Exercise 1 (Nature of an integral). Set $d = 1$. Let us consider the following integral, for $t \geq 0$,

$$I_t = \int_0^t B_s ds.$$ 

2. Show that $d(tB_t) = B_t dt + dB_t$.
3. Deduce from the preceding question that $I_t = \int_0^t (t-s)dB_s$ for all $t \geq 0$.
4. Deduce from the preceding question that $I_t \sim \mathcal{N}(0, \frac{1}{3}t^3)$ for all $t \geq 0$.
5. For all $t \geq 0$, $n \geq 1$, $0 \leq k \leq n$, let us define $t_k = \frac{k}{n} t$. Show that

$$\sum_{k=0}^{n-1} B_{t_k}(t_{k+1} - t_k) = \frac{t}{n} \sum_{j=0}^{n-2} (n-j-1)(B_{t_{j+1}} - B_{t_j}).$$

6. Deduce from the preceding question another proof that $I_t \sim \mathcal{N}(0, \frac{1}{3}t^3)$ for all $t \geq 0$.
7. Is the process $(I_t)_{t \geq 0}$ a martingale?

Exercise 2 (Study of a special process). Set $d = 2$. For all $t \geq 0$, we write $B_t = (X_t, Y_t)$ and

$$A_t = \int_0^t X_s dY_s - \int_0^t Y_s dX_s.$$ 

1. Show that $(A) = \int_0^t (X^2_s + Y^2_s) ds$ and that the process $A$ is a square integrable martingale.
2. From now on let $\lambda > 0$. Show that for all $t \geq 0$,

$$Ee^{\lambda A_t} = E\cos(\lambda A_t).$$

3. From now on, let $f : \mathbb{R}_+ \to \mathbb{R}$ be $\mathcal{C}^2$, and let us define the continuous semi-martingales

$$(Z_t)_{t \geq 0} = (\cos(\lambda A_t))_{t \geq 0} \quad \text{and} \quad (W_t)_{t \geq 0} = \left( - \frac{f'(t)}{2}(X_t^2 + Y_t^2) + f(t) \right)_{t \geq 0}.$$ 

Show that for all $t \geq 0$,

$$Z_t = 1 - \lambda \int_0^t \sin(\lambda A_s) dA_s - \frac{\lambda^2}{2} \int_0^t (X^2_s + Y^2_s) Z_s ds.$$ 

and

$$W_t = f(0) - \int_0^t f'(s) X_s dY_s - \int_0^t f'(s) Y_s dX_s - \frac{1}{2} \int_0^t f''(s)(X^2_s + Y^2_s) ds,$$

and deduce that

$$\langle Z, W \rangle = 0.$$
4. Show that if $f$ solves $f'' = f'^2 - \lambda^2$ then $Ze^W$ is a continuous local martingale and
\[
Ze^W = e^{f(0)} - \lambda \int_0^t \sin(\lambda A_s) e^{W_s} dA_s - \lambda \int_0^t f'(s) Z_s e^{W_s} X_s dX_s - \int_0^t f'(s) Z_s e^{W_s} Y_s dY_s.
\]

5. Let $r > 0$. By using $f(t) = -\log \cosh(\lambda(r - t))$ deduce from the previous question that
\[
\mathbb{E}e^{\lambda A_t} = \frac{1}{\cosh(\lambda r)}.
\]

Exercise 3 (Criterion for a stochastic differential equation). Set $d = 1$. Let $\sigma, b$ be two functions $\mathbb{R} \to \mathbb{R}$ such that for some finite constant $C < \infty$ and for all $x, y \in \mathbb{R}$,
\[
|\sigma(x) - \sigma(y)| \leq C\sqrt{x - y} \quad \text{and} \quad |b(x) - b(y)| \leq C|x - y|
\]
The goal of this exercise is to prove pathwise uniqueness for the stochastic differential equation
\[
dX_t = \sigma(X_t)dB_t + b(X_t)dt. \tag{SDE}
\]
A solution $X$ is a continuous semi-martingale with canonical decomposition $X = X_0 + M + V$ with $X_0 \in L^2$, local martingale part $M = \int_0^t \sigma(X_s)dB_s$, and finite variation part $V = \int_0^t b(X_s)ds$. Note that the continuity of $\sigma, X, b$ gives that almost surely, for all $t \geq 0$, $s - \sigma(X_s) + b(X_s)$ is locally bounded.

1. Let $Z$ be a continuous semi-martingale such that $\langle Z \rangle = \int_0^t \varphi_s ds$ for a progressive process $\varphi$ such that $0 \leq \varphi \leq C|Z|$ for some constant $C < \infty$. Prove that for all $t \geq 0$ and all $\alpha > 0$,
\[
\mathbb{E} \int_0^t \frac{1_{0 < |Z_s| \leq a}}{|Z_s|} d\langle Z \rangle_s \leq Ct.
\]

2. Deduce from the preceding question that for all $t \geq 0$,
\[
\lim_{n \to \infty} n\mathbb{E} \int_0^t 1_{0 < |Z_s| \leq \frac{1}{n}} d\langle Z \rangle_s = 0.
\]

3. For all $n \geq 1$, $x \in \mathbb{R}$, let us define $g_n(x) = 2n(1 + nx)1_{x \in [-\frac{1}{n}, 0]} + 2n1_{x=0} + 2n(1 - nx)1_{x \in [0, \frac{1}{n}]}$.
Let $f_n : \mathbb{R} \to \mathbb{R}$ be the twice differentiable function such that $f_n'' = g_n$ and $f_n(0) = f'_n(0) = 0$.
Show that for all $x \in \mathbb{R}$, the following properties hold true:
(a) $f_n''(x) \in [-1, 1]$ and $\lim_{n \to \infty} f_n'(x) = \text{sign}(x) = 1_{x > 0} - 1_{x < 0}$
(b) $|f_n(x)| \leq |x|$ and $\lim_{n \to \infty} f_n(x) = |x|$.

4. By using Itô formula, prove that for all continuous semi-martingale $Z = (Z_t)_{t \geq 0}$, all $t \geq 0$,
\[
\int_0^t 1_{Z_s=0} d\langle Z \rangle_s = 0.
\]

5. From now on, let $X$ and $X'$ be two solutions of (SDE) on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ and with respect to the Brownian motion $B$. Show that for all $t \geq 0$,
\[
\langle X - X' \rangle_t = \int_0^t (\sigma(X_s) - \sigma(X'_s))^2 ds.
\]

6. By using the assumption on $\sigma$, deduce from the preceding questions that for all $t \geq 0$,
\[
\lim_{n \to \infty} \mathbb{E} \int_0^t g_n(X_s - X'_s) d\langle X - X' \rangle_s = 0.
7. Set $Z = X - X'$. From now on, let $T$ be a stopping time such that the semi-martingale $(Z_{t∧T})_{t≥0}$ is bounded. By using notably the assumption on $σ$, prove that for all $t ≥ 0$, $n ≥ 1$,

$$E(f_n(Z_{t∧T})) = E(f_n(Z_0)) + E \int_0^{t∧T} f_n'(Z_s)(b(X_s) - b(X'_s))ds + \frac{1}{2} E \int_0^{t∧T} f_n''(Z_s)d\langle Z \rangle_s.$$

8. Deduce from the preceding questions and the assumption on $b$ that for all $t ≥ 0$,

$$E(|X_{t∧T} - X'_{t∧T}|) = E(|X_0 - X'_0|) + E \int_0^{t∧T} (b(X_s) - b(X'_s))\text{sign}(X_s - X'_s)ds.$$

9. By using the Grönwall lemma, deduce that if $X_0 = X'_0$ then $X_t = X'_t$ for all $t ≥ 0$. 