(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}) is a filtered probability space, with complete and right continuous filtration. 

\( B = (B_t)_{t \geq 0} \) is a \( d \)-dimensional Brownian motion issued from the origin, \( d \geq 1 \).

**Exercise 1** (Representation of a process). Take \( d = 1 \) and \( x \in \mathbb{R} \).

1. Recall the computations and reasoning showing that the process \((Z_t)_{t \geq 0}\) defined by

   \[
   Z_t = xe^{-t} + e^{-t}M_t \quad \text{where} \quad M_t = \sqrt{2} \int_0^t e^s dB_s
   \]

   is the unique solution of the stochastic differential equation \(Z_0 = x, dZ_t = \sqrt{2} dB_t - Z_t dt\).

2. Show that for all \( t \geq 0 \), \( Z_t \) law \( \sim x e^{-t} + e^{-t}B_{t}e^{-t-1} \).

3. Can we have, for all \( t \geq 0 \), \( Z_t = xe^{-t} + e^{-t}B_{t}e^{-t-1} \) ?

4. Show that the process \((M_t)_{t \geq 0}\) is a continuous local martingale with, for all \( t \geq 0 \), \( \langle M \rangle_t = e^{2t} - 1 \).

5. Deduce that there exists a Brownian motion \((W_t)_{t \geq 0}\) such that for all \( t \geq 0 \), \( Z_t = xe^{-t} + e^{-t}W_{e^{-t}-1} \).

**Elements of solution for Exercise 1.** Ornstein–Uhlenbeck and Dubins–Schwarz!

1. The Itô formula \(X_t Y_t = X_0 Y_0 + \int_0^t (X_s dY_s + Y_s dX_s)\) gives, with \((X_t, Y_t) = (e^{-t}, M_t)\),

   \[
   e^{-t} M_t = \sqrt{2} \int_0^t e^s dB_t - \int_0^t e^{-s} M_s ds = \sqrt{2} B_t - \int_0^t e^{-s} M_s ds
   \]

   which gives

   \[
   xe^{-t} + e^{-t} M_t = x + \sqrt{2} B_t - \int_0^t (xe^{-s} + e^{-s} M_s) ds.
   \]

   On the other hand, to establish the uniqueness, let us suppose that \((X_t)_{t \geq 0}\) is a continuous semi-martingale solution of the SDE. Then by combining the Itô formula above for the function \(f(u, v) = uv\) and the semi-martingale \((e^t, X_t)\), and the SDE satisfied by \(X\), we obtain

   \[
   e^t X_t - x = \int_0^t e^s dX_s + \int_0^t e^s X_s ds = \sqrt{2} \int_0^t e^s dB_s.
   \]

2. Since \(M_t\) is a Wiener integral, it is Gaussian with zero mean and variance \(2 \int_0^t e^{2s} ds = e^{2t} - 1\), which gives \(Z_t \sim \mathcal{N}(xe^{-t}, 1 - e^{-2t})\), and it turns out that this is also the law of \(xe^{-t} + e^{-t}B_{t}e^{-t-1}\).

3. No because the process on the right hand side would not be adapted. Indeed the random variable \(B_{e^{-t}-1}\) is \(\mathcal{F}_{e^{-t}-1}\) measurable instead of being \(\mathcal{F}_t\)-measurable.

4. The process \(M\) is a Wiener (–Itô) integral, in particular it is a Gaussian square integrable martingale, and in particular a local martingale. Moreover \(\langle M \rangle_t = \mathbb{E}(M_t^2) = 2 \int_0^t e^{2s} ds = e^{2t} - 1\).

5. The process \(M\) is a continuous local martingale with respect to the filtration \((\mathcal{F}_t)_{t \geq 0}\) with \(M_0 = 0\) and \(\langle M \rangle_{\infty} = \infty\). Therefore the Dubins–Schwarz theorem states that \((W_t)_{t \geq 0} = (M_t)_{t \geq 0}\) where \(T_t = \inf\{s \geq 0 : \langle M \rangle_s > t\}\) is a Brownian motion for the filtration \((\mathcal{F}_t)_{t \geq 0}\), and \((W_{\langle M \rangle_t})_{t \geq 0} = (M_t)_{t \geq 0}\).

**Exercise 2** (Study of a special process). Let \(d = 1, \alpha \geq 0, x \geq 0\). Let \(X\) be a continuous semi-martingale taking values in \(\mathbb{R}_+\) and solving the stochastic differential equation:

\[
X_t = x + 2 \int_0^t \sqrt{X_s} dB_s + \alpha t, \quad t \geq 0.
\]

Let \(f : [0, +\infty) \to [0, +\infty)\) be continuous and \(\varphi : [0, +\infty) \to (0, +\infty)\) be positive and \(\mathcal{C}^2\), solving the ordinary differential equation \(\varphi'' = 2f\varphi\) with boundary conditions \(\varphi(0) = 1\) and \(\varphi(1) = 0\). Note that \(\varphi > 0\).
1. Could you give an explicit example of process $X$ for special values of $\alpha$?
2. Show that $\varphi$ decreases on the interval $[0, 1]$
3. Show that $u = \varphi'/2\varphi$ solves the differential equation $u' + 2u^2 = f$
4. Show that for all $t \geq 0$,
   \[ u(t)X_t - \int_0^t f(s)X_s \, ds = u(0)x + \int_0^t u(s) \, dX_s - 2 \int_0^t u(s)^2 X_s \, ds. \]
5. For all $t \geq 0$, let us define $Y_t = u(t)X_t - \int_0^t f(s)X_s \, ds$. Show that
   \[ \varphi(t)^{-\frac{1}{2}} e^{Y_t} = e^{N_t - \frac{1}{2}(N)_t} \quad \text{where} \quad N_t = u(0)x + 2 \int_0^t u(s) \sqrt{X_s} dB_s. \]
6. Show that
   \[ \mathbb{E} \exp \left( -\int_0^1 f(s)X_s \, ds \right) = \varphi(1)^{\frac{1}{2}} e^{\hat{\varphi}(0)} \]
7. From now on, let $\lambda > 0$. Prove that
   \[ \mathbb{E} \exp \left( -\lambda \int_0^1 X_s \, ds \right) = \left( \cosh(\sqrt{2\lambda}) \right)^{-\frac{1}{2}} e^{-\frac{1}{2} \sqrt{\lambda} \tanh \sqrt{\lambda}}. \]
8. Prove that for all $\lambda > 0$ and $y \in \mathbb{R}$,
   \[ \mathbb{E} \exp \left( -\lambda \int_0^1 (y + B_s)^2 \, ds \right) = \left( \cosh(\sqrt{2\lambda}) \right)^{-\frac{1}{2}} e^{-\frac{1}{2} \sqrt{\lambda} \tanh \sqrt{\lambda}}. \]

**Elements of solution for Exercise 2.** This is on squared Bessel processes : [1, Exercise 5.31 p. 145–146].

1. We know from the course (Itô+Lévy) that if $\alpha = n \in \{1, 2, \ldots\}$ then $X$ has the law of $|x + W|^2$ where $W$ is a $n$-dimensional Brownian motion issued from the origin (squared Bessel process).
2. We have $\varphi'(1) = 0$ and $\varphi'' \geq 0$ hence $\varphi' \leq 0$ on $[0, 1]$.
3. We have $u' = \frac{\varphi'' \varphi - \varphi'^2}{2\varphi^2}$ thus
   \[ u' + 2u^2 = \frac{\varphi'' \varphi - \varphi'^2}{2\varphi^2} + \frac{\varphi'^2}{2\varphi^2} = \frac{\varphi''}{2\varphi} = f. \]
4. The Itô formula for $F(x_1, x_2) = x_1x_2$ and $(u(t), X_t)$ gives
   \[ u(t)X_t - u(0)x = \int_0^t X_s u'(s) \, ds + \int_0^t u(s) \, dX_s \]
   \[ = \int_0^t X_s (f(s) - 2u^2(s)) \, ds + \int_0^t u(s) \, dX_s \]
   and it remains to use the result of the previous question.
5. We have, using the SDE solved by $X$ for the second step, and in the third step the definition of $Y$ and the previous question,
   \[ N_t - \frac{1}{2}(N)_t = u(0)x + 2 \int_0^t u(s) \sqrt{X_s} dB_s - 2 \int_0^t u(s)^2 X_s \, ds \]
   \[ = u(0)x + \int_0^t u(s) \, dX_s - \alpha \int_0^t u(s) \, ds. \]
   Since
   \[ \int_0^t u(s) \, ds = \int_0^t \frac{\varphi'(s)}{2\varphi(s)} \, ds = \frac{\log \varphi(1) - \log \varphi(0)}{2} = \frac{\log \varphi(1)}{2}, \]
   we obtain
   \[ e^{N_t - \frac{1}{2}(N)_t} = e^{Y_t \varphi(1)^{-\alpha t}}. \]
6. The process $e^{N_t - \frac{1}{2} \langle N \rangle_t}$ is a Doléans-Dade exponential, hence a continuous local martingale. In order to show that it is a martingale for $t \in [0,1]$, it suffices to show that it is dominated by an integrable random variable. From the previous computations, for all $t \in [0,1]$,

\[ e^{N_t - \frac{1}{2} \langle N \rangle_t} = e^{Y_t - f_0^t u(s)ds}. \]

Now, $Y_t = u(t)X_t - \int_0^t f(s)X_sds \leq 0$ since $u \leq 0$ (recall that $\varphi' \leq 0$), while $X_t, f \geq 0$.

Hence $(e^{N_t - \frac{1}{2} \langle N \rangle_t})_{t \in [0,1]}$ is a martingale.

Next $u(1) = \varphi'(1)/(2\varphi(1)) = 0$, we get, from the previous question with $t = 1$,

\[ \varphi(1)^{-\frac{1}{2}} e^{\int_0^1 f(s)X_sds} = e^{N_0 - \frac{1}{2} \langle N \rangle_0} \]

\[ = e^{u(0)x} \]

\[ = e^{\frac{\varphi(0)}{2\varphi(0)}} \]

\[ = e^{\frac{1}{2} \varphi(0)}. \]

7. We take $f$ constant and equal to $\lambda > 0$. The differential equation solved by $\varphi$ writes $\varphi'' = 2\lambda \varphi$

with $\varphi(0) = 1$ and $\varphi'(1) = 0$. The associated equation has two roots $\pm \sqrt{2\lambda}$ hence $\varphi(x) = ae^{\sqrt{2\lambda}x} + be^{-\sqrt{2\lambda}x}$. The boundary conditions give $a + \beta = 1$ and $ae^{\sqrt{2\lambda}} = \beta$, hence

\[ a = \frac{1}{1 + e^{2\sqrt{2\lambda}}} = \frac{e^{-\sqrt{2\lambda}}}{2 \cosh(\sqrt{2\lambda})} \quad \text{and} \quad \beta = \frac{e^{\sqrt{2\lambda}}}{1 + e^{2\sqrt{2\lambda}}} = \frac{e^{\sqrt{2\lambda}}}{2 \cosh(\sqrt{2\lambda})}. \]

This gives

\[ \varphi(1) = \frac{1}{\cosh(\sqrt{2\lambda})} \quad \text{and} \quad \varphi'(0) = \sqrt{2\lambda}(\alpha - \beta) = -\sqrt{2\lambda} \tanh(\sqrt{2\lambda}). \]

8. If we take $\alpha = 1$ then $X$ has the law of the squared Bessel process $(x + B)^2$.

**Exercise 3** (Strict local martingales). We take $d = 3$, $X = x + B$, $0 < r < |x| < R < \infty$, and, for all $a \geq 0$,

\[ T_a = \inf\{t \geq 0 : |X_t| = a\}. \]

1. Show that if $M = (M_t)_{t \geq 0}$ is a continuous local martingale with for all $t \geq 0$, $|M_t| \leq U$ where $U \in L^1$, then $M$ is a martingale. Does it remain true if the domination condition is replaced by "$M$ is $u.i.$"?

2. Show that if $Z = (Z_t)_{t \geq 0}$ is $d$-dimensional, adapted, taking values in an open set $D \subset \mathbb{R}^d$, such that its components are continuous local martingales, and for all $1 \leq j, k \leq d$, $\langle Z^j, Z^k \rangle = \int \mathbf{1}_{j=k}$ for a finite variation process $V$, then, for all harmonic $u : D \to \mathbb{R}$, the process $u(Z)$ is a local martingale.

3. Show that $|\cdot|^{-1}$ is harmonic on $\mathbb{R}^3 \setminus \{0\}$.

4. Show that $T_R < \infty$ almost surely and

\[ \mathbb{P}(T_r < T_R) = \frac{R^{-1} - |x|^{-1}}{R^{-1} - r^{-1}}. \]

5. Deduce from the previous formula that a.s. for all $t \geq 0$, $X_t \neq 0$.

6. Show that a.s. $\lim_{t \to 0} |B_t| = +\infty$. Hint: show that $|X|^{-1}$ is a non-negative super-martingale.

7. Show that $|X|^{-1}$ is bounded in $L^2$. Hint: density of $B_t$ in spherical coordinates.

8. Show that $|X|^{-1}$ is a continuous local martingale, but is not a martingale.

**Elements of solution for Exercise 3.** The goal is to construct a local martingale which is u.i. but which is not a martingale. The example chosen here is a non-centered Bessel process. This example of a strict local martingale is quite classical, and corresponds for instance to [1, Exercise 5.33(8) page 148].
1. For all \( t \geq 0 \), the measurability and integrability of \( M_t \) comes from the adaptation of \( M \) and the domination assumption. Let \((T_n)_{n \geq 0}\) be a localizing sequence for \( M \). For all \( n \geq 0 \), \( M^{T_n} \) is a martingale: for all \( 0 \leq s \leq t \), \( E(M^{T_n}_{s\wedge T_s} \mid \mathcal{F}_s) = M^{T_n}_{s\wedge T_s} \). Now since \( \lim_{n \to \infty} T_n = +\infty \) a.s. and since \( M \) is continuous, if follows that a.s. for all \( t \geq 0 \), \( \lim_{n \to \infty} M^{T_n}_{t\wedge T_t} = M_t \). It remains to use dominated convergence to get that \( \lim_{n \to \infty} E(M^{T_n}_{t\wedge T_t} \mid \mathcal{F}_s) = E(M_t \mid \mathcal{F}_s) \). Hence \( M \) is a martingale. Finally, if \( M \) is u.i. instead of being dominated, then the argument is no longer valid since it does not give a way to handle \( M^{T_n}_{t\wedge T_t} \). The goal of the exercise is precisely the construction of an u.i. continuous local martingale which is not a martingale! Note: domination implies u.i. but the converse is wrong.

2. The Itô formula is licit on an open domain \( D \) for a process taking values in \( D \). It gives, for all \( t \geq 0 \),

\[
 u(X_t) = u(X_0) + \int_0^t \nabla u(X_s) dX_s + \frac{1}{2} \int_0^t \Delta u(X_s) dV_s.
\]

The last integral vanishes since \( \Delta u = 0 \) and \( X \) takes values on \( D \), thus \( u(X) \) is a local martingale.

3. For all \( y \in D \), denoting \( u = ||\cdot||^{-1} \), we obtain \( \Delta u(y) = 0 \) from

\[
 \partial_i u(y) = (2 - d) \frac{y_i}{|y|^d}, \quad \text{and} \quad \partial^2_{ij} u(y) = \frac{d(d - 2) |y|^d - dy_i^2 |y|^{d-2}}{2 |y|^{d+2}}.
\]

4. Set \( Z = X^{T_t} \). Then \( \langle Z^1_j, Z^k_j \rangle = \langle B^1_j, B^k_j \rangle_{t \wedge T_t} = (t \wedge T_t) 1_{j=k} \). Hence, by a previous question with the processes \( Z \) and \( V_t = t \wedge T_t \), the harmonic function \( u = ||\cdot||^{-1} \), and \( D = \{x \in \mathbb{R}^3 : x \neq 0\} \), we get that

\[
 u(Z) = (u(X_{t \wedge T_t}))_{t \geq 0}
\]

is a local martingale, and since it is bounded by \( r^{-1} \), it is a bounded martingale.

Next, since a 1-dimensional BM escapes almost surely from every finite interval, the first component of our 3-dimensional Brownian motion \( x + B \) escapes almost surely from \( [-R, R] \), and thus almost surely \( T_R < \infty \). In particular almost surely \( T_r < T_R \) or \( T_r > T_R \) and we cannot have \( T_r = T_R = \infty \). We have thus the immediate equation

\[
 1 = \mathbb{P}(T_r < T_R) + \mathbb{P}(T_r > T_R).
\]

By the Doob stopping theorem for the bounded martingale \( u(Z) \) and the finite stopping time \( T_R \),

\[
 |x|^{-1} = E(u(Z_0)) = E(u(Z_{T_R})) = E(|X_{T_R}|-1) = r^{-1} \mathbb{P}(T_r < T_R) + R^{-1} \mathbb{P}(T_r > T_R).
\]

It remains to solve the system of equations to get the desired formula.

5. We have \( T_r < T_R \) if \( R > \sup_{x \in [0,X_{T_r}]} |X_x| \), hence

\[
 \{ T_r < T_R \} \nearrow \{ T_r < \infty \}.
\]

It follows that

\[
 \mathbb{P}(T_r < T_R) \nearrow \mathbb{P}(T_r < \infty)
\]

and thus, from the formula provided by the previous question,

\[
 \mathbb{P}(T_r < \infty) = \lim_{R \to \infty} \mathbb{P}(T_r < T_R) = \frac{|x|^{-1}}{r^{-1}} = \frac{r}{|x|}.
\]

Now a.s. \( X \) is continuous and therefore

\[
 \{ T_r < \infty \} \nearrow \{ T_0 < \infty \}
\]

and thus

\[
 \mathbb{P}(T_0 < \infty) = \lim_{r \to 0^+} \mathbb{P}(T_r < \infty) = \lim_{r \to 0^+} \frac{r}{|x|} = 0.
\]
6. By the previous questions $T_0 < \infty$ a.s. thus $X$ remains a.s. in $D$, and since $u = \{\cdot\}^{-1}$ is harmonic on $D$, we get that $u(X) = |X|^{-1} = (|x + B|^{-1})_{t \geq 0}$ is a non-negative local martingale. By using a localizing sequence and the Fatou lemma, it is a non-negative super-martingale. Therefore it converges a.s. to an integrable random variable, hence, as $t \to \infty$, $|x + B_t|^{-1}$ converges a.s. and thus $|x + B_t|$ converges a.s. in $[0, +\infty]$, and since this convergence holds also in law, this law can only be $\delta_0$.

7. Let us show that $X = |x + B|^{-1}$ is bounded in $L^2$. By rotational invariance and scaling of $B_t$, we can assume without loss of generality that $x = (1, 0, 0)$. Since $x + B_t \sim \mathcal{N}(x, t I_3)$, using spherical coordinates $y_1 = r \cos(\theta) \sin(\varphi)$, $y_2 = r \sin(\theta) \sin(\varphi)$, $y_3 = r \cos(\varphi)$, with $r \in [0, \infty)$, $\theta \in [0, 2\pi)$, $\varphi \in [0, \pi)$, we have $dy = r^2 \sin(\varphi) dr d\theta d\varphi$, and for all $t > 0$,

$$E(|X_t|^2) = (2\pi t)^{-3/2} \int y^{-1} e^{-\frac{y^2 + 2^{3/2} + y^2 x^2}{2t}} dy$$

$$= (2\pi t)^{-3/2} \int_0^{2\pi} \int_0^{\infty} \int_0^{\pi} r^2 e^{-\frac{r^2 \sin^2(\varphi) \sin(\theta)}{2t}} \sin(\varphi) r^2 \sin(\varphi) dr d\theta d\varphi$$

$$= (2\pi)^{-1/2} t^{-3/2} \int_0^{\infty} \int_0^{\pi} e^{-\frac{r^2 \sin^2(\varphi) \sin(\theta)}{2t}} \sin(\varphi) r^2 \sin(\varphi) dr d\varphi$$

$$= 2(2\pi)^{-1/2} t^{-3/2} e^{-\frac{1}{4t}} \int_0^{\infty} e^{-\frac{r^2}{4t}} \left( \sum_{n=1}^{\infty} \left[ \frac{1}{r} \int e^{\frac{n^2}{r}} \right] \right) dr$$

$$= 2(2\pi)^{-1/2} t^{-3/2} e^{-\frac{1}{4t}} \int_0^{\infty} e^{-\frac{r^2}{4t}} \sinh(\frac{r}{2}) \frac{1}{r} dr$$

$$= 2(2\pi)^{-1/2} t^{-3/2} e^{-\frac{1}{4t}} \int_0^{\infty} e^{-\frac{r^2}{4t}} \frac{1}{r} \left( \sum_{n=0}^{\infty} \frac{1}{(2n + 1)!} \right) \int_0^{\infty} \frac{1}{t} e^{-2n t} dr$$

$$= r^{-1} e^{-\frac{1}{4t}} \sum_{n=0}^{\infty} \frac{r^{-2n}}{(2n + 1)!} \int_0^{\infty} \left( \frac{1}{t} \right)^{2n} e^{-\frac{x^2}{t}} dr$$

$$= r^{-1} e^{-\frac{1}{4t}} \sum_{n=0}^{\infty} \frac{r^{-2n}}{(2n + 1)!} \int_0^{\infty} \frac{2n t}{{r}^{2n}} e^{-\frac{x^2}{t}} dr$$

$$= r^{-1} e^{-\frac{1}{4t}} \sum_{n=0}^{\infty} \frac{r^{-2n}}{(2n + 1)!} \left( 2n \right)^{2n-1} (n + 1)!$$

$$= 2 e^{-\frac{1}{t}} \sum_{n=0}^{\infty} \frac{(2t)^{-n}}{(2n + 1)!} \left( 2n \right)^{2n-1} (n + 1)!$$

$$\leq 2 e^{-\frac{1}{t}} \sum_{n=0}^{\infty} \frac{(2t)^{-n}}{(2n + 1)!} \left( 2n \right)^{2n-1} (n + 1)!$$

8. By a previous question, a.s. $X$ takes its values in $[0, \infty)$ and $\{\cdot\}^{-1}$ is harmonic on this domain, and this implies that $|X|^{-1} = |x + B|^{-1}$ is a local martingale. Moreover $|X|^{-1}$ is u.i.

Now, suppose that $Y = |X|^{-1}$ is a martingale. Since it is u.i. $\lim_{t \to \infty} Y_t = Y_\infty$ a.s. and in $L^1$, with $Y_\infty \geq 0$ and $Y_\infty \in L^1$. Moreover $E(Y_\infty) = E(Y_0) = |x|^{-1} > 0$. But we know from a previous question that a.s. $\lim_{t \to \infty} B_t = +\infty$, which gives that a.s. $Y_\infty = 0$, thus $E(Y_\infty) = 0$, a contradiction.

Alternatively, we could use Doob stopping for u.i. martingales, with the u.i. martingale $Y$ and the stopping time $T_R$, which is a.s. finite, this gives $|x|^{-1} = E(Y_0) = E(Y_{T_R}) = R^{-1}$ which is impossible.

Note that from the first question, $Y$ cannot be dominated by an integrable random variable!

It can be shown that the process $Y$ solves the SDE $dY_t = -Y^2_t dW_t$.

Explicit computations show that $E(Y_t) \to 0$, and this is another way to show that $Y$ is not a martingale!
Exercise 4 (Strict local martingales and stochastic integrals).

1. Give an example of an Itô stochastic integral which is a local martingale but not a martingale, without using the previous exercise.

Elements of solution for Exercise 4.

1. Of course we could consider the trivial example \( \int_0^t dY_s = Y_t - Y_0 \) where \( Y \) is the strict local martingale considered in the previous exercise, but a deeper understanding is expected here!

A more interesting idea relies on the stochastic integral

\[
I_B(\varphi) = \int_0^* \varphi_s dB_s
\]

where \( \varphi \) is the single step function \( \varphi = U 1_{[0,1[} \) where \( U \) is an \( \mathcal{F}_0 \) measurable random variable. A property of the Itô stochastic integral for semi-martingale integrators (here \( B \)) gives

\[
I_B(\varphi) = UB_{\star \wedge 1} - UB_0 = UB_{\star \wedge 1}.
\]

Now if we take \( U \) independent of \( B \), then, in \([0, +\infty[\),

\[
\mathbb{E}(\left| I_B(\varphi)_1 \right|) = \mathbb{E}(\left| U \right|) \mathbb{E}(\left| B_1 \right|).
\]

Thus, if \( U \) is not integrable then \( I_B(\varphi)_1 \) is not integrable and thus \( I_B(\varphi) \) is not a martingale.

References