(Ω, ℱ, (ℱt)t≥0, P) is a filtered probability space, with complete and right continuous filtration. B = (Bt)t≥0 is a d-dimensional Brownian motion issued from the origin, d ≥ 1.

**Exercise 1** (Representation of a process). Take d = 1 and x ∈ ℝ.

1. Recall the computations and reasoning showing that the process (Zt)t≥0 defined by

   \[ Z_t = xe^{-t} + e^{-t}M_t \quad \text{where} \quad M_t = \sqrt{2} \int_0^t e^s dB_s \]

   is the unique solution of the stochastic differential equation Z0 = x, dZt = \sqrt{2}dBt - Zt dt.

2. Show that for all t ≥ 0, Zt law xe^{-t} + e^{-t}B_{e^2t-1}.

3. Can we have, for all t ≥ 0, Zt = xe^{-t} + e^{-t}B_{e^2t-1}?

4. Show that the process (Mt)t≥0 is a continuous local martingale with, for all t ≥ 0, (Mt)t = e^{2t} - 1.

5. Deduce that there exists a Brownian motion (Wt)t≥0 such that for all t ≥ 0, Zt = xe^{-t} + e^{-t}W_{e^2t-1}.

**Elements of solution for Exercise 1.** Ornstein–Uhlenbeck and Dubins–Schwarz!

1. The Itô formula XtYt = Xo Yo + \int_0^t (Xs ds + Ys dXs) gives, with (Xt, Yt) = (e^{-t}, Mt),

   \[ e^{-t}M_t = \sqrt{2} \int_0^t e^{-s} dB_s - \int_0^t e^{-s} M_s ds = \sqrt{2} B_t - \int_0^t e^{-s} M_s ds \]

   which gives

   \[ xe^{-t} + e^{-t}M_t = x + \sqrt{2} B_t - \int_0^t (xe^{-s} + e^{-s} M_s) ds. \]

   On the other hand, to establish the uniqueness, let us suppose that (Xt)t≥0 is a continuous semi-martingale solution of the SDE. Then by combining the Itô formula above for the function f(u, v) = uv and the semi-martingale (e^t, Xt), and the SDE satisfied by X, we obtain

   \[ e^t X_t - x = \int_0^t e^s dB_s + \int_0^t e^s X_s ds = \sqrt{2} \int_0^t e^s dB_s. \]

2. Since Mt is a Wiener integral, it is Gaussian with zero mean and variance 2 \int_0^t e^{2s} ds = e^{2t} - 1, which gives Zt ~ N(xe^{-t}, 1 - e^{-2t}), and it turns out that this is also the law of xe^{-t} + e^{-t}B_{e^2t-1}.

3. No because the process on the right hand side would not be adapted. Indeed the random variable B_{e^2t-1} is ℱ_{e^2t-1}-measurable instead of being ℱt-measurable.

4. The process M is a Wiener (– Itô) integral, in particular it is a Gaussian square integrable martingale, and in particular a local martingale. Moreover \( \langle M \rangle_t = \mathbb{E}(M_t^2) = 2 \int_0^t e^{2s} ds = e^{2t} - 1. \)

5. The process M is a continuous local martingale with respect to the filtration (ℱt)t≥0 with M0 = 0 and \( \langle M \rangle_\infty = \infty. \) Therefore the Dubins–Schwarz theorem states that (Wt)t≥0 = (Mt)t≥0 where \( T_t = \inf(s \geq 0 : \langle M \rangle_s > t) \) is a Brownian motion for the filtration (ℱt)t≥0, and (Wt)t≥0 = (Mt)t≥0.

**Exercise 2** (Study of a special process). Let d = 1, α ≥ 0, x ≥ 0. Let X be a continuous semi-martingale taking values in ℝ+ and solving the stochastic differential equation:

\[ X_t = x + 2 \int_0^t \sqrt{X_s} dB_s + \alpha t, \quad t \geq 0. \]

Let f : [0, +∞) → [0, +∞) be continuous and \( \varphi : [0, +\infty) \to (0, +\infty) \) be positive and \( \mathcal{C}^2 \), solving the ordinary differential equation \( \varphi'' = 2f \varphi \) with boundary conditions \( \varphi(0) = 1 \) and \( \varphi'(1) = 0. \) Note that \( \varphi > 0. \)
1. Could you give an explicit example of process $X$ for special values of $a$?
2. Show that $\varphi$ decreases on the interval $[0,1]$.
3. Show that $u = \varphi'/2\varphi$ solves the differential equation $u' + 2u^2 = f$.
4. Show that for all $t \geq 0$,  
   \[ u(t)X_t - \int_0^t f(s)X_s\,ds = u(0)x + \int_0^t u(s)dX_s - 2\int_0^t u(s)^2X_s\,ds. \]
5. For all $t \geq 0$, let us define $Y_t = u(t)X_t - \int_0^t f(s)X_s\,ds$. Show that  
   \[ \varphi(t)^{-\frac{1}{2}}e^{Y_t} = e^{N_t - \frac{1}{2}(N_t)} \quad \text{where} \quad N_t = u(0)x + 2\int_0^t u(s)\sqrt{X_s}\,dB_s. \]
6. Show that  
   \[ \mathbb{E}\exp\left(-\int_0^1 f(s)X_s\,ds\right) = \varphi(1)^{-\frac{1}{2}}e^{\frac{1}{2}\varphi'(0)}. \]
7. From now on, let $\lambda > 0$. Prove that  
   \[ \mathbb{E}\exp\left(-\lambda\int_0^1 X_s\,ds\right) = (\cosh(\sqrt{2}\lambda))^{-\frac{\lambda}{2}}e^{-\frac{\lambda^2}{2}\tanh(\sqrt{2}\lambda)}. \]
8. Prove that for all $\lambda > 0$ and $y \in \mathbb{R}$,  
   \[ \mathbb{E}\exp\left(-\lambda\int_0^1 (y + B_s)^2\,ds\right) = (\cosh(\sqrt{2}\lambda))^{-\frac{\lambda}{2}}e^{-\frac{\lambda^2}{2}\tanh(\sqrt{2}\lambda)}. \]

**Elements of solution for Exercise 2.** This is on squared Bessel processes: [1, Exercise 5.31 p. 145 – 146].

1. We know from the course (Itô+Lévy) that if $a = n \in \{1, 2, \ldots\}$ then $X$ has the law of $|x + W|^2$ where $W$ is a $n$-dimensional Brownian motion issued from the origin (squared Bessel process).
2. We have $\varphi'(1) = 0$ and $\varphi'' \geq 0$ hence $\varphi' \leq 0$ on $[0,1]$.
3. We have $u' = \frac{\varphi''\varphi - \varphi'^2}{2\varphi^2}$ thus  
   \[ u' + 2u^2 = \frac{\varphi''\varphi - \varphi'^2}{2\varphi^2} + \frac{\varphi'^2}{2\varphi} = \frac{\varphi''}{2\varphi} = f. \]
4. The Itô formula for $F(x_1, x_2) = x_1x_2$ and $(u(t), X_t)$ gives  
   \[ u(t)X_t - u(0)x = \int_0^t X_su'(s)\,ds + \int_0^t u(s)dX_s \]
   \[ = \int_0^t X_s(f(s) - 2u^2(s))\,ds + \int_0^t u(s)dX_s \]
and it remains to use the result of the previous question.
5. We have, using the SDE solved by $X$ for the second step, and in the third step the definition of $Y$ and the previous question,  
   \[ N_t - \frac{1}{2}(N_t) = u(0)x + 2\int_0^t u(s)\sqrt{X_s}\,dB_s - 2\int_0^t u(s)^2X_s\,ds \]
   \[ = u(0)x + \int_0^t u(s)(dX_s - \alpha\,ds) - 2\int_0^t u(s)^2X_s\,ds \]
   \[ = Y_t - \alpha\int_0^t u(s)\,ds. \]

Since  
   \[ \int_0^t u(s)\,ds = \int_0^t \frac{\varphi'(s)}{2\varphi(s)}\,ds = \frac{\log\varphi(1) - \log\varphi(0)}{2} = \frac{\log\varphi(1)}{2}, \]
we obtain  
\[ e^{N_t - \frac{1}{2}(N_t)} = e^{Y_t\varphi(1)^{-\alpha t}}. \]
6. The process $e^{N_t - \frac{1}{2}(N_t)}$ is a Doléans-Dade exponential, hence a continuous local martingale. In order to show that it is a martingale for $t \in [0, 1]$, it suffices to show that it is dominated by an integrable random variable. From the previous computations, for all $t \in [0, 1]$, 
    
    \[ e^{N_t - \frac{1}{2}(N_t)t} = e^{V_t - \int_0^t u(s)ds}. \]

Now, $Y_t = u(t)X_t - \int_0^t f(s)X_sds \leq 0$ since $u \leq 0$ (recall that $\varphi' \leq 0$), while $X, f \geq 0$. Hence $(e^{N_t - \frac{1}{2}(N_t)})_{t \in [0, 1]}$ is a martingale.

Next $u(1) = \varphi'(1)/(2\varphi(1)) = 0$, we get, from the previous question with $t = 1$,
    
    \[ \varphi(1)^{-\frac{1}{2}}e^{-\int_0^1 f(s)X_sds} = E(e^{N_1 - \frac{1}{2}(N_1)}) \]
    
    \[ = E(e^{N_0 - \frac{1}{2}(N_0)}) \]
    
    \[ = e^{u(0)x} \]
    
    \[ = e^{\frac{\varphi'(0)}{2\varphi(0)}} \]
    
    \[ = e^{\frac{1}{2}\varphi'(0)}. \]

7. We take $f$ constant and equal to $\lambda > 0$. The differential equation solved by $\varphi$ writes $\varphi'' = 2\lambda \varphi$ with $\varphi(0) = 1$ and $\varphi'(1) = 0$. The associated equation has two roots $\pm \sqrt{2\lambda}$ hence $\varphi(x) = a\text{e}^{\sqrt{2\lambda}x} + \beta e^{-\sqrt{2\lambda}x}$. The boundary conditions give $a + \beta = 1$ and $a\text{e}^{2\sqrt{2\lambda}} = \beta$, hence

    \[ a = \frac{1}{1 + e^{2\sqrt{2\lambda}}} = \frac{e^{-\sqrt{2\lambda}}}{2\cosh(\sqrt{2\lambda})} \]
    
    and
    
    \[ \beta = \frac{e^{2\sqrt{2\lambda}}}{1 + e^{2\sqrt{2\lambda}}} = \frac{e^{\sqrt{2\lambda}}}{2\cosh(\sqrt{2\lambda})}. \]

This gives
    
    \[ \varphi(1) = \frac{1}{\cosh(\sqrt{2\lambda})} \]
    
    and
    
    \[ \varphi'(0) = \sqrt{2\lambda}(\alpha - \beta) = -\sqrt{2\lambda}\tanh(\sqrt{2\lambda}). \]

8. If we take $a = 1$ then $X$ has the law of the squared Bessel process $(x + B)^2$.

Exercise 3 (Strict local martingales). We take $d = 3$, $X = x + B$, $0 < r < |x| < R < \infty$, and, for all $a \geq 0$,

\[ T_a = \inf\{t \geq 0 : |X_t| = a\}. \]

1. Show that if $M = (M_t)_{t \geq 0}$ is a continuous local martingale with for all $t \geq 0, |M_t| \leq U$ where $U \in L^1$, then $M$ is a martingale. Does it remain true if the domination condition is replaced by “$M$ is u.i.”?

2. Show that if $Z = (Z_t)_{t \geq 0}$ is $d$-dimensional, adapted, taking values in an open set $D \subset \mathbb{R}^d$, such that its components are continuous local martingales, and for all $1 \leq j, k \leq d$, $\langle Z^j, Z^k \rangle = \int_1^{\infty} V_{j-k}$ for a finite variation process $V$, then, for all harmonic $u : D \rightarrow \mathbb{R}$, the process $u(Z)$ is a local martingale.

3. Show that $|\cdot|^{-1}$ is harmonic on $\mathbb{R}^3 \setminus \{0\}$.

4. Show that $T_R < \infty$ almost surely and

\[ \mathbb{P}(T_r < T_R) = \frac{R^{-1} - |x|^{-1}}{R^{-1} - r^{-1}}. \]

5. Deduce from the previous formula that a.s. for all $t \geq 0$, $X_t \neq 0$.

6. Show that a.s. $\lim_{t \rightarrow \infty} |B_t| = +\infty$. Hint: show that $|X|^{-1}$ is a non-negative super-martingale.

7. Show that $|X|^{-1}$ is bounded in $L^2$. Hint: density of $B_t$ in spherical coordinates.

8. Show that $|X|^{-1}$ is a continuous local martingale, but is not a martingale.

Elements of solution for Exercise 3. The goal is to construct a local martingale which is u.i. but which is not a martingale. The example chosen here is a non-centered Bessel process. This example of a strict local martingale is quite classical, and corresponds for instance to [1, Exercise 5.33(8) page 148].
1. For all $t \geq 0$, the measurability and integrability of $M_t$ comes from the adaptation of $M$ and the domination assumption. Let $(T^n)_{n \geq 0}$ be a localizing sequence for $M$. For all $n \geq 0$, $M^{T^n}$ is a martingale: for all $0 \leq s \leq t$, $\mathbb{E}(M^{T^n}_{t} | \mathcal{F}_s) = M^{T^n}_s$. Now since $\lim_{n \to \infty} T^n = +\infty$ a.s. and since $M$ is continuous, if follows that a.s. for all $t \geq 0$, $\lim_{n \to \infty} M^{T^n}_{t} = M_t$. It remains to use dominated convergence to get that $\lim_{n \to \infty} \mathbb{E}(M^{T^n}_{t} | \mathcal{F}_s) = \mathbb{E}(M_t | \mathcal{F}_s)$. Hence $M$ is a martingale. Finally, if $M$ is u.i. instead of being dominated, then the argument is no longer valid since it does not give a way to handle $M^{T^n}_{t}$. The goal of the exercise is precisely the construction of an u.i. continuous local martingale which is not a martingale! Note: domination implies u.i. but the converse is wrong.

2. The Itô formula is licit on an open domain $D$ for a process taking values in $D$. It gives, for all $t \geq 0$,

$$u(X_t) = u(X_0) + \int_0^t \nabla u(X_s) dX_s + \frac{1}{2} \int_0^t \Delta u(X_s) dV_s.$$ 

The last integral vanishes since $\Delta u = 0$ and $X$ takes values on $D$, thus $u(X)$ is a local martingale.

3. For all $y \in D$, denoting $u = |\cdot|^{-1}$, we obtain $\Delta u(y) = 0$ from

$$\partial_i u(y) = (2 - d) \frac{y_i}{|y|^d}, \quad \text{and} \quad \partial^2_{ij} u(y) = \frac{d(d - 2)}{2} \frac{|y|^d - d y_i y_j |y|^{d-2}}{|y|^{d+2}}.$$ 

4. Set $Z = X^{T_1}$. Then $(Z^1, Z^k)_t = (B^1, B^k)_{t \wedge T_1} = (t \wedge T_1) 1_{j=k}$. Hence, by a previous question with the processes $Z$ and $V_t = t \wedge T_1$, the harmonic function $u = |\cdot|^{-1}$, and $D = \{x \in \mathbb{R}^3 : x \neq 0\}$, we get that

$$u(Z) = (u(X_{t \wedge T_1}))_{t \geq 0}$$ 

is a local martingale, and since it is bounded by $r^{-1}$, it is a bounded martingale.

Next, since a 1-dimensional BM escapes almost surely from every finite interval, the first component of our 3-dimensional Brownian motion $x + B$ escapes almost surely from $[-R, R]$, and thus almost surely $T_R < \infty$. In particular almost surely $T_r < T_R$ or $T_r > T_R$ and we cannot have $T_r = T_R = \infty$. We have thus the immediate equation

$$1 = \mathbb{P}(T_r < T_R) + \mathbb{P}(T_r > T_R).$$ 

By the Doob stopping theorem for the bounded martingale $u(Z)$ and the finite stopping time $T_R$, 

$$|x|^{-1} = \mathbb{E}(u(Z_0)) = \mathbb{E}(u(Z_{T_R})) = \mathbb{E}(|X_{T_r \wedge T_R}|^{-1}) = r^{-1} \mathbb{P}(T_r < T_R) + R^{-1} \mathbb{P}(T_r > T_R).$$ 

It remains to solve the system of equations to get the desired formula.

5. We have $T_r < T_R$ if $R > \sup_{x \in [0, X_{T_1}]} |X_s|$, hence

$$\{T_r < T_R\} \nearrow \{T_r < \infty\}.$$ 

It follows that

$$\mathbb{P}(T_r < T_R) \nearrow \mathbb{P}(T_r < \infty)$$ 

and thus, from the formula provided by the previous question,

$$\mathbb{P}(T_r < \infty) = \lim_{R \to \infty} \mathbb{P}(T_r < T_R) = \frac{|x|^{-1}}{r^{-1}} = \frac{r}{|x|}.$$ 

Now a.s. $X$ is continuous and therefore

$$\{T_r < \infty\} \searrow_{r \to 0^+} \{T_0 < \infty\}$$ 

and thus

$$\mathbb{P}(T_0 < \infty) = \lim_{r \to 0^+} \mathbb{P}(T_r < \infty) = \lim_{r \to 0^+} \frac{r}{|x|} = 0.$$
6. By the previous questions $T_0 < \infty$ a.s. thus $X$ remains a.s. in $D$, and since $u = |\cdot|^{-1}$ is harmonic on $D$, we get that $u(X) = |X|^{-1} = (|x + B|^1)_{t \geq 0}$ is a non-negative local martingale. By using a localizing sequence and the Fatou lemma, it is a non-negative super-martingale. Therefore it converges a.s. to an integrable random variable, hence, as $t \to \infty$, $|x + B_t|^{-1}$ converges a.s. and thus $|x + B_t|$ converges a.s. in $[0, +\infty]$, and since this convergence holds also in law, this law can only be $\delta_{0\infty}$.

7. Let us show that $X = |x + B|^{-1}$ is bounded in $L^2$. By rotational invariance and scaling of $B_t$, we can assume without loss of generality that $x = (1, 0, 0)$. Since $x + B_t \sim \mathcal{N}(x, t I_3)$, using spherical coordinates $y_1 = r \cos(\theta) \sin(\phi)$, $y_2 = r \sin(\theta) \sin(\phi)$, $y_3 = r \cos(\phi)$, with $r \in [0, \infty)$, $\theta \in [0, 2\pi)$, $\phi \in [0, \pi)$, we have $dy = r^2 \sin(\phi)d\phi d\theta d\phi$, and for all $t > 0$,

$$E(|X_t|^2) = (2\pi t)^{-3/2} \int_0^\infty |y|^{-2} e^{-\frac{r^2 y_1^2 + y_2^2 + y_3^2}{2t}} dy = (2\pi t)^{-3/2} \int_0^\infty \int_0^\pi \int_0^\pi r^{-2} e^{-\frac{r^2 \sin^2(\theta) \sin^2(\phi) + r^2 \cos^2(\theta) \sin^2(\phi)}{2t}} r^2 \sin(\phi) dr d\theta d\phi$$

$$= (2\pi t)^{-3/2} \int_0^\infty \int_0^\pi \int_0^\pi e^{-\frac{r^2 \sin^2(\theta) \sin^2(\phi) + r^2 \cos^2(\theta) \sin^2(\phi)}{2t}} \sin(\phi) dr d\theta d\phi$$

$$= (2\pi t)^{-1/2} t^{-3/2} \int_0^\infty \int_0^\pi e^{-\frac{r^2}{2t}} \sin(\phi) dr d\theta d\phi$$

$$= (2\pi t)^{-1/2} t^{-3/2} e^{-\frac{1}{2t}} \int_0^\infty \int_0^\pi e^{-\frac{r^2}{2t}} \left(\int_1^r \frac{e^{\nu^2}}{\nu} d\nu\right) d\nu d\theta d\phi$$

$$= (2\pi t)^{-1/2} t^{-3/2} e^{-\frac{1}{2t}} \int_0^\infty \int_0^\pi e^{-\frac{r^2}{2t}} \frac{\sin(\frac{r}{t})}{\frac{r}{t}} d\nu d\theta d\phi$$

$$= 2(2\pi t)^{-1/2} t^{-3/2} e^{-\frac{1}{2t}} \int_0^\infty \int_0^\pi e^{-\frac{r^2}{2t}} \sin(\frac{r}{t}) d\nu d\theta d\phi$$

$$= 2(2\pi t)^{-1/2} t^{-3/2} e^{-\frac{1}{2t}} \int_0^\infty \int_0^\pi e^{-\frac{r^2}{2t}} \frac{1}{n+1} \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} \int_0^\infty \left(\frac{r}{t}\right)^{2n} e^{-\frac{r^2}{t}} dr$$

$$= t^{-1} e^{-\frac{1}{2t}} \sum_{n=0}^{\infty} \frac{r^{-2n}}{(2n+1)!} \int_0^\infty \left(\frac{r}{t}\right)^{2n} e^{-\frac{r^2}{t}} dr$$

$$= t^{-1} e^{-\frac{1}{2t}} \sum_{n=0}^{\infty} \frac{r^{-2n}}{(2n+1)!} \int_0^\infty \left(\frac{r}{t}\right)^{2n} e^{-\frac{r^2}{t}} dr$$

$$= (2t)^{-1/2} \sum_{n=0}^{\infty} \frac{(2t)^{-(n+1)}}{(2n+1)!} \int_0^\infty \left(\frac{r}{t}\right)^{2n} e^{-\frac{r^2}{t}} dr$$

$$= 2e^{-\frac{1}{2t}} \sum_{n=0}^{\infty} \frac{(2t)^{-(n+1)}}{(2n+1)!} \int_0^\infty \left(\frac{r}{t}\right)^{2n} e^{-\frac{r^2}{t}} dr$$

$$\leq 2e^{-\frac{1}{2t}} \sum_{n=0}^{\infty} \frac{(2t)^{-(n+1)}}{(2n+1)!} \int_0^\infty \left(\frac{r}{t}\right)^{2n} e^{-\frac{r^2}{t}} dr$$

$$\leq 2e^{-\frac{1}{2t}} \sum_{n=0}^{\infty} \frac{(2t)^{-(n+1)}}{(2n+1)!} \int_0^\infty \left(\frac{r}{t}\right)^{2n} e^{-\frac{r^2}{t}} dr$$

8. By a previous question, a.s. $X$ takes its values in $[0, \infty)$ and $|\cdot|^{-1}$ is harmonic on this domain, and this implies that $|X|^{-1} = |x + B|^{-1}$ is a local martingale. Moreover $|X|^{-1}$ is u.i.

Now, suppose that $Y = |X|^{-1}$ is a martingale. Since it is u.i. $\lim_{t \to \infty} Y_t = Y_\infty$ a.s. and in $L^1$, with $Y_\infty \geq 0$ and $Y_\infty \in L^1$. Moreover $E(Y_\infty) = E(Y_0) = |x|^{-1} > 0$. But we know from a previous question that a.s. $\lim_{t \to \infty} |B_t| = +\infty$, which gives that a.s. $Y_\infty = 0$, thus $E(Y_\infty) = 0$, a contradiction.

Alternatively, we could use Doob stopping for u.i. martingales, with the u.i. martingale $Y$ and the stopping time $T_R$, which is a.s. finite, this gives $|x|^{-1} = E(Y_t) = E(Y_{T_R}) = R^{-1}$ which is impossible.

Note that from the first question, $Y$ cannot be dominated by an integrable random variable!

It can be shown that the process $Y$ solves the SDE $dY_t = -Y_t^2 dW_t$.

Explicit computations show that $E(Y_t) \to 0$, and this is another way to show that $Y$ is not a martingale!
Exercise 4 (Strict local martingales and stochastic integrals).

1. Give an example of an Itô stochastic integral which is a local martingale but not a martingale, without using the previous exercise.

Elements of solution for Exercise 4.

1. Of course we could consider the trivial example \[ \int_{0}^{t} dY_s = Y_t - Y_0 \] where \( Y \) is the strict local martingale considered in the previous exercise, but a deeper understanding is expected here!

A more interesting idea relies on the stochastic integral

\[ I_B(\varphi) = \int_{0}^{\bullet} \varphi_s dB_s \]

where \( \varphi \) is the single step function \( \varphi = U1_{(0,1]} \) where \( U \) is an \( \mathcal{F}_0 \) measurable random variable. A property of the Itô stochastic integral for semi-martingale integrators (here \( B \)) gives

\[ I_B(\varphi) = UB_{\bullet^+1} - UB_0 = UB_{\bullet^+1}. \]

Now if we take \( U \) independent of \( B \), then, in \([0, +\infty)\),

\[ \mathbb{E}(|I_B(\varphi)_1|) = \mathbb{E}(|U|)\mathbb{E}(|B_1|). \]

Thus, if \( U \) is not integrable then \( I_B(\varphi)_1 \) is not integrable and thus \( I_B(\varphi) \) is not a martingale.

References