Exercise 1. Assume that \( d = 1 \). Let \( \sigma > 0, \rho \in \mathbb{R} \), and \( x \in \mathbb{R} \) be fixed parameters.

1. Solve the ODE \( X_0 = x \) and \( X'(t) = \rho X(t) \) and discuss its sign depending on \( x \).
2. Solve the SDE \( X_0 = x \) and \( dX_t = \rho X_t \, dt + \sigma X_t \, dB_t \) (existence, uniqueness, explicit formula).

Exercise 2. Let \( \theta > 0, \rho \in \mathbb{R}, z \in \mathbb{R}^d \) be parameters, and let \( Z^z \) be the solution of

\[
Z^z_0 = z, \quad dZ^z_t = \theta dB_t - \rho Z^z_t \, dt
\]

1. Why this SDE has a pathwise unique solution? What is the name of the process \( Z^z \)?
2. Show that the process \( W_t = \int_0^t Z^z_s \, dB_s \) with the convention \( 0/0 = 1 \) is a Brownian motion.
3. Let us define \( x = |z|^2 \). Show that the process \( X^x_t = |Z^z_t|^2 \) solves the stochastic differential equation

\[
X^x_0 = x, \quad dX^x_t = \sigma \sqrt{X^x_t} \, dW_t + (a - bX^x_t) \, dt \quad \text{where} \quad \sigma = 2\theta, \ a = \theta^2 d, \ b = 2\rho.
\]

4. Show that if \( \rho > 0 \) then \( X^x_t \xrightarrow{\text{law}} \Gamma(d/2, 2b/\sigma^2) \). What happens when \( b \leq 0 \)?
5. From now on, we assume that \( X^x \) solves the SDE above for \( x \geq 0 \) and an arbitrary real parameter \( d > 0 \), without relation to \( Z^z \). Our goal is to evaluate \( P(T^x_0 < \infty), T^x_c = \inf \{ t \geq 0 : X^x_t = c \} \). Show that

\[
u \in (0, +\infty) \rightarrow \varphi(u) = \int_0^u v^{2\eta/\sigma^2} e^{\sigma^2 v} \, dv \quad \text{satisfies} \quad \frac{\sigma^2}{2} u \varphi''(u) + (a - bu) \varphi'(u) = 0.
\]

6. From now on, we take \( x > 0 \) and \( 0 < \epsilon < x < R \). Let us define \( T^x_{\epsilon, R} = T^x_\epsilon \wedge T^x_R \). Show that for all \( t > 0 \),

\[
\varphi(X^x_{\epsilon \wedge T^x_{\epsilon, R}}) = \varphi(x) + \int_0^{T^x_{\epsilon, R}} \varphi'(X^x_s) \sigma \sqrt{X^x_s} \, dW_s.
\]

7. Show that \( \mathbb{E}(T^x_{\epsilon, R}) < \infty \), which gives \( T_{\epsilon, R} < \infty \) a.s. (hint: use an isometry, and a lower bound on \( \varphi' \)).
8. Show that

\[
\varphi(x) = \varphi(\epsilon) P(T^x_\epsilon < T^x_R) + \varphi(R) P(T^x_R > T^x_\epsilon).
\]

9. Show that if \( a \geq \frac{\sigma^2}{2} \) then \( P(T^x_0 < \infty) = 0 \) (hint: use \( \lim_{u \to 0} \varphi(u) = -\infty \)).
10. Show that if \( 0 \leq a < \frac{\sigma^2}{2} \) and \( b \geq 0 \) then \( P(T^x_0 < \infty) = 1 \) (hint: use \( \lim_{R \to +\infty} \varphi(R) = +\infty \)).
11. Show that if \( 0 \leq a < \frac{\sigma^2}{2} \) and \( b < 0 \) then \( P(T^x_0 < \infty) \in (\varphi(\infty) - \varphi(x))/(|\varphi(\infty) - \varphi(0)|) \in (0, 1) \).

– The third and last exercise is on the opposite side of this page –

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1. If \( G \sim \mathcal{N}(0, \lambda) \) then \( |G|^2 \sim \chi^2(d) = \Gamma(d/2, 1/2) \). The law \( \Gamma(a, \lambda) \) has density \( u \mapsto \frac{\lambda^a}{\Gamma(a)} u^{a-1} e^{-\lambda u}, u \geq 0 \).
**Exercise 3.** Let $U \in \mathcal{C}^2(\mathbb{R}^d, \mathbb{R})$. In particular $-\nabla U$ is locally Lipschitz but is not globally Lipschitz in general.

Let us fix $x \in \mathbb{R}^d$. From the lecture notes, we recall and admit that there exists an adapted process $X$ with values in $\mathbb{R}^d \cup \{\infty\}$ and a stopping time $T$ with values in $(0, +\infty]$ such that

- $X_t \in \mathbb{R}^d$ if $t < T$ while $X_t = \infty$ if $t \geq T$, and $\lim_{t \to T^-} |X_t| = \infty$ on $\{T < \infty\}$
- $t \mapsto X_t \in \mathbb{R}^d$ is continuous
- $X_t = x + B_t - \int_0^t \nabla U(X_s) ds$ on the (maximal) time interval $[0, T)$

We study now a couple of sufficient criteria on $U$ in order to get $\mathbb{P}(T < \infty) = 0$ (no explosion in finite time).

1. Suppose that
   \[
   \lim_{|x| \to \infty} U(x) = +\infty \quad \text{and} \quad C_2 = \sup_{x \in \mathbb{R}^d} \left( \frac{1}{2} \Delta U - |\nabla U|^2 \right) < \infty.
   \]
   (a) Show that $T_R = \inf\{t \geq 0 : U(X_t) > R\} \nearrow T$.
   (b) Show that $Y = X_T = (X_{t \wedge T_R})_{t \geq 0}$ solves the following SDE
      \[
      Y_t = x + \int_0^t \mathbf{1}_{s \leq T_R} dB_s - \int_0^t \mathbf{1}_{s \leq T_R} \nabla U(X_s) ds, \quad t \geq 0.
      \]
   (c) Show that for all $R > 0$ and $t > 0$,
      \[
      \mathbb{E}(U(X_{t \wedge T_R})) = U(x) + \mathbb{E}\left( \int_0^{t \wedge T_R} \left( \frac{1}{2} \Delta U - |\nabla U|^2 \right)(X_s) ds \right).
      \]
   (d) Show that $C_1 = \inf_{R \geq 0} U > -\infty$ and, for all $R > 0$ and $t > 0$,
      \[
      R1_{T_R \leq t} - |C_1| \leq U(X_{t \wedge T_R}).
      \]
   (e) Show that for all $R > 0$ and $t > 0$,
      \[
      \mathbb{E}(R1_{T_R \leq t} - |C_1| - U(x)) \leq \mathbb{E}(C_2 (t \wedge T_R)) \leq t.
      \]
   (f) Show that $\mathbb{P}(T < \infty) = 0$.

2. Suppose that for some $a, b \in \mathbb{R}$ and all $x \in \mathbb{R}^d$,
   \[
   \langle x, \nabla U(x) \rangle \geq -a|x|^2 - b.
   \]
   (a) Show that
      \[
      T_n = \inf\{t \geq 0 : |X_t|^2 > n\} \nearrow T.
      \]
   (b) Show that $Y = X_T = (X_{t \wedge T_n})_{t \geq 0}$ solves the following SDE
      \[
      Y_t = x + \int_0^t \mathbf{1}_{s \leq T_n} dB_s - \int_0^t \mathbf{1}_{s \leq T_n} \nabla U(X_s) ds, \quad t \geq 0.
      \]
   (c) Show that for all $t \geq 0$ and $n \geq 1$,
      \[
      \mathbb{E}(|X_{t \wedge T_n}|^2) \leq |x|^2 + (1 + 2|b|) t + 2a \int_0^t \mathbb{E}(|X_{s \wedge T_n}|^2) ds.
      \]
   (d) Show that for all $t \geq 0$ and $n \geq 1$,
      \[
      \mathbb{E}(|X_{t \wedge T_n}|^2) \leq (|x|^2 + (1 + 2|b|) t) e^{2|a| t}.
      \]
   (e) Show that $\mathbb{P}(T < \infty) = 0$.

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