Exam 2020/2021

October 28, 2020, from 13:45 to 16:45
Documents allowed, Internet not allowed
Do what you can, and do not worry

$(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$ is a filtered probability space, with complete and right continuous filtration. $B = (B_t)_{t\geq 0}$ is a d-dimensional Brownian motion issued from the origin, $d \geq 1$.

Exercise 1 (Representation of a process)**.** Take $d = 1$ and $x \in \mathbb{R}$.

1. Recall the computations and reasoning showing that the process $(Z_t)_{t\geq 0}$ defined by
$$Z_t = x\mathrm{e}^{-t} + \mathrm{e}^{-t} M_t \quad \text{where} \quad M_t = \sqrt{2}\int_0^t \mathrm{e}^s \mathrm{d}B_s$$
is the unique solution of the stochastic differential equation $Z_0 = x$, $\mathrm{d}Z_t = \sqrt{2}\mathrm{d}B_t - Z_t \mathrm{d}t$.
2. Show that for all $t \geq 0$, $Z_t \overset{\text{law}}{=} x\mathrm{e}^{-t} + \mathrm{e}^{-t} B_{\mathrm{e}^{2t}-1}$.
3. Can we have, for all $t \geq 0$, $Z_t = x\mathrm{e}^{-t} + \mathrm{e}^{-t} B_{\mathrm{e}^{2t}-1}$?
4. Show that the process $(M_t)_{t\geq 0}$ is a continuous local martingale with, for all $t \geq 0$, $\langle M\rangle_t = \mathrm{e}^{2t} - 1$.
5. Deduce that there exists a Brownian motion $(W_t)_{t\geq 0}$ such that for all $t \geq 0$, $Z_t = x\mathrm{e}^{-t} + \mathrm{e}^{-t} W_{\mathrm{e}^{2t}-1}$.

Elements of solution for Exercise 1. Ornstein – Uhlenbeck and Dubins – Schwarz!

1. The Itô formula $X_t Y_t = X_0 Y_0 + \int_0^t (X_s \mathrm{d}Y_s + Y_s \mathrm{d}X_s)$ gives, with $(X_t, Y_t) = (\mathrm{e}^{-t}, M_t)$,
$$\mathrm{e}^{-t} M_t = \sqrt{2}\int_0^t \mathrm{e}^{-s}\mathrm{e}^s \mathrm{d}B_s - \int_0^t \mathrm{e}^{-s} M_s \mathrm{d}s = \sqrt{2}B_t - \int_0^t \mathrm{e}^{-s} M_s \mathrm{d}s$$
which gives
$$x\mathrm{e}^{-t} + \mathrm{e}^{-t} M_t = x + \sqrt{2}B_t - \int_0^t (x\mathrm{e}^{-s} + \mathrm{e}^{-s} M_s)\mathrm{d}s.$$
On the other hand, to establish the uniqueness, let us suppose that $(X_t)_{t\geq 0}$ is a continuous semi-martingale solution of the SDE. Then by combining the Itô formula above for the function $f(u, v) = uv$ and the semi-martingale (e^t, X_t), and the SDE satisfied by X, we obtain
$$\mathrm{e}^t X_t - x = \int_0^t \mathrm{e}^s \mathrm{d}X_s + \int_0^t \mathrm{e}^s X_s \mathrm{d}s = \sqrt{2}\int_0^t \mathrm{e}^s \mathrm{d}B_s.$$
2. Since M_t is a Wiener integral, it is Gaussian with zero mean and variance $2\int_0^t \mathrm{e}^{2s}\mathrm{d}s = \mathrm{e}^{2t} - 1$, which gives $Z_t \sim \mathcal{N}(x\mathrm{e}^{-t}, 1 - \mathrm{e}^{-2t})$, and it turns out that this is also the law of $x\mathrm{e}^{-t} + \mathrm{e}^{-t} B_{\mathrm{e}^{2t}-1}$.
3. No because the process on the right hand side would not be adapted. Indeed the random variable $B_{\mathrm{e}^{2t}-1}$ is $\mathscr{F}_{\mathrm{e}^{2t}-1}$-measurable instead of being $\mathscr{F}_t$-measurable.
4. The process M is a Wiener(–Itô) integral, in particular it is a Gaussian square integrable martingale, and in particular a local martingale. Moreover $\langle M\rangle_t = \mathbb{E}(M_t^2) = 2\int_0^t \mathrm{e}^{2s}\mathrm{d}s = \mathrm{e}^{2t} - 1$.
5. The process M is a continuous local martingale with respect to the filtration $(\mathscr{F}_t)_{t\geq 0}$ with $M_0 = 0$ and $\langle M\rangle_\infty = \infty$. Therefore the Dubins – Schwarz theorem states that $(W_t)_{t\geq 0} = (M_{T_t})_{t\geq 0}$ where $T_t = \inf\{s \geq 0 : \langle M\rangle_s > t\}$ is a Brownian motion for the filtration $(\mathscr{F}_{T_t})_{t\geq 0}$, and $(W_{\langle M\rangle_t})_{t\geq 0} = (M_t)_{t\geq 0}$.

Exercise 2 (Study of a special process)**.** Let $d = 1$, $\alpha \geq 0$, $x \geq 0$. Let X be a continuous semi-martingale taking values in $\mathbb{R}_+$ and solving the stochastic differential equation:
$$X_t = x + 2\int_0^t \sqrt{X_s}\mathrm{d}B_s + \alpha t, \quad t \geq 0.$$
Let $f : [0, +\infty) \to [0, +\infty)$ be continuous and $\varphi : [0, +\infty) \to (0, +\infty)$ be positive and $\mathscr{C}^2$, solving the ordinary differential equation $\varphi'' = 2f\varphi$ with boundary conditions $\varphi(0) = 1$ and $\varphi'(1) = 0$. Note that $\varphi > 0$.

1. Could you give an explicit example of process X for special values of α?
2. Show that φ decreases on the interval $[0,1]$
3. Show that $u=\varphi'/(2\varphi)$ solves the differential equation $u'+2u^2=f$
4. Show that for all $t\geq 0$,
$$u(t)X_t-\int_0^t f(s)X_s\mathrm{d}s=u(0)x+\int_0^t u(s)\mathrm{d}X_s-2\int_0^t u(s)^2X_s\mathrm{d}s.$$
5. For all $t\geq 0$, let us define $Y_t=u(t)X_t-\int_0^t f(s)X_s\mathrm{d}s$. Show that
$$\varphi(t)^{-\frac{\alpha}{2}}\mathrm{e}^{Y_t}=\mathrm{e}^{N_t-\frac{1}{2}\langle N\rangle_t}\quad\text{where}\quad N_t=u(0)x+2\int_0^t u(s)\sqrt{X_s}\mathrm{d}B_s$$
6. Show that
$$\mathbb{E}\exp\Big(-\int_0^1 f(s)X_s\mathrm{d}s\Big)=\varphi(1)^{\frac{\alpha}{2}}\mathrm{e}^{\frac{x}{2}\varphi'(0)}$$
7. From now on, let $\lambda>0$. Prove that
$$\mathbb{E}\exp\Big(-\lambda\int_0^1 X_s\mathrm{d}s\Big)=(\cosh(\sqrt{2\lambda}))^{-\frac{\alpha}{2}}\mathrm{e}^{-\frac{x}{2}\sqrt{2\lambda}\tanh\sqrt{2\lambda}}$$
8. Prove that for all $\lambda>0$ and $y\in\mathbb{R}$,
$$\mathbb{E}\exp\Big(-\lambda\int_0^1(y+B_s)^2\mathrm{d}s\Big)=(\cosh(\sqrt{2\lambda}))^{-\frac{1}{2}}\mathrm{e}^{-\frac{y^2}{2}\sqrt{2\lambda}\tanh\sqrt{2\lambda}}$$

Elements of solution for Exercise 2. This is on squared Bessel processes : [1, Exercise 5.31 p. 145–146].

1. We know from the course (Itô+Lévy) that if $\alpha=n\in\{1,2,\ldots\}$ then X has the law of $|x+W|^2$ where W is a n-dimensional Brownian motion issued from the origin (squared Bessel process).
2. We have $\varphi'(1)=0$ and $\varphi''\geq 0$ hence $\varphi'\leq 0$ on $[0,1]$.
3. We have $u'=\frac{\varphi''\varphi-\varphi'^2}{2\varphi^2}$ thus
$$u'+2u^2=\frac{\varphi''\varphi-\varphi'^2}{2\varphi^2}+\frac{\varphi'^2}{2\varphi^2}=\frac{\varphi''}{2\varphi}=f.$$
4. The Itô formula for $F(x_1,x_2)=x_1x_2$ and $(u(t),X_t)$ gives
$$\begin{aligned}u(t)X_t-u(0)x&=\int_0^t X_su'(s)\mathrm{d}s+\int_0^t u(s)\mathrm{d}X_s\\&=\int_0^t X_s(f(s)-2u^2(s))\mathrm{d}s+\int_0^t u(s)\mathrm{d}X_s\end{aligned}$$
and it remains to use the result of the previous question.
5. We have, using the SDE solved by X for the second step, and in the third step the definition of Y and the previous question,
$$\begin{aligned}N_t-\frac{1}{2}\langle N\rangle_t&=u(0)x+2\int_0^t u(s)\sqrt{X_s}\mathrm{d}B_s-2\int_0^t u(s)^2X_s\mathrm{d}s\\&=u(0)x+\int_0^t u(s)(\mathrm{d}X_s-\alpha\mathrm{d}s)-2\int_0^t u(s)^2X_s\mathrm{d}s\\&=Y_t-\alpha\int_0^t u(s)\mathrm{d}s.\end{aligned}$$
Since
$$\int_0^t u(s)\mathrm{d}s=\int_0^t\frac{\varphi'(s)}{2\varphi(s)}\mathrm{d}s=\frac{\log\varphi(1)-\log\varphi(0)}{2}=\frac{\log\varphi(1)}{2},$$
we obtain
$$\mathrm{e}^{N_t-\frac{1}{2}\langle N\rangle_t}=\mathrm{e}^{Y_t}\varphi(1)^{-\alpha t}.$$

6. The process $\mathrm{e}^{N-\frac{1}{2}\langle N\rangle}$ is a Doléans-Dade exponential, hence a continuous local martingale. In order to show that it is a martingale for $t\in[0,1]$, it suffices to show that it is dominated by an integrable random variable. From the previous computations, for all $t\in[0,1]$,

$$\mathrm{e}^{N_t-\frac{1}{2}\langle N\rangle_t}=\mathrm{e}^{Y_t-\alpha\int_0^t u(s)\mathrm{d}s}.$$

Now, $Y_t=u(t)X_t-\int_0^t f(s)X_s\mathrm{d}s\leq 0$ since $u\leq 0$ (recall that $\varphi'\leq 0$), while $X,f\geq 0$.
Hence $(\mathrm{e}^{N_t-\frac{1}{2}\langle N\rangle_t})_{t\in[0,1]}$ is a martingale.
Next $u(1)=\varphi'(1)/(2\varphi(1))=0$, we get, from the previous question with $t=1$,

$$\begin{aligned}\varphi(1)^{-\frac{\alpha}{2}}\mathbb{E}\mathrm{e}^{-\int_0^1 f(s)X_s\mathrm{d}s}&=\mathbb{E}(\mathrm{e}^{N_t-\frac{1}{2}\langle N\rangle_t})\\&=\mathbb{E}(\mathrm{e}^{N_0-\frac{1}{2}\langle N\rangle_0})\\&=\mathrm{e}^{u(0)x}\\&=\mathrm{e}^{x\frac{\varphi'(0)}{2\varphi(0)}}\\&=\mathrm{e}^{\frac{x}{2}\varphi'(0)}.\end{aligned}$$

7. We take f constant and equal to $\lambda>0$. The differential equation solved by φ writes $\varphi''=2\lambda\varphi$ with $\varphi(0)=1$ and $\varphi'(1)=0$. The associated equation has two roots $\pm\sqrt{2\lambda}$ hence $\varphi(x)=\alpha\mathrm{e}^{\sqrt{2\lambda}x}+\beta\mathrm{e}^{-\sqrt{2\lambda}x}$. The boundary conditions give $\alpha+\beta=1$ and $\alpha\mathrm{e}^{2\sqrt{2\lambda}}=\beta$, hence

$$\alpha=\frac{1}{1+\mathrm{e}^{2\sqrt{2\lambda}}}=\frac{\mathrm{e}^{-\sqrt{2\lambda}}}{2\cosh(\sqrt{2\lambda})}\quad\text{and}\quad\beta=\frac{\mathrm{e}^{2\sqrt{2\lambda}}}{1+\mathrm{e}^{2\sqrt{2\lambda}}}=\frac{\mathrm{e}^{\sqrt{2\lambda}}}{2\cosh(\sqrt{2\lambda})}.$$

This gives

$$\varphi(1)=\frac{1}{\cosh(\sqrt{2\lambda})}\quad\text{and}\quad\varphi'(0)=\sqrt{2\lambda}(\alpha-\beta)=-\sqrt{2\lambda}\tanh(\sqrt{2\lambda}).$$

8. If we take $\alpha=1$ then X has the law of the squared Bessel process $(x+B)^2$.

Exercise 3 (Strict local martingales). We take $d=3$, $X=x+B$, $0<r<|x|<R<\infty$, and, for all $a\geq 0$,

$$T_a=\inf\{t\geq 0:|X_t|=a\}.$$

1. Show that if $M=(M_t)_{t\geq 0}$ is a continuous local martingale with for all $t\geq 0$, $|M_t|\leq U$ where $U\in\mathrm{L}^1$, then M is a martingale. Does it remain true if the domination condition is replaced by "M is u.i."?
2. Show that if $Z=(Z_t)_{t\geq 0}$ is d-dimensional, adapted, taking values in an open set $D\subset\mathbb{R}^d$, such that its components are continuous local martingales, and for all $1\leq j,k\leq d$, $\langle Z^j,Z^k\rangle=V\mathbf{1}_{j=k}$ for a finite variation process V, then, for all harmonic $u:D\to\mathbb{R}$, the process $u(Z)$ is a local martingale.
3. Show that $|\bullet|^{-1}$ is harmonic on $\mathbb{R}^3\setminus\{0\}$.
4. Show that $T_R<\infty$ almost surely and

$$\mathbb{P}(T_r<T_R)=\frac{R^{-1}-|x|^{-1}}{R^{-1}-r^{-1}}.$$

5. Deduce from the previous formula that a.s. for all $t\geq 0$, $X_t\neq 0$.
6. Show that a.s. $\lim_{t\to\infty}|B_t|=+\infty$. Hint: show that $|X|^{-1}$ is a non-negative super-martingale.
7. Show that $|X|^{-1}$ is bounded in L^2. Hint: density of B_t in spherical coordinates.
8. Show that $|X|^{-1}$ is a continuous local martingale, but is not a martingale.

Elements of solution for Exercise 3. The goal is to construct a local martingale which is u.i. but which is not a martingale. The example chosen here is a non-centered Bessel process. This example of a strict local martingale is quite classical, and corresponds for instance to [1, Exercise 5.33(8) page 148].

1. For all $t \geq 0$, the measurability and integrability of M_t comes from the adaptation of M and the domination assumption. Let $(T_n)_{n\geq 0}$ be a localizing sequence for M. For all $n \geq 0$, M^{T_n} is a martingale: for all $0 \leq s \leq t$, $\mathbb{E}(M_{T_n \wedge t} \mid \mathscr{F}_s) = M_{T_n \wedge s}$. Now since $\lim_{n\to\infty} T_n = +\infty$ a.s. and since M is continuous, if follows that a.s. for all $t \geq 0$, $\lim_{n\to\infty} M_{T_n\wedge t} = M_t$. It remains to use dominated convergence to get that $\lim_{n\to\infty} \mathbb{E}(M_{T_n\wedge t} \mid \mathscr{F}_s) = \mathbb{E}(M_t \mid \mathscr{F}_s)$. Hence M is a martingale. Finally, if M is u.i. instead of being dominated, then the argument is no longer valid since it does not give a way to handle $M_{T_n\wedge t}$. The goal of the exercise is precisely the construction of an u.i. continuous local martingale which is not a martingale! Note: domination implies u.i. but the converse is wrong.
2. The Itô formula is licit on an open domain D for a process taking values in D. It gives, for all $t \geq 0$,
$$u(X_t) = u(X_0) + \int_0^t \nabla u(X_s)\mathrm{d}X_s + \frac{1}{2}\int_0^t \Delta u(X_s)\mathrm{d}V_s.$$
The last integral vanishes since $\Delta u = 0$ and X takes values on D, thus $u(X)$ is a local martingale.
3. For all $y \in D$, denoting $u = |\bullet|^{-1}$, we obtain $\Delta u(y) = 0$ from
$$\partial_i u(y) = (2-d)\frac{y_i}{|y|^d}, \quad \text{and} \quad \partial^2_{i,i} u(y) = \frac{d(d-2)}{2}\frac{|y|^d - dy_i^2|y|^{d-2}}{|y|^{d+2}}.$$
4. Set $Z = X^{T_r}$. Then $\langle Z^j, Z^k\rangle_t = \langle B^j, B^k\rangle_{t\wedge T_r} = (t\wedge T_r)\mathbf{1}_{j=k}$. Hence, by a previous question with the processes Z and $V_t = t \wedge T_r$, the harmonic function $u = |\bullet|^{-1}$, and $D = \{x \in \mathbb{R}^3 : x \neq 0\}$, we get that
$$u(Z) = (u(X_{t\wedge T_r}))_{t\geq 0}$$
is a local martingale, and since it is bounded by r^{-1}, it is a bounded martingale.
Next, since a 1-dimensional BM escapes almost surely from every finite interval, the first component of our 3-dimensional Brownian motion $x + B$ escapes almost surely from $[-R, R]$, and thus almost surely $T_R < \infty$. In particular almost surely $T_r < T_R$ or $T_r > T_R$ and we cannot have $T_r = T_R = \infty$. We have thus the immediate equation
$$1 = \mathbb{P}(T_r < T_R) + \mathbb{P}(T_r > T_R).$$
By the Doob stopping theorem for the bounded martingale $u(Z)$ and the finite stopping time T_R,
$$|x|^{-1} = \mathbb{E}(u(Z_0)) = \mathbb{E}(u(Z_{T_R})) = \mathbb{E}(|X_{T_r\wedge T_R}|^{-1}) = r^{-1}\mathbb{P}(T_r < T_R) + R^{-1}\mathbb{P}(T_r > T_R).$$
It remains to solve the system of equations to get the desired formula.
5. We have $T_r < T_R$ if $R > \sup_{s\in[0,X_{T_r}]} |X_s|$, hence
$$\{T_r < T_R\} \underset{R\to\infty}{\nearrow} \{T_r < \infty\}.$$
It follows that
$$\mathbb{P}(T_r < T_R) \underset{R\nearrow\infty}{\nearrow} \mathbb{P}(T_r < \infty)$$
and thus, from the formula provided by the previous question,
$$\mathbb{P}(T_r < \infty) = \lim_{R\to\infty} \mathbb{P}(T_r < T_R) = \frac{|x|^{-1}}{r^{-1}} = \frac{r}{|x|}.$$
Now a.s. X is continuous and therefore
$$\{T_r < \infty\} \underset{r\searrow 0^+}{\searrow} \{T_0 < \infty\}$$
and thus
$$\mathbb{P}(T_0 < \infty) = \lim_{r\to 0^+} \mathbb{P}(T_r < \infty) = \lim_{r\to 0^+} \frac{r}{|x|} = 0.$$

6. By the previous questions $T_0 < \infty$ a.s. thus X remains a.s. in D, and since $u = |\bullet|^{-1}$ is harmonic on D, we get that $u(X) = |X|^{-1} = (|x+B|^{-1})_{t\geq 0}$ is a non-negative local martingale. By using a localizing sequence and the Fatou lemma, it is a non-negative super-martingale. Therefore it converges a.s. to an integrable random variable, hence, as $t \to \infty$, $|x+B_t|^{-1}$ converges a.s. and thus $|x+B_t|$ converges a.s. in $[0,+\infty]$, and since this convergence holds also in law, this law can only be δ_∞.
7. Let us show that $X = |x+B|^{-1}$ is bounded in L^2. By rotational invariance and scaling of B_t, we can assume without loss of generality that $x = (1,0,0)$. Since $x + B_t \sim \mathcal{N}(x, tI_3)$, using spherical coordinates $y_1 = r\cos(\theta)\sin(\varphi)$, $y_2 = r\sin(\theta)\sin(\varphi)$, $y_3 = r\cos(\varphi)$, with $r \in [0,\infty)$, $\theta \in [0,2\pi)$, $\varphi \in [0,\pi)$, we have $\mathrm{d}y = r^2\sin(\varphi)\mathrm{d}r\mathrm{d}\theta\mathrm{d}\varphi$, and for all $t > 0$,

$$
\begin{aligned}
\mathbb{E}(|X_t|^2) &= (2\pi t)^{-3/2}\int_{\mathbb{R}^3}|y|^{-2}\mathrm{e}^{-\frac{y_1^2+y_2^2+(y_3-1)^2}{2t}}\mathrm{d}y\\
&= (2\pi t)^{-3/2}\int_0^\infty\int_0^{2\pi}\int_0^\pi r^{-2}\mathrm{e}^{-\frac{r^2\sin(\varphi)^2+(r\cos(\varphi)-1)^2}{2t}}r^2\sin(\varphi)\mathrm{d}r\mathrm{d}\theta\mathrm{d}\varphi\\
&= (2\pi)^{-1/2}t^{-3/2}\int_0^\infty\int_0^\pi \mathrm{e}^{-\frac{r^2\sin(\varphi)^2+(r\cos(\varphi)-1)^2}{2t}}\sin(\varphi)\mathrm{d}r\mathrm{d}\varphi\\
&= (2\pi)^{-1/2}t^{-3/2}\int_0^\infty\int_0^\pi \mathrm{e}^{-\frac{r^2-2r\cos(\varphi)+1}{2t}}\sin(\varphi)\mathrm{d}r\mathrm{d}\varphi\\
&= (2\pi)^{-1/2}t^{-3/2}\mathrm{e}^{-\frac{1}{2t}}\int_0^\infty \mathrm{e}^{-\frac{r^2}{2t}}\Big(\int_{-1}^1\mathrm{e}^{\frac{ru}{t}}\mathrm{d}u\Big)\mathrm{d}r\\
&= (2\pi)^{-1/2}t^{-3/2}\mathrm{e}^{-\frac{1}{2t}}\int_0^\infty \mathrm{e}^{-\frac{r^2}{2t}}\Big[\frac{t}{r}\mathrm{e}^{\frac{ru}{t}}\Big]_{u=-1}^{u=1}\mathrm{d}r\\
&= 2(2\pi)^{-1/2}t^{-3/2}\mathrm{e}^{-\frac{1}{2t}}\int_0^\infty \mathrm{e}^{-\frac{r^2}{2t}}\frac{\sinh(\frac{r}{t})}{\frac{r}{t}}\mathrm{d}r\\
&= 2(2\pi)^{-1/2}t^{-3/2}\mathrm{e}^{-\frac{1}{2t}}\sum_{n=0}^\infty\frac{1}{(2n+1)!}\int_0^\infty\Big(\frac{r}{t}\Big)^{2n}\mathrm{e}^{-\frac{r^2}{2t}}\mathrm{d}r\\
&= t^{-1}\mathrm{e}^{-\frac{1}{2t}}\sum_{n=0}^\infty\frac{t^{-2n}}{(2n+1)!}(2\pi t)^{-1/2}\int_{-\infty}^\infty r^{2n}\mathrm{e}^{-\frac{r^2}{2t}}\mathrm{d}r\\
&= t^{-1}\mathrm{e}^{-\frac{1}{2t}}\sum_{n=0}^\infty\frac{t^{-2n}}{(2n+1)!}t^n\frac{(2n-1)!}{2^{n-1}(n-1)!}\\
&= t^{-1}\mathrm{e}^{-\frac{1}{2t}}\sum_{n=0}^\infty\frac{(2t)^{-n}}{(2n+1)n!}\\
&= 2\mathrm{e}^{-\frac{1}{2t}}\sum_{n=0}^\infty\frac{(2t)^{-(n+1)}}{(2n+1)n!}\\
&\leq 2\mathrm{e}^{-\frac{1}{2t}}\sum_{n=0}^\infty\frac{(2t)^{-(n+1)}}{(n+1)!} = 2\mathrm{e}^{-\frac{1}{2t}}(\mathrm{e}^{\frac{1}{2t}}-1)\leq 2.
\end{aligned}
$$

8. By a previous question, a.s. X takes its values in $\mathbb{R}^3\setminus\{0\}$ and $|\bullet|^{-1}$ is harmonic on this domain, and this implies that $|X|^{-1} = |x+B|^{-1}$ is a local martingale. Moreover $|X|^{-1}$ is u.i.

   Now, suppose that $Y = |X|^{-1}$ is a martingale. Since it is u.i. $\lim_{t\to\infty} Y_t = Y_\infty$ a.s. and in L^1, with $Y_\infty \geq 0$ and $Y_\infty \in \mathrm{L}^1$. Moreover $\mathbb{E}(Y_\infty) = \mathbb{E}(Y_0) = |x|^{-1} > 0$. But we know from a previous question that a.s. $\lim_{t\to\infty}|B_t| = +\infty$, which gives that a.s. $Y_\infty = 0$, thus $\mathbb{E}(Y_\infty) = 0$, a contradiction.

   Alternatively, we could use Doob stopping for u.i. martingales, with the u.i. martingale Y and the stopping time T_R, which is a.s. finite, this gives $|x|^{-1} = \mathbb{E}(Y_0) = \mathbb{E}(Y_{T_R}) = R^{-1}$ which is impossible.

   Note that from the first question, Y cannot be dominated by an integrable random variable!

It can be shown that the process Y solves the SDE $\mathrm{d}Y_t = -Y_t^2\mathrm{d}W_t$.
Explicit computations show that $\mathbb{E}(Y_t) \underset{t\to\infty}{\searrow} 0$, and this is another way to show that Y is not a martingale!

Exercise 4 (Strict local martingales and stochastic integrals)**.**

1. Give an example of an Itô stochastic integral which is a local martingale but not a martingale, without using the previous exercise.

Elements of solution for Exercise 4.

1. Of course we could consider the trivial example $\int_0^t \mathrm{d}Y_s = Y_t - Y_0$ where Y is the strict local martingale considered in the previous exercise, but a deeper understanding is expected here!

   A more interesting idea relies on the stochastic integral

   $$I_B(\varphi) = \int_0^\bullet \varphi_s \mathrm{d}B_s$$

   where φ is the single step function $\varphi = U\mathbf{1}_{(0,1]}$ where U is an $\mathscr{F}_0$ measurable random variable. A property of the Itô stochastic integral for semi-martingale integrators (here B) gives

   $$I_B(\varphi) = UB_{\bullet\wedge 1} - UB_0 = UB_{\bullet\wedge 1}.$$

   Now if we take U independent of B, then, in $[0,+\infty]$,

   $$\mathbb{E}(|I_B(\varphi)_1|) = \mathbb{E}(|U|)\mathbb{E}(|B_1|).$$

   Thus, if U is not integrable then $I_B(\varphi)_1$ is not integrable and thus $I_B(\varphi)$ is not a martingale.

– oOo –

References

[1] Jean-François Le Gall. *Brownian motion, martingales, and stochastic calculus*, volume 274 of *Graduate Texts in Mathematics*. Springer, french edition, 2016.