Exam 2018/2019

January 9, 2019, from 09:00 to 12:00
Documents allowed, Internet not allowed
Do what you can, and do not worry

\((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})\) is a filtered probability space, with complete and right continuous filtration.
\(B = (B_t)_{t \geq 0}\) is a \(d\)-dimensional Brownian motion issued from the origin, \(d \geq 1\).
If \(Z\) is a semi-martingale, we denote by \(\langle Z \rangle\) the increasing process of its local martingale part.
If \(Z = Z_0 + \mathcal{M} + \mathcal{V}\), do not confuse \(\langle Z \rangle = \langle \mathcal{M} \rangle\) with the finite variation part \(\mathcal{V}\) of \(Z\).

**Exercise 1** (Nature of an integral). Set \(d = 1\). Let us consider the following integral, for \(t \geq 0\),

\[ I_t = \int_0^t B_s ds. \]

2. Show that \(d(tB_t) = B_t dt + t dB_t\);
3. Deduce from the preceding question that \(I_t = \int_0^t (t - s) dB_s\) for all \(t \geq 0\);
4. Deduce from the preceding question that \(I_t \sim \mathcal{N}(0, \frac{1}{3} t^3)\) for all \(t \geq 0\);
5. For all \(t \geq 0\), \(n \geq 1\), \(0 \leq k \leq n\), let us define \(t_k = \frac{k}{n} t\). Show that
\[
\sum_{k=0}^{n-1} B_{t_k} (t_{k+1} - t_k) = \frac{t}{n} \sum_{j=0}^{n-2} (n - j - 1)(B_{t_{j+1}} - B_{t_j}).
\]
6. Deduce from the preceding question another proof that \(I_t \sim \mathcal{N}(0, \frac{1}{3} t^3)\) for all \(t \geq 0\);
7. Is the process \((I_t)_{t \geq 0}\) a martingale?

**Exercise 2** (Study of a special process). Set \(d = 2\). For all \(t \geq 0\), we write \(B_t = (X_t, Y_t)\) and

\[ A_t = \int_0^t X_s dY_s - \int_0^t Y_s dX_s. \]

1. Show that \(\langle A \rangle = \int_0^t (X_s^2 + Y_s^2) ds\) and that the process \(A\) is a square integrable martingale;
2. From now on let \(\lambda > 0\). Show that for all \(t \geq 0\),
\[
\mathbb{E} e^{i \lambda A_t} = \mathbb{E} \cos(\lambda A_t).
\]
3. From now on, let \(f : \mathbb{R}^+ \to \mathbb{R}\) be \(\mathcal{C}^2\), and let us define the continuous semi-martingales
\[
(Z_t)_{t \geq 0} = (\cos(\lambda A_t))_{t \geq 0} \quad \text{and} \quad (W_t)_{t \geq 0} = \left(- \frac{f'(t)}{2} (X_t^2 + Y_t^2) + f(t) \right)_{t \geq 0}.
\]
Show that for all \(t \geq 0\),
\[
Z_t = 1 - \lambda \int_0^t \sin(\lambda A_s) dA_s - \frac{\lambda^2}{2} \int_0^t (X_s^2 + Y_s^2) Z_s ds.
\]
and
\[
W_t = f(0) - \int_0^t f'(s) X_s dX_s - \int_0^t f'(s) Y_s dY_s - \frac{1}{2} \int_0^t f''(s) (X_s^2 + Y_s^2) ds,
\]
and deduce that
\[
\langle Z, W \rangle = 0.
\]
4. Show that if \( f \) solves \( f'' = f'^2 - \lambda^2 \) then \( Z e^{W_t} \) is a continuous local martingale and
\[
Z e^{W_t} = e^{f(0)} - \lambda \int_0^t \sin(\lambda A_s) e^{W_s} dA_s - \int_0^t f'(s) Z_s e^{W_s} X_s dX_s - \int_0^t f'(s) Z_s e^{W_s} Y_s dY_s.
\]

5. Let \( r > 0 \). By using \( f(t) = -\log \cosh(\lambda (r - t)) \) deduce from the previous question that
\[
\mathbb{E} e^{\lambda A_t} = \frac{1}{\cosh(\lambda r)}.
\]

**Exercise 3** (Criterion for a stochastic differential equation). Set \( d = 1 \). Let \( \sigma, b \) be two functions \( \mathbb{R} \to \mathbb{R} \) such that for some finite constant \( C < \infty \) and for all \( x, y \in \mathbb{R} \),
\[
|\sigma(x) - \sigma(y)| \leq C \sqrt{x - y} \quad \text{and} \quad |b(x) - b(y)| \leq C|x - y|
\]
The goal of this exercise is to prove pathwise uniqueness for the stochastic differential equation
\[
dX_t = \sigma(X_t) dB_t + b(X_t) dt.
\]
A solution \( X \) is a continuous semi-martingale with canonical decomposition \( X = X_0 + M + V \) with \( X_0 \in L^2 \), local martingale part \( M = \int_0^t \sigma(X_s) dB_s \), and finite variation part \( V = \int_0^t b(X_s) ds \). Note that the continuity of \( \sigma, X, b \) gives that almost surely, for all \( t \geq 0, s - \sigma(X_s) + b(X_s) \) is locally bounded.

1. Let \( Z \) be a continuous semi-martingale such that \( \langle Z \rangle = \int_0^t \varphi_s ds \) for a progressive process \( \varphi \) such that \( 0 \leq \varphi \leq C|Z| \) for some constant \( C < \infty \). Prove that for all \( t \geq 0 \) and all \( a > 0 \),
\[
\mathbb{E} \int_0^t \mathbf{1}_{0 < |Z| \leq a} \frac{1}{|Z|} d\langle Z \rangle_s \leq Ct.
\]

2. Deduce from the preceding question that for all \( t \geq 0 \),
\[
\lim_{n \to \infty} n \mathbb{E} \int_0^t \mathbf{1}_{0 < |Z| \leq \frac{1}{n}} d\langle Z \rangle_s = 0.
\]

3. For all \( n \geq 1, x \in \mathbb{R} \), let us define \( g_n(x) = 2n(1 + nx) \mathbf{1}_{x \in [-\frac{1}{n}, 0)} + 2n \mathbf{1}_{x = 0} + 2n(1 - nx) \mathbf{1}_{x \in (0, \frac{1}{n})} \).

Let \( f_n : \mathbb{R} \to \mathbb{R} \) be the twice differentiable function such that \( f_n'' = g_n \) and \( f_n(0) = f'_n(0) = 0 \).
Show that for all \( x \in \mathbb{R} \), the following properties hold true:

(a) \( f_n''(x) \in [-1, 1] \) and \( \lim_{n \to \infty} f_n''(x) = \text{sign}(x) = \mathbf{1}_{x > 0} - \mathbf{1}_{x < 0} \);

(b) \( |f_n(x)| \leq |x| \) and \( \lim_{n \to \infty} f_n(x) = |x| \).

4. By using Itô formula, prove that for all continuous semi-martingale \( Z = (Z_t)_{t \geq 0} \), all \( t \geq 0 \),
\[
\int_0^t \mathbf{1}_{Z_s = 0} d\langle Z \rangle_s = 0.
\]

5. From now on, let \( X \) and \( X' \) be two solutions of (SDE) on \( (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}) \) and with respect to the Brownian motion \( B \). Show that for all \( t \geq 0 \),
\[
\langle X - X' \rangle_t = \int_0^t (\sigma(X_s) - \sigma(X'_s))^2 ds.
\]

6. By using the assumption on \( \sigma \), deduce from the preceding questions that for all \( t \geq 0 \),
\[
\lim_{n \to \infty} \mathbb{E} \int_0^t g_n(X_s - X'_s) d\langle X - X' \rangle_s = 0.
\]
7. Set $Z = X - X'$. From now on, let $T$ be a stopping time such that the semi-martingale $(Z_{t∧T})_{t≥0}$ is bounded. By using notably the assumption on $σ$, prove that for all $t ≥ 0, n ≥ 1$,

$$E(f_n(Z_{t∧T})) = E(f_n(Z_0)) + E\int_0^{t∧T} f'_n(Z_s)(b(X_s) - b(X'_s))ds + \frac{1}{2} E\int_0^{t∧T} f''_n(Z_s)d\langle Z \rangle_s.$$ 

8. Deduce from the preceding questions and the assumption on $b$ that for all $t ≥ 0$,

$$E(|X_{t∧T} - X'_{t∧T}|) = E(|X_0 - X'_0|) + E\int_0^{t∧T} (b(X_s) - b(X'_s))\text{sign}(X_s - X'_s)ds.$$ 

9. By using the Grönwall lemma, deduce that if $X_0 = X'_0$ then $X_t = X'_t$ for all $t ≥ 0$. 

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