We use the notations of the lecture notes.

Exercise 1. Assume that $d = 1$. Let $\sigma > 0$, $\rho \in \mathbb{R}$, and $x \in \mathbb{R}$ be fixed parameters.

1. Solve the ODE $X_0 = x$ and $X'(t) = \rho X(t)$ and discuss its sign depending on $x$.
2. Solve the SDE $X_0 = x$ and $dX_t = \rho X_t dt + \sigma X_t dB_t$ (existence, uniqueness, explicit formula).

Exercise 2. Let $\theta > 0$, $\rho \in \mathbb{R}$, $z \in \mathbb{R}^d$ be parameters, and let $Z^z$ be the solution of

$$Z^z_0 = z, \quad dZ^z_t = \theta dB_t - \rho Z^z_t dt$$

1. Why this SDE has a pathwise unique solution? What is the name of the process $Z^z$?
2. Show that the process $W_t = \int_0^t \frac{Z^z_s}{|Z^z_s|^2} dB_s$ with the convention $0/0 = 1$ is a Brownian motion.
3. Let us define $x = |z|^2$. Show that the process $X^x_t = |Z^z_t|^2$ solves the stochastic differential equation

$$X^x_0 = x, \quad dX^x_t = \sigma \sqrt{X^x_t} dW_t + (a - b X^x_t) dt \quad \text{where} \quad \sigma = 2\theta, \ a = \theta^2 d, \ b = 2\rho.$$  

4. Show that if $\rho > 0$ then $X^x_t \xrightarrow{\text{law}}_{t\to\infty} \text{Gamma}(d/2, 2b/\sigma^2)$. What happens when $b \leq 0$?
5. From now on, we assume that $X^x$ solves the SDE above for $x \geq 0$ and an arbitrary real parameter $d > 0$, without relation to $Z^z$.

Our goal is to evaluate $P(T^x_0 < \infty)$, $T^x_\infty = \inf\{t \geq 0 : X^x_t = c\}$. Show that

$$u \in (0, +\infty) \to \varphi(u) = \int_1^u v - \frac{2a}{v} \ e^{\frac{2b}{v}} v \ dV \quad \text{satisfies} \quad \frac{\sigma^2}{2} u \varphi''(u) + (a - bu) \varphi'(u) = 0.$$  

6. From now on, we take $x > 0$ and $0 < \epsilon < x < R$. Let us define $T^x_{\epsilon, R} = T^x_\epsilon \wedge T^x_R$. Show that for all $t > 0$,

$$\varphi(X^x_{t \wedge T^x_{\epsilon, R}}) = \varphi(x) + \int_0^{t \wedge T^x_{\epsilon, R}} \varphi'(X^x_s) \sigma \sqrt{X^x_s} dW_s.$$  

7. Show that $\mathbb{E}(T^x_{\epsilon, R}) < \infty$, which gives $T_{\epsilon, R} < \infty$ a.s. (hint: use an isometry, and a lower bound on $\varphi'$).
8. Show that

$$\varphi(x) = \varphi(\epsilon)\mathbb{P}(T^x_\epsilon < T^x_R) + \varphi(R)\mathbb{P}(T^x_\epsilon > T^x_R).$$  

9. Show that if $a \geq \frac{\sigma^2}{2}$ then $\mathbb{P}(T^x_0 < \infty) = 0$ (hint: use $\lim_{u \to 0} \varphi(u) = -\infty$).
10. Show that if $0 \leq a < \frac{\sigma^2}{2}$ and $b \geq 0$ then $\mathbb{P}(T^x_0 < \infty) = 1$ (hint: use $\lim_{R \to +\infty} \varphi(R) = +\infty$).
11. Show that if $0 \leq a < \frac{\sigma^2}{2}$ and $b < 0$ then $\mathbb{P}(T^x_0 < \infty) = (\varphi(\infty) - \varphi(x))/(\varphi(\infty) - \varphi(0)) \in (0, 1)$.

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1. If $G \sim \mathcal{N}(0, I_d)$ then $|G|^2 \sim \chi^2(d) = \text{Gamma}(d/2, 1/2)$. The law $\Gamma(\alpha, \lambda)$ has density $u \mapsto \frac{\lambda^\alpha}{\Gamma(\alpha)} u^{\alpha-1} e^{-\lambda u} 1_{u \geq 0}$.
Exercise 3. Let $U \in \mathcal{C}^2(\mathbb{R}^d, \mathbb{R})$. In particular $-\nabla U$ is locally Lipschitz but is not globally Lipschitz in general. Let us fix $x \in \mathbb{R}^d$. From the lecture notes, we recall and admit that there exists an adapted process $X$ with values in $\mathbb{R}^d \cup \{\infty\}$ and a stopping time $T$ with values in $(0, +\infty)$ such that
- $X_t \in \mathbb{R}^d$ if $t < T$ while $X_t = \infty$ if $t \geq T$, and $\lim_{t \to -T} |X_t| = \infty$ on $\{T < \infty\}$
- $t \mapsto X_t \in \mathbb{R}^d$ is continuous
- $X_t = x + B_t - \int_0^t \nabla U(X_s) \, ds$ on the (maximal) time interval $[0, T)$

We study now a couple of sufficient criteria on $U$ in order to get $\mathbb{P}(T < \infty) = 0$ (no explosion in finite time).

1. Suppose that

\[ \lim_{|x| \to \infty} U(x) = +\infty \quad \text{and} \quad C_2 = \sup_{x \in \mathbb{R}^d} \left( \frac{1}{2} \Delta U - |\nabla U|^2 \right) < \infty. \]

(a) Show that $T_R = \inf\{t \geq 0 : U(X_t) > R\} / \sim \sim \sim \sim T$.

(b) Show that $Y = X^{Tr} = (X_{t \wedge T})_{t \geq 0}$ solves the following SDE

\[ Y_t = x + \int_0^t 1_{s \leq T} \, dB_s - \int_0^t 1_{s \leq T} \nabla U(X_s) \, ds, \quad t \geq 0. \]

(c) Show that for all $R > 0$ and $t > 0$,

\[ \mathbb{E}(U(X_{t \wedge T})) = U(x) + \mathbb{E}\left( \int_0^{t \wedge T} \left( \frac{1}{2} \Delta U - |\nabla U|^2 \right)(X_s) \, ds \right). \]

(d) Show that $C_1 = \inf_{\mathbb{R}^d} U > -\infty$ and, for all $R > 0$ and $t > 0$,

\[ R1_{T_R \leq t} - |C_1| \leq U(X_{t \wedge T_R}). \]

(e) Show that for all $R > 0$ and $t > 0$,

\[ \mathbb{E}(R1_{T_R \leq t} - |C_1| - U(x)) \leq \mathbb{E}(C_2 (t \wedge T_R)) \leq t. \]

(f) Show that $\mathbb{P}(T < \infty) = 0$.

2. Suppose that for some $a, b \in \mathbb{R}$ and all $x \in \mathbb{R}^d$,

\[ \langle x, \nabla U(x) \rangle \geq -a|x|^2 - b. \]

(a) Show that $T_n = \inf\{t \geq 0 : |X_t|^2 > n\} / \sim \sim \sim \sim T$.

(b) Show that $Y = X^{T_n} = (X_{t \wedge T_n})_{t \geq 0}$ solves the following SDE

\[ Y_t = x + \int_0^t 1_{s \leq T_n} \, dB_s - \int_0^t 1_{s \leq T_n} \nabla U(X_s) \, ds, \quad t \geq 0. \]

(c) Show that for all $t \geq 0$ and $n \geq 1$,

\[ \mathbb{E}(|X_{t \wedge T_n}|^2) \leq |x|^2 + (1 + 2|b|)t + 2|a| \int_0^t \mathbb{E}(|X_{s \wedge T_n}|^2) \, ds. \]

(d) Show that for all $t \geq 0$ and $n \geq 1$,

\[ \mathbb{E}(|X_{t \wedge T_n}|^2) \leq (|x|^2 + (1 + 2|b|)t)e^{2|a|t}. \]

(e) Show that $\mathbb{P}(T < \infty) = 0$. 

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