic $u : D \to \mathbb{R}$, the process $u(Z)$ is a local martingale.

3. Show that $|\cdot|^{-1}$ is harmonic on $\mathbb{R}^3 \setminus \{0\}$.

4. Show that $T_R < \infty$ almost surely and

$$\mathbb{P}(T_R < T_R) = \frac{R^{-1} - |x|^{-1}}{R^{-1} - r^{-1}}.$$

5. Deduce from the previous formula that a.s. for all $t \geq 0$, $X_t \neq 0$.

6. Show that a.s. $\lim_{t \to \infty} |B_t| = +\infty$. Hint: show that $|X|^{-1}$ is a non-negative super-martingale.

7. Show that $|X|^{-1}$ is bounded in $L^2$. Hint: density of $B_t$ in spherical coordinates.

8. Show that $|X|^{-1}$ is a continuous local martingale, but is not a martingale.

**Elements of solution for Exercise 3.** The goal is to construct a local martingale which is u.i. but which is not a martingale. The example chosen here is a non-centered Bessel process. This example of a strict local martingale is quite classical, and corresponds for instance to [1, Exercise 5.33(8) page 148].
1. For all \( t \geq 0 \), the measurability and integrability of \( M_t \) comes from the adaptation of \( M \) and the domination assumption. Let \((T_n)_{n \geq 0}\) be a localizing sequence for \( M \). For all \( n \geq 0 \), \( \langle M^{T_n} \rangle \) is a martingale: for all \( 0 \leq s \leq t \), \( \mathbb{E}(\langle M^{T_n} \rangle_s) = M^{T_n}_s \). Now since \( \lim_{n \to \infty} T_n = +\infty \) a.s. and since \( M \) is continuous, if follows that a.s. for all \( t \geq 0 \), \( \lim_{n \to \infty} M^{T_n} = M_t \). It remains to use dominated convergence to get that \( \lim_{n \to \infty} \mathbb{E}(\langle M^{T_n} \rangle_s) = \mathbb{E}(M_s) \). Hence \( M \) is a martingale. Finally, if \( M \) is u.i. instead of being dominated, then the argument is no longer valid since it does not give a way to handle \( M^{T_n} \). The goal of the exercise is precisely the construction of an u.i. continuous local martingale which is not a martingale! Note: domination implies u.i. but the converse is wrong.

2. The Itô formula is licit on an open domain \( D \) for a process taking values in \( D \). It gives, for all \( t \geq 0 \),

\[
u(X_t) = \nu(X_0) + \int_0^t \nabla \nu(X_s) dX_s + \frac{1}{2} \int_0^t \Delta \nu(X_s) dV_s.
\]

The last integral vanishes since \( \Delta \nu = 0 \) and \( X \) takes values on \( D \), thus \( \nu(X) \) is a local martingale.

3. For all \( y \in D \), denoting \( u = |\bullet|^{-1} \), we obtain \( \Delta \nu(y) = 0 \) from

\[
\partial_i u(y) = \frac{(2-d) y_i}{|y|^d}, \quad \text{and} \quad \partial_{ij}^2 u(y) = \frac{(d(d-2) y_i^2 y_j^2 - d y_i y_j^2 y_i y_j - 2 d y_j y_i y_i y_j)}{|y|^{d+2}}.
\]

4. Set \( Z = X_T \). Then \( \langle Z^1, Z^k \rangle_t = \langle B^1, B^k \rangle_{t \wedge T} = \langle B^1, B^k \rangle_{T} 1_{j=k} \). Hence, by a previous question with the processes \( Z \) and \( V_t = t \wedge T_r \), the harmonic function \( u = |\bullet|^{-1} \), and \( D = \{ x \in \mathbb{R}^3 : x \neq 0 \} \), we get that

\[
u(Z) = \nu(X_{T^1})
\]
is a local martingale, and since it is bounded by \( r^{-1} \), it is a bounded martingale.

Next, since a 1-dimensional BM escapes almost surely from every finite interval, the first component of our 3-dimensional Brownian motion \( x + B \) escapes almost surely from \([-R, R] \), and thus almost surely \( T_R < \infty \). In particular almost surely \( T_r < T_R \) or \( T_r > T_R \) and we cannot have \( T_r = T_R = \infty \). We have thus the immediate equation

\[1 = \mathbb{P}(T_r < T_R) + \mathbb{P}(T_r > T_R).
\]

By the Doob stopping theorem for the bounded martingale \( \nu(Z) \) and the finite stopping time \( T_R \),

\[
|x|^{-1} = \mathbb{E}(\nu(Z_0)) = \mathbb{E}(\nu(Z_{T_R})) = \mathbb{E}(\nu(X_{T^1 \wedge T_R})) - 1^{-1} = r^{-1} \mathbb{P}(T_r < T_R) + R^{-1} \mathbb{P}(T_r > T_R).
\]

It remains to solve the system of equations to get the desired formula.

5. We have \( T_r < T_R \) if \( R > \sup_{t \in [0, T_r]} |X_s| \), hence

\[
\{ T_r < T_R \} \, \Rightarrow \, \{ T_r < \infty \}.
\]

It follows that

\[
\mathbb{P}(T_r < T_R) \, \Rightarrow \, \mathbb{P}(T_r < \infty)
\]

and thus, from the formula provided by the previous question,

\[
\mathbb{P}(T_r < \infty) = \lim_{R \to \infty} \mathbb{P}(T_r < T_R) = \frac{|x|^{-1}}{|x|} = \frac{r}{|x|}.
\]

Now a.s. \( X \) is continuous and therefore

\[
\{ T_r < \infty \} \, \Rightarrow \, \{ T_0 < \infty \}
\]

and thus

\[
\mathbb{P}(T_0 < \infty) = \lim_{r \to 0^+} \mathbb{P}(T_r < \infty) = \lim_{r \to 0^+} \frac{r}{|x|} = 0.
\]
6. By the previous questions $T_0 < \infty$ a.s. thus $X$ remains a.s. in $D$, and since $u = |\cdot|^{-1}$ is harmonic on $D$, we get that $u(X) = |X|^{-1} = (|x + B|^{-1})_{t \geq 0}$ is a non-negative local martingale. By using a localizing sequence and the Fatou lemma, it is a non-negative super-martingale. Therefore it converges a.s. to an integrable random variable, hence, as $t \to \infty$, $|x + B_t|^{-1}$ converges a.s. and thus $|x + B_t|$ converges a.s. in $[0, +\infty]$, and since this convergence holds also in law, this law can only be $\delta_0$.

7. Let us show that $X = |x + B|^{-1}$ is bounded in $L^2$. By rotational invariance and scaling of $B_t$, we can assume without loss of generality that $x = (1,0,0)$. Since $x + B_t \sim \mathcal{N}(x, tI_3)$, using spherical coordinates $y_1 = r \cos(\theta) \sin(\varphi)$, $y_2 = r \sin(\theta) \sin(\varphi)$, $y_3 = r \cos(\varphi)$, with $r \in [0,\infty)$, $\theta \in [0,2\pi)$, $\varphi \in [0,\pi)$, we have $dy = r^2 \sin(\varphi) dr d\theta d\varphi$, and for all $t > 0$,

\[
E(|X_t|^2) = (2\pi t)^{-3/2} \int_{[0,2\pi]} |y|^2 e^{-\frac{r^2 + x^2 y_3^2 y_1^2}{2t}} \, dy
\]

\[
= (2\pi t)^{-3/2} \int_0^\infty \int_0^\pi \int_0^\infty r^{-2} e^{-\frac{r^2 \sin^2(\theta) \sin(\varphi)^2 + x^2 y_3^2 y_1^2}{2t}} r^2 \sin(\varphi) dr d\theta d\varphi
\]

\[
= (2\pi)^{-1/2} t^{-3/2} \int_0^\infty \int_0^\pi \int_0^\infty e^{-\frac{r^2 \sin^2(\theta) \sin(\varphi)^2}{2t}} \sin(\varphi) dr d\theta d\varphi
\]

\[
= (2\pi)^{-1/2} t^{-3/2} \int_0^\infty \int_0^\pi \int_0^\infty e^{-\frac{r^2 \sin^2(\theta) \sin(\varphi)^2}{2t}} \sin(\varphi) dr d\theta d\varphi
\]

\[
= 2(2\pi)^{-1/2} t^{-3/2} e^{-\frac{r^2}{2t}} \int_0^\infty \int_0^\pi \int_0^\infty e^{-\frac{r^2}{2t}} \sin\left(\frac{\varphi}{r}\right) dr d\theta d\varphi
\]

\[
= 2(2\pi)^{-1/2} t^{-3/2} e^{-\frac{r^2}{2t}} \int_0^\infty \int_0^\pi \int_0^\infty e^{-\frac{r^2}{2t}} \sin\left(\frac{\varphi}{r}\right) dr d\theta d\varphi
\]

\[
= r t^{-3/2} e^{-\frac{r^2}{2t}} \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} \int_0^\infty \left(\frac{r}{t}\right)^{2n} e^{-\frac{r^2}{t}} dr
\]

\[
= t^{-1} e^{-\frac{r^2}{2t}} \sum_{n=0}^{\infty} \frac{r^{-2n} (2\pi t)^{-1/2} \int_0^\infty r^{2n} e^{-\frac{r^2}{t}} dr}{(2n+1)!} t^{2n}
\]

\[
= t^{-1} e^{-\frac{r^2}{2t}} \sum_{n=0}^{\infty} \frac{r^{-2n} (2\pi t)^{-1/2} \int_0^\infty r^{2n} e^{-\frac{r^2}{t}} dr}{(2n+1)!} t^{2n}
\]

\[
= t^{-1} e^{-\frac{r^2}{2t}} \sum_{n=0}^{\infty} \frac{(2t)^{-n}}{(2n+1)!} t^{2n}
\]

\[
= 2^{-1/2} \sum_{n=0}^{\infty} \frac{(2t)^{-n}}{(2n+1)!} t^{2n}
\]

\[
\leq 2^{-1/2} \sum_{n=0}^{\infty} \frac{(2t)^{-n}}{(2n+1)!} t^{2n} e^{-\frac{r^2}{2t}} \leq 2.
\]

8. By a previous question, a.s. $X$ takes its values in $\mathbb{R}^3 \setminus \{0\}$ and $|\cdot|^{-1}$ is harmonic on this domain, and this implies that $|X|^{-1} = |x + B|^{-1}$ is a local martingale. Moreover $|X|^{-1}$ is u.i.

Now, suppose that $Y = |X|^{-1}$ is a martingale. Since it is u.i. $\lim_{t \to \infty} Y_t = Y_\infty$ a.s. and in $L^1$, with $Y_\infty \geq 0$ and $Y_\infty \in L^1$. Moreover $\mathbb{E}(Y_\infty) = \mathbb{E}(Y_0) = |x|^{-1} > 0$. But we know from a previous question that a.s. $\lim_{t \to \infty} |B_t| = +\infty$, which gives that a.s. $Y_\infty = 0$, thus $\mathbb{E}(Y_\infty) = 0$, a contradiction.

Alternatively, we could use Doob stopping for u.i. martingales, with the u.i. martingale $Y$ and the stopping time $T_R$, which is a.s. finite, this gives $|x|^{-1} = \mathbb{E}(Y_0) = \mathbb{E}(Y_{T_R}) = R^{-1}$ which is impossible.

Note that from the first question, $Y$ cannot be dominated by an integrable random variable!

It can be shown that the process $Y$ solves the SDE $dY_t = -Y_t^2 dW_t$.

Explicit computations show that $\lim_{t \to \infty} E(Y_t) \not\rightarrow 0$, and this is another way to show that $Y$ is not a martingale!
**Exercise 4** (Strict local martingales and stochastic integrals).

1. Give an example of an Itô stochastic integral which is a local martingale but not a martingale, without using the previous exercise.

**Elements of solution for Exercise 4.**

1. Of course we could consider the trivial example $\int_0^t dY_s = Y_t - Y_0$ where $Y$ is the strict local martingale considered in the previous exercise, but a deeper understanding is expected here!

A more interesting idea relies on the stochastic integral

$$I_B(\varphi) = \int_0^\cdot \varphi_s dB_s$$

where $\varphi$ is the single step function $\varphi = U1_{[0,1]}$ where $U$ is an $\mathcal{F}_0$ measurable random variable. A property of the Itô stochastic integral for semi-martingale integrators (here $B$) gives

$$I_B(\varphi) = UB_{\cdot \wedge 1} - UB_0 = UB_{\cdot \wedge 1}.$$

Now if we take $U$ independent of $B$, then, in $[0, +\infty]$,

$$E(|I_B(\varphi)_1|) = E(|U|)E(|B_1|).$$

Thus, if $U$ is not integrable then $I_B(\varphi)_1$ is not integrable and thus $I_B(\varphi)$ is not a martingale.

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**References**