Exam 2019/2020

December 4, 2019, from 13:45 to 16:45
Documents allowed, Internet not allowed
Do what you can, and do not worry

We use the notations of the lecture notes.
$B = (B_t)_{t\geq 0}$ is a d-dimensional Brownian motion issued from the origin, $d \geq 1$.

Exercise 1. Assume that $d = 1$. Let $\sigma > 0$, $\rho \in \mathbb{R}$, and $x \in \mathbb{R}$ be fixed parameters.

1. Solve the ODE $X_0 = x$ and $X'(t) = \rho X(t)$ and discuss its sign depending on x.
2. Solve the SDE $X_0 = x$ and $\mathrm{d}X_t = \rho X_t \mathrm{d}t + \sigma X_t \mathrm{d}B_t$ (existence, uniqueness, explicit formula).

Elements of solution for Exercise 1 (Geometric Brownian motion)**.**

1. The unique solution of the ODE is $X_t = x\mathrm{e}^{\rho t}$ for all $t \geq 0$. It keeps for all times the sign of the initial condition, namely if $x = 0$ (respectively < 0, > 0), then $X_t = 0$ (respectively < 0, > 0) for all $t \geq 0$.
2. The coefficients of the SDE are Lipschitz: there exists a solution, which pathwise uniqueness. If $x = 0$ then $X = 0$ is a solution. If $x < 0$ then $-X$ solves the SDE where X solves the SDE with initial condition $-x$. We consider now the case $x > 0$. Suppose that X solves the SDE, with $X_t > 0$ for all $t \geq 0$. Then
$$\frac{\mathrm{d}X_t}{X_t} = \rho \mathrm{d}t + \sigma \mathrm{d}B_t.$$
This suggests to use the Itô formula for $\log(X_t)$. The semi-martingale X_t has local martingale part $\sigma \int_0^t X_s \mathrm{d}B_s$ and finite variation part $\rho \int_0^t X_s \mathrm{d}s$. The Itô formula gives
$$\log(X_t) - \log(x) = \sigma \int_0^t \frac{X_s}{X_s}\mathrm{d}B_s + \rho \int_0^t \frac{X_s}{X_s}\mathrm{d}s - \frac{\sigma^2}{2}\int_0^t \frac{X_s^2}{X_s^2}\mathrm{d}s = \sigma B_t + \rho t - \frac{\sigma^2}{2}t,$$
hence
$$X_t = x\mathrm{e}^{\sigma B_t + (\rho - \frac{\sigma^2}{2})t}, \quad t \geq 0.$$
We can check now by Itô formula that this well defined semi-martingale solves indeed the SDE, regardless of x. The process X is known as *geometric Brownian motion*. This basic process is studied in all courses and books about stochastic calculus, for instance in [2, Section 8.4.2 page 226].

Exercise 2. Let $\theta > 0$, $\rho \in \mathbb{R}$, $z \in \mathbb{R}^d$ be parameters, and let Z^z be the solution of
$$Z_0^z = z, \quad \mathrm{d}Z_t^z = \theta \mathrm{d}B_t - \rho Z_t^z \mathrm{d}t$$

1. Why this SDE has a pathwise unique solution? What is the name of the process Z^z?
2. Show that the process $W_t = \displaystyle\int_0^t \frac{Z_s^z}{|Z_s^z|}\mathrm{d}B_s$ with the convention $0/0 = 1$ is a Brownian motion.
3. Let us define $x = |z|^2$. Show that the process $X_t^x = |Z_t^z|^2$ solves the stochastic differential equation
$$X_0^x = x, \quad \mathrm{d}X_t^x = \sigma\sqrt{X_t^x}\mathrm{d}W_t + (a - bX_t^x)\mathrm{d}t \quad \text{where} \quad \sigma = 2\theta,\ a = \theta^2 d,\ b = 2\rho.$$
4. Show that if $\rho > 0$ then[1] $X_t^x \xrightarrow[t\to\infty]{\text{law}} \mathrm{Gamma}(d/2, 2b/\sigma^2)$. What happens when $b \leq 0$?
5. **From now on**, we assume that X^x solves the SDE above for $x \geq 0$ and an arbitrary **real parameter** $d > 0$, without relation to Z^z. Our goal is to evaluate $\mathbb{P}(T_0^x < \infty)$, $T_c^x = \inf\{t \geq 0 : X_t^x = c\}$. Show that
$$u \in (0, +\infty) \mapsto \varphi(u) = \int_1^u v^{-\frac{2a}{\sigma^2}}\mathrm{e}^{\frac{2b}{\sigma^2}v}\mathrm{d}v \quad \text{satisfies} \quad \frac{\sigma^2}{2}u\varphi''(u) + (a - bu)\varphi'(u) = 0.$$

1. If $G \sim \mathcal{N}(0, I_d)$ then $|G|^2 \sim \chi^2(d) = \mathrm{Gamma}(d/2, 1/2)$. The law $\mathrm{Gamma}(a, \lambda)$ has density $u \mapsto \frac{\lambda^a}{\Gamma(a)}u^{a-1}\mathrm{e}^{-\lambda u}\mathbf{1}_{u\geq 0}$.

6. **From now on**, we take $x>0$ and $0<\varepsilon<x<R$. Let us define $T^x_{\varepsilon,R}=T^x_\varepsilon\wedge T^x_R$. Show that for all $t>0$,
$$\varphi(X^x_{t\wedge T^x_{\varepsilon,R}})=\varphi(x)+\int_0^{t\wedge T^x_{\varepsilon,R}}\varphi'(X^x_s)\sigma\sqrt{X^x_s}\mathrm{d}W_s.$$
7. Show that $\mathbb{E}(T^x_{\varepsilon,R})<\infty$, which gives $T_{\varepsilon,R}<\infty$ a.s. (hint: use an isometry, and a lower bound on φ').
8. Show that
$$\varphi(x)=\varphi(\varepsilon)\mathbb{P}(T^x_\varepsilon<T^x_R)+\varphi(R)\mathbb{P}(T^x_\varepsilon>T^x_R).$$
9. Show that if $a\geq\frac{\sigma^2}{2}$ then $\mathbb{P}(T^x_0<\infty)=0$ (hint: use $\lim_{u\to0}\varphi(u)=-\infty$).
10. Show that if $0\leq a<\frac{\sigma^2}{2}$ and $b\geq0$ then $\mathbb{P}(T^x_0<\infty)=1$ (hint: use $\lim_{R\to+\infty}\varphi(R)=+\infty$).
11. Show that if $0\leq a<\frac{\sigma^2}{2}$ and $b<0$ then $\mathbb{P}(T^x_0<\infty)=(\varphi(\infty)-\varphi(x))/(\varphi(\infty)-\varphi(0))\in(0,1)$.

Elements of solution for Exercise 2 (Cox–Ingersoll–Ross processes)**.** CIR processes, see [1, Section 6.2.2].

1. The existence and pathwise uniqueness of the solution follows from the fact that we have here a constant diffusion coefficient θI_d and a non-random constant in time Lipschitz drift $b(z)=-\rho z$. From the lecture notes Z^z is an Ornstein–Uhlenbeck (OU) process.
2. The function $z\mapsto z/|z|$ is a multivariate analogue of “sign”. Since Z^z is an OU process, the process $(Z^z_t/|Z^z_t|)_{t\geq0}$ is progressive and square integrable. Also W is a continuous local martingale, and since $\langle W\rangle_t=\int_0^t\frac{|Z^z_s|^2}{|Z^z_s|^2}\mathrm{d}B_s=t$, it follows by a famous criterion that W is a Brownian motion.
3. The semi-martingale Z^z has local martingale part θB_t and finite-variation part $-\rho\int_0^t Z^z_s\mathrm{d}s$. The Itô formula for the continuous semi-martingale Z^z and the $\mathscr{C}^2$ function $|\cdot|^2$ gives
$$\begin{aligned}\mathrm{d}X^x_t=\mathrm{d}(|Z^z_t|^2)&=2Z_t\mathrm{d}Z^z_t+\frac{1}{2}\theta^2(2d)\mathrm{d}t\\&=2\theta Z^z_t\mathrm{d}B_t-2\rho|Z^z_t|^2\mathrm{d}t+\theta^2d\mathrm{d}t\\&=2\theta|Z^z_t|\frac{Z^z_t}{|Z^z_t|}\mathrm{d}B_t+(\theta^2d-2\rho|Z^z_t|^2)\mathrm{d}t\\&=2\theta\sqrt{X^x_t}\mathrm{d}W_t+(\theta^2d-2\rho X^x_t)\mathrm{d}t\end{aligned}$$
We say that $X^x=|Z^z|^2$ is a Cox–Ingersoll–Ross (CIR) process. CIR processes are for OU processes what square Bessel processes are for BM. They include square Bessel processes as special cases. They are used in mathematical finance for the modeling of interest rates, see for instance [1, Section 6.2.2].
4. Since Z^z is OU with $Z_0=z$, by the Mehler formula, $Z^z_t=z\mathrm{e}^{-\rho t}+\sqrt{\frac{\theta^2(1-\mathrm{e}^{-2\rho t})}{2\rho}}G$, $G\sim\mathcal{N}(0,I_d)$. Now $|G|^2\sim\chi^2(d)=\mathrm{Gamma}(d/2,1/2)$ and $r|G|^2\sim\mathrm{Gamma}(d/2,1/(2r))$. Now $\rho>0$ gives $\lim_{t\to\infty}z\mathrm{e}^{-\rho t}=0$ and $\lim_{t\to\infty}(1-\mathrm{e}^{-2\rho t})=1$, hence the result. When $\rho\leq0$ then the law of X_t degenerates as $t\to\infty$.
5. These last questions are taken from [1, Proposition 6.2.3 and Exercise 37]. The result follows from
$$\varphi'(u)=u^{-\frac{2a}{\sigma^2}}\mathrm{e}^{\frac{2b}{\sigma^2}u}\quad\text{and}\quad\varphi''(u)=-\frac{2a}{\sigma^2}u^{-\frac{2a}{\sigma^2}-1}\mathrm{e}^{\frac{2b}{\sigma^2}u}+\frac{2b}{\sigma^2}u^{-\frac{2a}{\sigma^2}}\mathrm{e}^{\frac{2b}{\sigma^2}u}.$$
6. The continuous semi-martingale X^x has local martingale part $\sigma\int_0^\bullet\sqrt{X^x_s}\mathrm{d}W_s$, thus $\langle X^x\rangle=\sigma^2\int_0^\bullet X^x_s\mathrm{d}s$, and finite variation process $\int_0^\bullet(a-bX^x_s)\mathrm{d}s$. The Itô formula for X^x and the $\mathscr{C}^2$ function φ gives
$$\varphi(X^x_t)=\varphi(x)+\sigma\int_0^t\varphi'(X^x_s)\sqrt{X^x_s}\mathrm{d}W_s+\int_0^t\varphi'(X^x_s)(a-bX^x_s)\mathrm{d}s+\frac{\sigma^2}{2}\int_0^t\varphi''(X^x_s)X^x_s\mathrm{d}s$$
The desired result follows by stopping at $T^x_{\varepsilon,R}$ and then using the ODE on φ (previous question).
7. By definition of $T^x_{\varepsilon,R}$, the random variable $Y=\varphi(T^x_{\varepsilon,R})-\varphi(x)$ is bounded. By using the Itô isometry,
$$\mathbb{E}(Y^2)=\mathbb{E}\Big(\Big(\int_0^t\mathbf{1}_{s\leq T^x_{\varepsilon,R}}\varphi'(X^x_s)\sigma\sqrt{X^x_s}\mathrm{d}W_s\Big)^2\Big)=\sigma^2\mathbb{E}\int_0^t\mathbf{1}_{s\leq T^x_{\varepsilon,R}}(\varphi'(X^x_s))^2X^x_s\mathrm{d}s.$$
Now on $\{s\leq T^x_{\varepsilon,R}\}$, $\inf((\varphi'(X^x_s))^2X^x_s)\geq\ell>0$ for a deterministic ℓ, thus $+\infty>\mathbb{E}(Y^2)\geq\sigma^2\ell\mathbb{E}(T^x_{\varepsilon,R})$.

8. Let us consider the expression of $\varphi(X^x_{T^x_{\varepsilon,R}})$ obtained already. The stochastic integral in the result is centered because it is a martingale (put the stopping time inside or use the Doob stopping theorem) issued from the origin. By taking the expectation, we get $\varphi(x) = \mathbb{E}(\varphi(X^x_{t\wedge T^x_{\varepsilon,R}}))$. By the previous question, $T^x_{\varepsilon,R} < \infty$ almost surely, and since $t \mapsto \varphi(X^x_{t\wedge T^x_{\varepsilon,R}})$ is bounded, we get, by dominated convergence as $t \to \infty$, that $\varphi(x) = \mathbb{E}(\varphi(X^x_{T^x_{\varepsilon,R}}))$. It remains to consider the partition $\{T^x_\varepsilon < T^x_R\} \cup \{T^x_\varepsilon > T^x_R\}$.
9. If $a \geq \frac{\sigma^2}{2}$ then (Riemann criterion) $\lim_{u\to 0}\varphi(u) = -\infty$. Now T^x_ε grows if ε decreases, thus $\mathbb{P}(T^x_\varepsilon < T^x_R)$ decreases when ε decreases, and from the previous question, we get $\lim_{\varepsilon\to 0}\mathbb{P}(T^x_\varepsilon < T^x_R) = 0$ (contradiction otherwise). Next $T^x_\varepsilon < T^x_0$ gives $\mathbb{P}(T^x_0 < T^x_R) \leq \mathbb{P}(T^x_\varepsilon < T^x_R) \underset{\varepsilon\to 0}{\longrightarrow} 0$. It remains to let $R \to \infty$ to get the desired result by monotone or dominated convergence.
10. Since $0 \leq a < \frac{\sigma^2}{2}$, then $\varphi(0) = \lim_{u\to 0}\varphi(u)$ is finite. By monotone or dominated convergence,
$$\varphi(x) = \varphi(0)\mathbb{P}(T^x_0 < T^x_R) + \varphi(R)\mathbb{P}(T^x_0 > T^x_R).$$
Since $b \geq 0$ we get $\lim_{R\to+\infty}\varphi(R) = +\infty$ and thus $\mathbb{P}(T^x_0 = \infty) = 0$ (contradiction otherwise).
11. Since $0 \leq a < \frac{\sigma^2}{2}$ we get that $\varphi(0)$ is finite. Moreover $b < 0$ gives $\varphi(\infty) = \lim_{R\to+\infty}\varphi(R) \in (0, +\infty)$ and we get $\varphi(x) = \varphi(0)\mathbb{P}(T^x_0 < \infty) + \varphi(\infty)\mathbb{P}(T^x_0 = \infty)$ hence the formula. Note: it is still valid when $x = 0$.

   Note that for $\theta = 1$ and $\rho = 0$, we have $\sigma = 2$, $a = d$, $b = 0$, and $X = |z + B|^2$ is a squared Bessel process of dimension d started from z. The condition $a \geq \frac{\sigma^2}{2}$ reads $d \geq 2$, and it follows that almost surely a Bessel process of dimension d hits the origin if $d < 2$ and never hits the origin if $d \geq 2$, and this remains valid if we define the Bessel process with real dimension parameter $d > 0$ via an SDE.

Exercise 3. Let $U \in \mathscr{C}^2(\mathbb{R}^d, \mathbb{R})$. In particular $-\nabla U$ is locally Lipschitz but is not globally Lipschitz in general. Let us fix $x \in \mathbb{R}^d$. From the lecture notes, we recall and admit that there exists an adapted process X with values in $\mathbb{R}^d \cup \{\infty\}$ and a stopping time T with values in $(0, +\infty]$ such that
- $X_t \in \mathbb{R}^d$ if $t < T$ while $X_t = \infty$ if $t \geq T$, and $\lim_{t\to_< T}|X_t| = \infty$ on $\{T < \infty\}$
- $t \in [0, T) \mapsto X_t \in \mathbb{R}^d$ is continuous
- $X_t = x + B_t - \int_0^t \nabla U(X_s)\mathrm{d}s$ on the (maximal) time interval $[0, T)$

We study now a couple of sufficient criteria on U in order to get $\mathbb{P}(T < \infty) = 0$ (no explosion in finite time).

1. Suppose that
$$\lim_{|x|\to\infty} U(x) = +\infty \quad \text{and} \quad C_2 = \sup_{x\in\mathbb{R}^d}\Big(\frac{1}{2}\Delta U - |\nabla U|^2\Big) < \infty.$$
   (a) Show that $T_R = \inf\{t \geq 0 : U(X_t) > R\} \underset{R\to\infty}{\nearrow} T$.

   (b) Show that $Y = X^{T_R} = (X_{t\wedge T_R})_{t\geq 0}$ solves the following SDE
$$Y_t = x + \int_0^t \mathbf{1}_{s\leq T_R}\mathrm{d}B_s - \int_0^t \mathbf{1}_{s\leq T_R}\nabla U(X_s)\mathrm{d}s, \quad t \geq 0.$$
   (c) Show that for all $R > 0$ and $t > 0$,
$$\mathbb{E}(U(X_{t\wedge T_R})) = U(x) + \mathbb{E}\Big(\int_0^{t\wedge T_R}\Big(\frac{1}{2}\Delta U - |\nabla U|^2\Big)(X_s)\mathrm{d}s\Big).$$
   (d) Show that $C_1 = \inf_{\mathbb{R}^d} U > -\infty$ and, for all $R > 0$ and $t > 0$,
$$R\mathbf{1}_{T_R\leq t} - |C_1| \leq U(X_{t\wedge T_R}).$$
   (e) Show that for all $R > 0$ and $t > 0$,
$$\mathbb{E}(R\mathbf{1}_{T_R\leq t} - |C_1| - U(x)) \leq \mathbb{E}(C_2(t \wedge T_R)) \leq t.$$
   (f) Show that $\mathbb{P}(T < \infty) = 0$.

2. Suppose that for some $a,b\in\mathbb{R}$ and all $x\in\mathbb{R}^d$,
$$\langle x,\nabla U(x)\rangle\geq -a|x|^2-b.$$
   (a) Show that
   $$T_n=\inf\{t\geq 0:|X_t|^2>n\}\underset{n\to\infty}{\nearrow}T.$$
   (b) Show that $Y=X^{T_n}=(X_{t\wedge T_n})_{t\geq 0}$ solves the following SDE
   $$Y_t=x+\int_0^t\mathbf{1}_{s\leq T_n}\mathrm{d}B_s-\int_0^t\mathbf{1}_{s\leq T_n}\nabla U(X_s)\mathrm{d}s,\quad t\geq 0.$$
   (c) Show that for all $t\geq 0$ and $n\geq 1$,
   $$\mathbb{E}(|X_{t\wedge T_n}|^2)\leq |x|^2+(1+2|b|)t+2|a|\int_0^t\mathbb{E}(|X_{s\wedge T_n}|^2)\mathrm{d}s.$$
   (d) Show that for all $t\geq 0$ and $n\geq 1$,
   $$\mathbb{E}(|X_{t\wedge T_n}|^2)\leq (|x|^2+(1+2|b|)t)\mathrm{e}^{2|a|t}.$$
   (e) Show that $\mathbb{P}(T<\infty)=0$.

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Elements of solution for Exercise 3 (Non-explosion criteria for Kolmogorov diffusions). From [3, Th. 2.2.19].

1. (a) On $\{T=\infty\}$ we have $\lim_{R\to\infty}T_R=\infty=T$ since $U(X_t)$ takes then its values in $\mathbb{R}$ for all t. On $\{T<\infty\}$ we have $\lim_{R\to\infty}T_R=T$ since $\lim_{t\overset{<}{\to}T}U(X_t)=+\infty$.
   (b) Follows from the property of X, the fact that $T_R<T$, and the properties of the stochastic integral and the Lebesgue–Stieltjes integral with respect to stopping times.
   (c) In the SDE in the previous question, by using cutoff and regularization, we could replace $-\nabla U$ by a globally Lipschitz coefficient, because $-\nabla U$ is seen only on $\{U\leq R\}$ and U is $\mathscr{C}^2$ and thus locally Lipschitz. It follows that X^{T_R} solves an SDE with globally Lipschitz coefficients, and is a continuous semi-martingale, with local martingale and finite variations parts $B_{\bullet\wedge T_R}$ and $-\int_0^{\bullet\wedge T_R}\nabla U(X_s)\mathrm{d}s$ respectively. By the Itô formula and the properties of stopped integrals/brackets,
   $$U(X_{t\wedge T_R})=U(x)+\int_0^{t\wedge T_R}\nabla U(X_s)\mathrm{d}B_t-\int_0^{t\wedge T_R}|\nabla U(X_s)|^2\mathrm{d}s+\frac{1}{2}\int_0^{t\wedge T_R}(\Delta U)(X_s)\mathrm{d}s.$$
   By definition of T_R and the regularity of U the stochastic integral with respect to BM is a martingale, and since it is issued from the origin, its has zero expectation, hence the desired result.
   (d) Since $U\in\mathscr{C}^2(\mathbb{R}^d,\mathbb{R})$ and $\lim_{|x|\to\infty}U(x)=+\infty$ it follows that $C_1=\inf_{\mathbb{R}^d}U>-\infty$.
   Next, the inequality $R\mathbf{1}_{T_R\leq t}-|C_1|\leq U(X_{t\wedge T_R})$ holds because on $\{T_R>t\}$, it reads $-|C_1|\leq U(X_t)$, while on $\{T_R\leq t\}$, it reads $R-|C_1|\leq R\leq U(X_{T_R})=U(X_{t\wedge T_R})$.
   (e) The desired result follows by using the couple of previous questions.
   (f) From the previous question we get $\lim_{R\to\infty}\mathbb{P}(T_R\leq t)=0$. The desired result follows from the facts $\mathbb{P}(T<\infty)=\lim_{t\to\infty}\mathbb{P}(T\leq t)$ and $\mathbb{P}(T\leq t)\leq\lim_{R\to\infty}\mathbb{P}(T_R\leq t)=0$.
2. (a) On $\{T=\infty\}$, we have $\lim_{n\to\infty}T_n=\infty=T$ thanks to the fact that X takes then its values in $\mathbb{R}^d$, while on $\{T<\infty\}$, we have $\lim_{n\to\infty}T_n=T$ since X is continuous on $[0,T)$ and $\lim_{t\overset{<}{\to}T}|X_t|=+\infty$
   (b) Follows from the property of X, the fact that $T_n<T$, and the properties of the stochastic integral and the Lebesgue–Stieltjes integral with respect to stopping times.

(c) In the SDE satisfies by X^{T_n}, by using cutoff and regularization, we could replace $-\nabla U$ by a globally Lipschitz coefficient, because $-\nabla U$ is seen only on $D(0,R)$ and U is $\mathscr{C}^2$ and thus locally Lipschitz. It follows that X^{T_n} solves an SDE with globally Lipschitz coefficients, and is a continuous semi-martingale, with local martingale and finite variations parts $B_{\bullet\wedge T_n}$ and $-\int_0^{\bullet\wedge T_n}\nabla U(X_s)\mathrm{d}s$ respectively. By the Itô formula and the properties of stopped integrals/brackets,

$$|X_{t\wedge T_n}|^2 = |x|^2 + 2\int_0^{t\wedge T_n} X_s \mathrm{d}B_t - 2\int_0^{t\wedge T_n}\langle X_s, \nabla U(X_s)\rangle \mathrm{d}s + (t\wedge T_n).$$

By definition of T_n, the stochastic integral with respect to BM is a martingale, and since it is issued from zero, its has zero expectation. Taking expectations and using the assumption on U give

$$\mathbb{E}(|X_{t\wedge T_n}|^2) \le |x|^2 + 2|a|\int_0^t \mathbb{E}(|X_{s\wedge T_n}|^2)\mathrm{d}s + (1+2|b|)(t\wedge T_n).$$

(d) The Grönwall lemma for $f(t) = \mathbb{E}(|X_{t\wedge T_n}|^2)$ yields the desired result.

(e) We have $\lim_{n\to\infty}|X_{t\wedge T_n}|^2 = |X_{t\wedge T}|^2$ a.s. Moreover $\mathbb{P}(T<t)\nearrow_{t\to\infty}\mathbb{P}(T<\infty)$, and thus if we have $\mathbb{P}(T<\infty)>0$ then $\mathbb{P}(T<t)>0$ for t large enough, and for such a t, the Fatou lemma would gives

$$+\infty = \mathbb{E}(|X_{t\wedge T}|^2\mathbf{1}_{T<t}) \le \mathbb{E}(|X_{t\wedge T}|^2) = \mathbb{E}(\varliminf_{n\to\infty}|X_{t\wedge T_n}|^2) \le \varliminf_{n\to\infty}\mathbb{E}(|X_{t\wedge T_n}|^2) \le c(x,t) < \infty$$

where the last inequality follows from the previous question, which is impossible. Note that we could replace the Fatou lemma by the dominated convergence theorem since the previous questions gives that for all $t\ge 0$, the sequence $(|X_{t\wedge T_n}|)_{n\ge 1}$ is bounded in L^2 and thus u.i.

References

[1] Damien Lamberton and Bernard Lapeyre. *Introduction to stochastic calculus applied to finance*. Chapman & Hall/CRC Financial Mathematics Series. Chapman & Hall/CRC, second edition, 2008.

[2] Jean-François Le Gall. *Brownian motion, martingales, and stochastic calculus*, volume 274 of *Graduate Texts in Mathematics*. Springer, french edition, 2016.

[3] Gilles Royer. *An initiation to logarithmic Sobolev inequalities*, volume 14 of *SMF/AMS Texts and Monographs*. AMS-SMF, 2007. Translated from the 1999 French original by Donald Babbitt.