We use the notations of the lecture notes. 
\( B = (B_t)_{t \geq 0} \) is a \( d \)-dimensional Brownian motion issued from the origin, \( d \geq 1 \).

**Exercise 1.** Assume that \( d = 1 \). Let \( \sigma > 0, \rho \in \mathbb{R}, \) and \( x \in \mathbb{R} \) be fixed parameters.

1. Solve the ODE \( X_0 = x \) and \( X'(t) = \rho X(t) \) and discuss its sign depending on \( x \).
2. Solve the SDE \( X_0 = x \) and \( dX_t = \rho X_t dt + \sigma X_t dB_t \) (existence, uniqueness, explicit formula).

**Elements of solution for Exercise 1 (Geometric Brownian motion).**

1. The unique solution of the ODE is \( X_t = xe^{\rho t} \) for all \( t \geq 0 \). It keeps for all times the sign of the initial condition, namely if \( x = 0 \) (respectively \( < 0, > 0 \)), then \( X_t = 0 \) (respectively \( < 0, > 0 \)) for all \( t \geq 0 \).
2. The coefficients of the SDE are Lipschitz: there exists a solution, which pathwise uniqueness. If \( x < 0 \) then \( X = 0 \) is a solution. If \( x > 0 \) then \( -X \) solves the SDE where \( X \) solves the SDE with initial condition \( -x \). We consider now the case \( x > 0 \). Suppose that \( X \) solves the SDE, with \( X_t > 0 \) for all \( t \geq 0 \). Then
   \[
   \frac{dX_t}{X_t} = \rho dt + \sigma dB_t.
   \]
   This suggests to use the Itô formula for \( \log(X_t) \). The semi-martingale \( X_t \) has local martingale part \( \sigma \int_0^t X_s dB_s \) and finite variation part \( \rho \int_0^t X_s ds \). The Itô formula gives
   \[
   \log(X_t) - \log(x) = \sigma \int_0^t \frac{X_s}{X_t} dB_s + \rho \int_0^t \frac{X_s}{X_t} ds - \frac{\sigma^2}{2} \int_0^t \frac{X_s^2}{X_t^2} ds = \sigma B_t + \rho t - \frac{\sigma^2}{2} t,
   \]
   hence
   \[
   X_t = xe^{\sigma B_t + (\rho - \frac{\sigma^2}{2}) t}, \quad t \geq 0.
   \]
   We can check now by Itô formula that this well defined semi-martingale solves indeed the SDE, regardless of \( x \). The process \( X \) is known as geometric Brownian motion. This basic process is studied in all courses and books about stochastic calculus, for instance in [2, Section 8.4.2 page 226].

**Exercise 2.** Let \( \theta > 0, \rho \in \mathbb{R}, z \in \mathbb{R}^d \) be parameters, and let \( Z^z \) be the solution of
\[
Z^z_0 = z, \quad dZ^z_t = \theta dB_t - \rho Z^z_t dt
\]
1. Why this SDE has a pathwise unique solution? What is the name of the process \( Z^z \)?
2. Show that the process \( W_t = \int_0^t \frac{Z^z_s}{|Z^z_s|} dB_s \) with the convention \( 0/0 = 1 \) is a Brownian motion.
3. Let us define \( x = |z|^2 \). Show that the process \( X^x_t = |Z^z_t|^2 \) solves the stochastic differential equation
   \[
   X^x_0 = x, \quad dX^x_t = \sigma \sqrt{X^x_t} dB_t + (a - \rho X^x_t) dt \quad \text{where} \quad \sigma = 2 \theta, \ a = \theta^2 d, \ b = 2 \rho.
   \]
4. Show that if \( \rho > 0 \) then \( X^x_t \) \( \xrightarrow{\text{law}} \) \Gamma(d/2, 2b/\sigma^2). What happens when \( b \leq 0 ? \)
5. From now on, we assume that \( X^x \) solves the SDE above for \( x \geq 0 \) and an arbitrary real parameter \( d > 0 \), without relation to \( Z^z \). Our goal is to evaluate \( \mathbb{P}(T^x_0 < \infty), T^x_c = \inf\{t \geq 0 : X^x_t = c\} \). Show that
   \[
   u \in (0, +\infty) \to \varphi(u) = \int_0^u \frac{u^{2b}}{\sqrt{v} \ e^{\frac{\sigma^2 v}{2}}} dv \quad \text{satisfies} \quad \frac{\sigma^2}{2} u \varphi''(u) + (a - bu) \varphi'(u) = 0.
   \]

1. If \( G \sim \mathcal{N}(0, I_d) \) then \( |G|^2 \sim \chi^2(d) = \Gamma(d/2, 1/2) \). The law \( \Gamma(a, \lambda) \) has density \( u \to \frac{\lambda^a}{\Gamma(a)} u^{a-1} e^{-\lambda u} 1_{u > 0} \).
Elements of solution for Exercise 2 (Cox–Ingersoll–Ross processes). CIR processes, see [1, Section 6.2.2].

1. The existence and pathwise uniqueness of the solution follows from the fact that we have here a constant diffusion coefficient \( \theta l_i \) and a non-random constant in time Lipschitz drift \( b(z) = -\rho z \). From the lecture notes Z^2 is an Ornstein–Uhlenbeck (OU) process.

2. The function \( z \rightarrow z/|z| \) is a multivariate analogue of “sign”. Since \( Z^2 \) is an OU process, the process \( (Z_t^2/|Z_t|^2)_{t \geq 0} \) is progressive and square integrable. Also \( W \) is a continuous local martingale, and since \( \langle W \rangle_t = \int_0^t |Z_s^2|/|Z_s|^2 \, dB_s = t \), it follows by a famous criterion that \( W \) is a Brownian motion.

3. The semi-martingale \( Z^2 \) has local martingale part \( \partial B_t \) and finite-variation part \( -\rho \int_0^t Z_s^2 \, ds \). The Itô formula for the continuous semi-martingale \( Z^2 \) and the \( \mathcal{C}^2 \) function \( |\cdot|^2 \) gives

\[
\mathrm{d}X_t^2 = \mathrm{d}(|Z_t^2|^2) = 2Z_t \, dZ_t^2 + \frac{1}{2} \theta^2 (2d) \, dt
\]

\[
= 2\theta Z_t^2 \, dB_t - 2\rho |Z_t|^2 \, dt + \theta^2 \, dt
\]

\[
= 2\theta |Z_t|^2 \, dB_t + (\theta^2 - 2\rho |Z_t|^2) \, dt
\]

\[
= 2\theta \sqrt{X_t^2} \, dW_t + (\theta^2 - 2\rho X_t^2) \, dt
\]

We say that \( X^2 = |Z_t|^2 \) is a Cox–Ingersoll–Ross (CIR) process. CIR processes are for OU processes what square Bessel processes are for BM. They include square Bessel processes as special cases. They are used in mathematical finance for the modeling of interest rates, see for instance [1, Section 6.2.2].

4. Since \( Z^2 \) is OU with \( Z_0 = z \), by the Mehler formula, \( Z_t^2 = ze^{-\rho t} + \sqrt{\frac{\rho^2(1-e^{-2\rho t})}{2\rho}} G, \ G \sim \mathcal{N}(0, l_i) \). Now \( |G|^2 \sim \chi^2(d) = \Gamma(d/2, 1/2) \) and \( \rho |G|^2 \sim \Gamma(d/2, 1/(2\rho)) \). Now \( \rho > 0 \) gives \( \lim_{t \to \infty} |ze^{-\rho t}| = 0 \) and \( \lim_{t \to \infty} (1 - e^{-2\rho t}) = 1 \), hence the result. When \( \rho \leq 0 \) then the law of \( X_t \) degenerates as \( t \to \infty \).

5. These last questions are taken from [1, Proposition 6.2.3 and Exercise 37]. The result follows from

\[
\varphi'(u) = u \frac{-2a}{\sigma^2} \frac{e^{u}}{\sigma^2} \quad \text{and} \quad \varphi''(u) = \frac{-2a}{\sigma^2} \frac{e^{u}}{\sigma^2} - \frac{-2a}{\sigma^2} \frac{e^{u}}{\sigma^2} - \frac{2b}{\sigma^2} \frac{e^{u}}{\sigma^2} \frac{e^{u}}{\sigma^2}.
\]

6. The continuous semi-martingale \( X^2 \) has local martingale part \( \sigma \int_0^t \sqrt{X_s^2} \, dW_s \), thus \( \langle X^2 \rangle = \sigma^2 \int_0^t X_s^2 \, ds \), and finite variation process \( \int_0^t (a - bX_s^2) \, ds \). The Itô formula for \( X^2 \) and the \( \mathcal{C}^2 \) function \( |\cdot|^2 \) gives

\[
\varphi(X_t^2) = \varphi(x) + \sigma \int_0^t \varphi'(X_s^2) \sqrt{X_s^2} \, dW_s + \int_0^t \varphi'(X_s^2) (a - bX_s^2) \, ds + \frac{\sigma^2}{2} \int_0^t \varphi''(X_s^2) X_s^2 \, ds
\]

The desired result follows by stopping at \( T_{x,R}^2 \) and then using the ODE on \( \varphi \) (previous question).

7. By definition of \( T_{x,R}^2 \) the random variable \( Y = \varphi(T_{x,R}^2) - \varphi(x) \) is bounded. By using the Itô isometry,

\[
\mathbb{E}(Y^2) = \mathbb{E} \left( \left( \int_0^t 1_{s \leq T_{x,R}^2} \varphi'(X_s^2) \sqrt{X_s^2} \, dW_s \right)^2 \right) = \sigma^2 \mathbb{E} \left( \int_0^t 1_{s \leq T_{x,R}^2} (\varphi'(X_s^2)) X_s^2 \, ds \right).
\]

Now \( \ell = \inf_{s \leq T_{x,R}^2} (\varphi'(X_s^2) X_s^2) > 0 \) and thus \( +\infty > \mathbb{E}(Y^2) \geq \sigma^2 \ell \mathbb{E}(T_{x,R}^2) \) hence the result.
8. Let us consider the expression of $\varphi(X_{t}^{x})$ obtained already. The stochastic integral in the result is centered because it is a martingale (put the stopping time inside or use the Doob stopping theorem) issued from the origin. By taking the expectation, we get $\varphi(x) = E(\varphi(X_{t}^{x}))$. By the previous question, $T_{t,R}^{x} < \infty$ almost surely, and since $t \to \varphi(X_{t}^{x})$ is bounded, we get, by dominated convergence as $t \to \infty$, that $\varphi(x) = E(\varphi(X_{t}^{x}))$. It remains to consider the partition $(T_{t}^{x} < T_{R}^{x}) \cup \{T_{t}^{x} > T_{R}^{x}\}$. 

9. If $a \geq \frac{a^{2}}{2}$ then (Riemann criterion) $\lim_{\varepsilon \to 0} \varphi(u) = -\infty$. Now $T_{t}^{x}$ grows if $\varepsilon$ decreases, thus $\mathbb{P}(T_{t}^{x} < T_{R}^{x})$ decreases when $\varepsilon$ decreases, and from the previous question, we get $\lim_{\varepsilon \to 0} \mathbb{P}(T_{t}^{x} < T_{R}^{x}) = 0$ (contradiction otherwise). Next $T_{t}^{x} < T_{R}^{x}$ gives $\mathbb{P}(T_{t}^{0} < T_{R}^{0}) \leq \mathbb{P}(T_{t}^{x} < T_{R}^{x}) \to 0$. It remains to let $R \to \infty$ to get the desired result by monotone or dominated convergence. 

10. Since $0 \leq a < \frac{a^{2}}{2}$, then $\varphi(0) = \lim_{\varepsilon \to 0} \varphi(u)$ is finite. By monotone or dominated convergence, 
$$
\varphi(x) = \varphi(0)P(T_{t}^{x} < T_{R}^{x}) + \varphi(R)P(T_{t}^{x} > T_{R}^{x}).
$$
Since $b \geq 0$ we get $\lim_{R \to +\infty} \varphi(R) = +\infty$ and thus $\mathbb{P}(T_{t}^{x} = \infty) = 0$ (contradiction otherwise).

11. Since $0 \leq a < \frac{a^{2}}{2}$ we get that $\varphi(0)$ is finite. Moreover $b < 0$ gives $\varphi(\infty) = \lim_{R \to +\infty} \varphi(R) \in (0, +\infty)$ and we get $\varphi(x) = \varphi(0)P(T_{t}^{0} < \infty) + \varphi(\infty)P(T_{t}^{0} = \infty)$ hence the formula. Note: it is still valid when $x = 0$.

Note that for $\theta = 1$ and $\rho = 0$, we have $\sigma = 2$, $a = d$, $b = 0$, and $X = |z + B|^{2}$ is a squared Bessel process of dimension $d$ started from $z$. The condition $a \geq \frac{a^{2}}{2}$ reads $d \geq 2$, and it follows that almost surely a Bessel process of dimension $d$ hits the origin if $d < 2$ and never hits the origin if $d \geq 2$, and this remains valid if we define the Bessel process with real dimension parameter $d > 0$ via an SDE.

**Exercise 3.** Let $U \in \mathcal{C}^{2}(\mathbb{R}^{d}, \mathbb{R})$. In particular $-\nabla U$ is locally Lipschitz but is not globally Lipschitz in general. Let us fix $x \in \mathbb{R}^{d}$. From the lecture notes, we recall and admit that there exists an adapted process $X$ with values in $\mathbb{R}^{d} \cup \{\infty\}$ and a stopping time $T$ with values in $(0, +\infty)$ such that

- $X_{t} \in \mathbb{R}^{d}$ if $t < T$ while $X_{t} = \infty$ if $t \geq T$, and $\lim_{t \to - T} |X_{t}| = \infty$ on $\{T < \infty\}$
- $t \to (0, T) \to X_{t} \in \mathbb{R}^{d}$ is continuous
- $X_{t} = x + B_{t} - \int_{0}^{t} \nabla U(X_{s}) \, ds$ on the (maximal) time interval $(0, T)$

We study now a couple of sufficient criteria on $U$ in order to get $\mathbb{P}(T < \infty) = 0$ (no explosion in finite time).

1. Suppose that 
$$
\lim_{|x| \to \infty} U(x) = +\infty \quad \text{and} \quad C_{2} = \sup_{x \in \mathbb{R}^{d}} \left\{ \frac{1}{2} \Delta U - |\nabla U|^{2} \right\} < \infty.
$$
(a) Show that $T_{R} = \inf\{t \geq 0 : U(X_{t}) > R\} \to T$.

(b) Show that $Y = X_{t}^{T_{R}} = (X_{t \wedge T_{R}})_{t \geq 0}$ solves the following SDE
$$
Y_{t} = x + \int_{0}^{t} 1_{s \leq T_{R}} dB_{s} - \int_{0}^{t} 1_{s \leq T_{R}} \nabla U(X_{s}) \, ds, \quad t \geq 0.
$$

(c) Show that for all $R > 0$ and $t > 0$,
$$
\mathbb{E}(U(X_{t \wedge T_{R}})) = U(x) + \mathbb{E}\left[ \int_{0}^{t \wedge T_{R}} \left\{ \frac{1}{2} \Delta U - |\nabla U|^{2} \right\} (X_{s}) \, ds \right].
$$

(d) Show that $C_{1} = \inf_{R \geq 0} U > -\infty$ and, for all $R > 0$ and $t > 0$,
$$
R1_{T_{R} \leq t} - |C_{1}| \leq U(X_{t \wedge T_{R}}).
$$

(e) Show that for all $R > 0$ and $t > 0$,
$$
\mathbb{E}(R1_{T_{R} \leq t} - |C_{1}| - U(x)) \leq \mathbb{E}(C_{2} (t \wedge T_{R})) \leq t.
$$

(f) Show that $\mathbb{P}(T < \infty) = 0$. 

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Elements of solution for Exercise 3 (Non-explosion criteria for Kolmogorov diffusions). From [3, Th. 2.2.19].

1. (a) Let \( \langle x, \nabla U(x) \rangle \geq -a|x|^2 - b \).

   Show that
   \[
   T_n = \inf\{t \geq 0 : |X_t|^2 > n\} \nearrow T.
   \]

   (b) Follows from the property of \( X \).

   (c) In the SDE in the previous question, by using cutoff and regularization, we could replace \( -\nabla U \) by a globally Lipschitz coefficient, because \( \nabla U \) is seen only on \( \{U \leq R\} \) and \( U \) is \( \mathcal{C}^2 \) and thus locally Lipschitz. It follows that \( X^{T_n} \) solves an SDE with globally Lipschitz coefficients, and is a continuous semi-martingale, with local martingale and finite variations parts \( B_{t \wedge T_n} \) and \( \int_0^{t \wedge T_n} \nabla U(X_s) \) respectively. By the Itô formula and the properties of stopped integrals/brackets,
   \[
   U(X_{t \wedge T_n}) = U(x) + \int_0^{t \wedge T_n} \nabla U(X_s)dB_s - \int_0^{t \wedge T_n} |\nabla U(X_s)|^2 ds + \frac{1}{2} \int_0^{t \wedge T_n} (\Delta U)(X_s) ds.
   \]

   By definition of \( T_n \) and the regularity of \( U \) the stochastic integral with respect to BM is a martingale, and since it is issued from the origin, its has zero expectation, hence the desired result.

   (d) Since \( U \in \mathcal{C}^2([0,d],\mathbb{R}) \) and \( \lim_{|x| \to \infty} U(x) = +\infty \) it follows that \( C_1 = \inf_{R \in \mathbb{R}} U > -\infty \).

   Next, the inequality \( R1_{t < s} - |C_1| \leq U(X_{t \wedge T_n}) \) holds because on \( \{T_n > t\} \), it reads \( -|C_1| \leq U(X_t) \), while on \( \{T_n \leq t\} \), it reads \( \lim_{t \to T_n} -|C_1| \leq U(X_{t \wedge T_n}) = U(X_{T_n}) \).

   (e) The desired result follows by using the couple of previous questions.

   (f) From the previous question we get \( \lim_{R \to \infty} \mathbb{P}(T_n \leq t) = 0 \). The desired result follows from the facts \( \mathbb{P}(T < \infty) = \lim_{t \to \infty} \mathbb{P}(T \leq t) \) and \( \mathbb{P}(T \leq t) \leq \lim_{R \to \infty} \mathbb{P}(T \leq t) = 0 \).

2. (a) On \( \{T = \infty\} \), we have \( \lim_{n \to \infty} T_n = \infty = T \) thanks to the fact that \( X \) takes then its values in \( \mathbb{R}^d \), while on \( \{T < \infty\} \), we have \( \lim_{n \to \infty} T_n = T \) since \( X \) is continuous on \( [0,T] \) and \( \lim_{t \to T} |X_t| = +\infty \).

   (b) Follows from the property of \( X \), the fact that \( T_n < T \), and the properties of the stochastic integral and the Lebesgue–Stieltjes integral with respect to stopping times.
(c) In the SDE satisfies by $X_{T_n}$, by using cutoff and regularization, we could replace $-\nabla U$ by a globally Lipschitz coefficient, because $-\nabla U$ is seen only on $D(0,R)$ and $U$ is $\mathcal{C}^2$ and thus locally Lipschitz. It follows that $X_{T_n}$ solves an SDE with globally Lipschitz coefficients, and is a continuous semi-martingale, with local martingale and finite variations parts $B_{\land T_n}$ and $-\int_0^{\land T_n} \nabla U(X_s)\,ds$ respectively. By the Itô formula and the properties of stopped integrals/brackets,

$$|X_{\land T_n}|^2 = |x|^2 + 2 \int_0^{\land T_n} X_s \,dB_t - 2 \int_0^{\land T_n} \langle X_s, \nabla U(X_s) \rangle \,ds + (t \land T_n).$$

By definition of $T_n$, the stochastic integral with respect to BM is a martingale, and since it is issued from zero, its has zero expectation. Taking expectations and using the assumption on $U$ give

$$\mathbb{E}(|X_{\land T_n}|^2) \leq |x|^2 + 2|a| \int_0^t \mathbb{E}(|X_{\land T_n}|^2) \,ds + (1 + 2|b|)(t \land T_n).$$

(d) The Grönwall lemma for $f(t) = \mathbb{E}(|X_{\land T_n}|^2)$ yields the desired result.

(e) We have $\lim_{n \to \infty} |X_{\land T_n}|^2 = |X_{\land T}|^2$ a.s. Moreover $P(T < t) \xrightarrow{t \to \infty} P(T < \infty)$, and thus if we have $P(T < \infty) > 0$ then $P(T < t) > 0$ for $t$ large enough, and for such a $t$, the Fatou lemma would gives

$$+\infty = \mathbb{E}(|X_{\land T}|^2 1_{T < t}) \leq \mathbb{E}(|X_{\land T}|^2) \leq \mathbb{E}(\lim_{n \to \infty} |X_{\land T_n}|^2) \leq \lim_{n \to \infty} \mathbb{E}(|X_{\land T_n}|^2) \leq c(x, t) < \infty$$

where the last inequality follows from the previous question, which is impossible. Note that we could replace the Fatou lemma by the dominated convergence theorem since the previous questions gives that for all $t \geq 0$, the sequence $(|X_{\land T_n}|^2)_{n \geq 1}$ is bounded in $L^2$ and thus u.i.

References

