17. ON THE THEORY OF CONTINUOUS RANDOM PROCESSES *

Let $\mathfrak{S}$ be a physical system with n degrees of freedom; this means that the admissible states x of $\mathfrak{S}$ form a Riemannian manifold $\mathfrak{R}$ of dimension n. The process of variation of $\mathfrak{S}$ is said to be *stochastically determined* if under an arbitrary choice of x, the region $\mathfrak{E}$ (in $\mathfrak{R}$) and times t' and t'' ($t' < t''$), the probability $P(t', x, t'', \mathfrak{E})$ that the system in state x at time t' will be in one of the states of $\mathfrak{E}$ at time t'' is defined. It is further assumed that the probability $P(t', x, t'', \mathfrak{E})$ can be given by the formula

$$P(t', x, t'', \mathfrak{E}) = \int_{\mathfrak{E}} f(t', x, t'', y) dV_y, \tag{1}$$

where dV_y denotes the volume element. Here $f(t', x, t'', y)$ has to satisfy the following fundamental equations:

$$\int_{\mathfrak{R}} f(t', x, t'', y) dV_y = 1, \tag{2}$$

$$f(t_1, x, t_3, y) = \int_{\mathfrak{R}} f(t_1, x, t_2, z) f(t_2, z, t_3, y) dV_z, \quad t_1 < t_2 < t_3. \tag{3}$$

The integral equation (3) was studied by Smolukhovskii and then by other authors.[1] In the paper 'Über die analytischen Methoden in der Wahrscheinlichkeitsrechnung'[2] I have proved that, under certain additional conditions, $f(t', x, t'', y)$ satisfies certain differential equations of parabolic type.[3] But in A.M. there was no answer to the question[4] as to what extent $f(t', x, t'', y)$ is uniquely determined by the coefficients $A(t, x)$ and $B(t, x)$. In this paper the theory is developed in the general case of a Riemannian manifold $\mathfrak{R}$ and the question of uniqueness is answered affirmatively for a closed manifold $\mathfrak{R}$.

§1. The first differential equation

Let $\mathfrak{R}$ be a Riemannian manifold of dimension n. Because of the assumptions made, $f(t', x, t'', y)$ is defined for $t' < t''$ and any pair of points x, y. Moreover,

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* 'Zur Theorie der stetigen zufälligen Prozesse', *Math. Ann.* **108** (1933), 149–160.

[1] See bibliography in: B. Hostinsky, 'Méthodes générales du calcul des probabilités', *Mem. Sci. Math.* **52** (1931).

[2] *Math. Ann.* **104** (1931), 415–458. Referred to in the present paper as A.M. (see No. 9 of this book).

[3] These differential equations were introduced by Fokker and Planck independently of Smolukhovskii's integral equation. See: A. Fokker, *Ann. Phys.* **43** (1914), 812; M. Planck, *Sitzungsber. Preuss. Acad. Wiss.* (1917) 10 May.

[4] See A.M. §15.

we assume that $f(t', x, t'', y)$ has continuous derivatives up to the third order with respect to all the arguments (t', t'' and the coordinates $x_1, x_2, \dots, x_n$, $y_1, y_2, \dots, y_n$ of the points x and y) and satisfies the continuity condition

$$\frac{\int_{\mathfrak{R}} f(t, x, t+\Delta, z)\rho^3(x, z)dV_z}{\int_{\mathfrak{R}} f(t, x, t+\Delta, z)\rho^2(x, z)dV_z} \to 0 \quad \text{as } \Delta \to 0, \tag{4}$$

where $\rho(x, z)$ denotes the geodesic distance[5] between x and z.

We choose a coordinate system $z = (z_1, \dots, z_n)$ in a neighbourhood $\mathfrak{A}$ of x. Then we set

$$\int_{\mathfrak{A}} f(s, x, s+\Delta, z)(z_i - x_i)dV_z = a_i(s, x, \Delta), \tag{5}$$

$$\int_{\mathfrak{A}} f(s, x, s+\Delta, z)(z_i - x_i)(z_j - x_j)dV_z = b_{ij}(s, x, \Delta), \tag{6}$$

$$\int_{\mathfrak{R}} f(s, x, s+\Delta, z)\rho^2(x, z)dV_z = \beta(s, x, \Delta), \tag{7}$$

$$\int_{\mathfrak{R}} f(s, x, s+\Delta, z)\rho^3(x, z)dV_z = \nu(s, x, \Delta). \tag{8}$$

Our purpose is to prove that the ratios

$$a_i(s, x, \Delta)/\Delta, \quad b_{ij}(s, x, \Delta)/\Delta$$

tend to limits $A_i(s, x)$ and $B_{ij}(s, x)$ as $\Delta \to 0$, independently of $\mathfrak{A}$. Below this is proved under the assumption that all $N = n + n(n+1)/2$ functions

$$\frac{\partial}{\partial x_i} f(s, x, t, y), \quad \frac{\partial^2}{\partial x_i \partial x_j} f(s, x, t, y)$$

of y and t (for fixed s and x) are linearly independent, that is, that $t_1, y_1, t_2, y_2, \dots$ $\dots, t_k, y_k, \dots, t_N, y_N$ can be chosen so that the determinant

$$D^N(s, x) = \begin{vmatrix} \dfrac{\partial}{\partial x_i} f(s, x, t_k, y_k) \\ \dfrac{\partial^2}{\partial x_i \partial x_j} f(s, x, t_k, y_k) \end{vmatrix} \tag{9}$$

is non-zero.[6]

[5] See A.M., §13, formula (112).
[6] See A.M., §13, determinant (119).

In $\mathfrak{A}$ we have

$$\rho^2(x,z) = \sum g_{ij}(z_i - x_i)(z_j - x_j) + \Theta\rho^3(x,z), \quad |\Theta| \leq C,$$

while outside $\mathfrak{A}$ we clearly have

$$\rho^2(x,z) = \Theta'\rho^3(x,z), \quad |\Theta'| \leq C',$$

where C' and C are constants independent of z. Hence

$$\begin{aligned} \beta(s,x,\Delta) = \int_{\mathfrak{R}} f(s,x,s+\Delta,z)\rho^2(x,z)dV_z = \\ = \sum g_{ij} \int_{\mathfrak{A}} f(s,x,s+\Delta,z)(z_i - x_i)(z_j - x_j)dV_z + \\ + \int_{\mathfrak{A}} f(s,x,s+\Delta,z)\Theta\rho^3(x,z)dV_z + \\ + \int_{\mathfrak{R}-\mathfrak{A}} f(s,x,s+\Delta,z)\Theta'\rho^3(x,z)dV_z = \\ \sum g_{ij}b_{ij}(s,x,\Delta) + \Theta''\nu(s,x,\Delta), \quad |\Theta''| \leq C''. \end{aligned} \tag{10}$$

But since, by the continuity condition (4),

$$\frac{\beta(s,x,\Delta)}{\nu(s,x,\Delta)} \to +\infty \quad \text{as } \Delta \to 0, \tag{11}$$

formula (10) implies that

$$\frac{\sum g_{ij}b_{ij}(s,x,\Delta)}{\nu(s,x,\Delta)} \to +\infty \quad \text{as } \Delta \to 0. \tag{12}$$

Now, for fixed x, y, s, τ, t, $s < \tau < t$, we consider only Δ so small that $s + \Delta < \tau$. Then $f(s+\Delta, z, t, y)$ and its derivatives with respect to z up to the fourth order are uniformly bounded and continuous in $\mathfrak{A}$ (we assume that $\mathfrak{A}$ is compact). Hence, for every point z in $\mathfrak{A}$ we have

$$\begin{aligned} f(s+\Delta,z,t,y) - f(s+\Delta,x,t,y) = \sum(z_i - x_i)\frac{\partial}{\partial x_i}f(s+\Delta,x,t,y) + \\ +\tfrac{1}{2}\sum(z_i - x_i)(z_j - x_j)\frac{\partial^2}{\partial x_i \partial x_j}f(s+\Delta,x,t,y) + \Theta\rho^3(x,z), \quad |\Theta| \leq C, \end{aligned} \tag{13}$$

where C does not depend on Δ or z. On the other hand, the fundamental equation (3) implies that

$$\begin{aligned} f(s,x,t,y) &= \int_{\mathfrak{R}} f(s,x,s+\Delta,z)f(s+\Delta,z,t,y)dV_z = \\ &= \int_{\mathfrak{R}} f(s,x,s+\Delta,z)f(s+\Delta,x,t,y)dV_z + \\ &+ \int_{\mathfrak{A}} f(s,x,s+\Delta,z)\{f(s+\Delta,z,t,y)-f(s+\Delta,x,t,y)\}dV_z + \\ &+ \int_{\mathfrak{R}-\mathfrak{A}} f(s,x,s+\Delta,z)\{f(s+\Delta,z,t,y)-f(s+\Delta,x,t,y)\}dV_z = \\ &= I_1+I_2+I_3. \end{aligned} \tag{14}$$

By (2),

$$\begin{aligned} I_1 &= \int_{\mathfrak{R}} f(s,x,s+\Delta,z)f(s+\Delta,x,t,y)dV_z = \\ &= f(s+\Delta,x,t,y)\int_{\mathfrak{R}} f(s,x,s+\Delta,z)dV_z = f(s+\Delta,x,t,y). \end{aligned} \tag{15}$$

Then (13), (5) and (6) imply that

$$\begin{aligned} I_2 &= \int_{\mathfrak{A}} f(s,x,s+\Delta,z)\{f(s+\Delta,z,t,y)-f(s+\Delta,x,t,y)\}dV_z = \\ &= \int_{\mathfrak{A}} f(s,x,s+\Delta,z)\Big\{\sum(z_i-x_i)\frac{\partial}{\partial x_i}f(s+\Delta,x,t,y) + \\ &+ \tfrac{1}{2}\sum(z_i-x_i)(z_j-x_j)\frac{\partial^2}{\partial x_i\partial x_j}f(s+\Delta,x,t,y) + \\ &+ \Theta\rho^3(x,z)\Big\}dV_z = \sum a_i(s,x,\Delta)\frac{\partial}{\partial x_i}f(s+\Delta,x,t,y) + \\ &+ \tfrac{1}{2}\sum b_{ij}(s,x,\Delta)\frac{\partial^2}{\partial x_i\partial x_j}f(s+\Delta,x,t,y) + \\ &+ \int_{\mathfrak{A}} f(s,x,s+\Delta,z)\Theta\rho^3(x,z)dV_z. \end{aligned} \tag{16}$$

Finally, since throughout $\mathfrak{R}-\mathfrak{A}$ we have

$$\rho^3(x,z) > K > 0,$$

where K does not depend on z, in $\mathfrak{R}-\mathfrak{A}$ we can set

$$f(s+\Delta,z,t,y) - f(s+\Delta,x,t,y) = \Theta'\rho^3(x,z).$$

Then

$$I_3 = \int_{\mathfrak{R}-\mathfrak{A}} f(s,x,s+\Delta,z)\{f(s+\Delta,z,t,y) - f(s+\Delta,x,t,y)\}dV_z =$$
$$= \int_{\mathfrak{R}-\mathfrak{A}} f(s,x,s+\Delta,z)\Theta'\rho^3(x,z)dV_z, \quad |\Theta'| \le C' = \frac{1}{K}. \tag{17}$$

Substituting (15)–(17) into (14) we finally obtain

$$f(s,x,t,y) = f(s+\Delta,x,t,y) + \sum a_i(s,x,\Delta)\frac{\partial}{\partial x_i}f(s+\Delta,x,t,y)+$$
$$+\tfrac{1}{2}\sum b_{ij}(s,x,\Delta)\frac{\partial^2}{\partial x_i \partial x_j}f(s+\Delta,x,t,y)+$$
$$+\int_{\mathfrak{R}} f(s,x,s+\Delta,z)\Theta''\rho^3(x,z)dV_z, \quad |\Theta''| \le C''. \tag{18}$$

If we also take into account the obvious equality

$$\int_{\mathfrak{R}} f(s,x,s+\Delta,z)\Theta''\rho^3(x,z)dV_z = \Theta''' \int_{\mathfrak{R}} f(s,x,s+\Delta,z)\rho^3(x,z)dV_z =$$
$$= \Theta'''\nu(s,x,\Delta), \quad |\Theta'''| \le C''',$$

then (18) can be rewritten as follows:

$$\frac{f(s+\Delta,x,t,y) - f(s,x,t,y)}{\Delta} = -\sum \frac{a_i(s,x,\Delta)}{\Delta}\frac{\partial}{\partial x_i}f(s+\Delta,x,t,y)-$$
$$-\sum \frac{b_{ij}(s,x,\Delta)}{2\Delta}\frac{\partial^2}{\partial x_i \partial x_j} - f(s+\Delta,x,t,y) - \Theta'''\frac{\nu(s,x,\Delta)}{\Delta}. \tag{19}$$

The left-hand side in (19) tends to $\frac{\partial}{\partial s}f(s,x,t,y)$ as $\Delta \to 0$.

Suppose that the determinant $D^N(s,x)$ is non-zero for $t_1, y_1, t_2, y_2, \ldots$ $\ldots, t_N, y_N$. Then $D^N(s+\Delta,x) \neq 0$ for sufficiently small Δ. Hence, there exist $\lambda_k(\Delta)$, $k = 1,2,\ldots,N$, such that

$$\begin{aligned} &\sum_k \lambda_k(\Delta)\frac{\partial}{\partial x_i}f(s+\Delta,x,t_k,y_k) = \alpha_i, \\ &\sum_k \lambda_k(\Delta)\frac{\partial^2}{\partial x_i \partial x_j}f(s+\Delta,x,t_k,y_k) = \alpha_{ij}. \end{aligned} \tag{20}$$

If we multiply (19) by $\lambda_k(\Delta)$ with $t = t_k$ and $y = y_k$ and sum all the N equalities thus obtained, then we have

$$\sum_k \lambda_k(\Delta)\frac{f(s+\Delta,x,t_k,y_k) - f(s,x,t_k,y_k)}{\Delta} =$$
$$= -\sum_i \alpha_i \frac{a_i(s,x,\Delta)}{\Delta} - \sum_{i,j} \alpha_{ij}\frac{b_{ij}(s,x,\Delta)}{2\Delta} - \sum_k \lambda_k(\Delta)\Theta_k'''\frac{\nu(s,x,\Delta)}{\Delta}. \tag{21}$$

If Δ tends to zero, then the $\lambda_k(\Delta)$, as solutions of (20), tend to the solution $\lambda_k(0)$ of the equations

$$\begin{aligned}\sum_k \lambda_k(0)\frac{\partial}{\partial x_i} f(s,x,t_k,y_k) &= \alpha_i,\\ \sum_k \lambda_k(0)\frac{\partial^2}{\partial x_i \partial x_j} f(s,x,t_k,y_k) &= \alpha_{ij}.\end{aligned} \tag{22}$$

Hence, the left-hand side of (21) has a finite limit

$$\Lambda_0 = \sum_k \lambda_k(0)\frac{\partial}{\partial s} f(s,x,t_k,y_k) \tag{23}$$

as $\Delta \to 0$.

In particular, if we set $\alpha_i = 0$, $\alpha_{ij} - g_{ij}$, then

$$\frac{\sum g_{ij} b_{ij}(s,x,\Delta)}{2\Delta} + \sum \lambda_k(\Delta)\Theta_k''' \frac{\nu(s,x,\Delta)}{\Delta} \to \Lambda_0 \quad \text{as } \Delta \to 0. \tag{24}$$

By (12), the second term in (24) is infinitesimally small as compared with the first one (since the $\lambda_k(\Delta)$ are bounded). Hence we have

$$\sum g_{ij} b_{ij}(s,x,\Delta)/2\Delta \to \Lambda_0 \quad \text{as } \Delta \to 0. \tag{25}$$

But (25) and (12) imply

$$\nu(s,x,\Delta)/\Delta \to 0 \quad \text{as } \Delta \to 0. \tag{26}$$

If we now equate all but one of the coefficients α_i and α_{ij} in (21) to zero, then a similar passage to the limit using (26) shows that all the limits

$$A_i(s,x) = \lim \frac{a_i(s,x,\Delta)}{\Delta} \quad \text{as } \Delta \to 0, \tag{27}$$

$$B_{ij}(s,x) = \lim \frac{b_{ij}(s,x,\Delta)}{2\Delta} \quad \text{as } \Delta \to 0, \tag{28}$$

exist and do not depend on the choice[7] of $\mathfrak{A}$. Then (27), (28), (26) and (19) immediately imply the *first differential equation*

$$\frac{\partial}{\partial s} f(s,x,t,y) = -\sum A_i(s,x)\frac{\partial}{\partial x_i} f(s,x,t,y) - \sum B_{ij}(s,x)\frac{\partial^2}{\partial x_i \partial x_j} f(s,x,t,y). \tag{29}$$

[7] See A.M., §13, formulas (122)–(124).

Certainly the condition that $D_N(s,x)$ does not vanish identically can be replaced by the direct requirement that the limits (27) and (28) exist, since (28) implies the existence of a finite limit (25) and therefore of (26).

At certain exceptional points the limits (27) and (28) need not exist. This was illustrated in A.M.[8] by the following example: $\mathfrak{R}$ is the ordinary number axis and

$$f(s,x,t,y) = \frac{3y^2}{2\sqrt{\pi(t-s)}}\exp\Big[-\frac{(y^3-x^3)^2}{4(t-s)}\Big]; \tag{30}$$

for $x = 0$ we easily obtain

$$b(s,x,\Delta)/2\Delta \to +\infty \quad \text{as } \Delta \to 0.$$

Hence there is no finite limit $B(s,x)$.

§2. The second differential equation

Assume now that in a neighbourhood $\mathfrak{A}$ of the point y_0 for a given t the limits $A_i(t,y)$ and $B_{ij}(t,y)$ exist uniformly and that $\nu(t,y,\Delta)/\Delta$ tends uniformly to 0 in $\mathfrak{A}$. Suppose further that $R(y)$ is a non-negative function vanishing outside $\mathfrak{A}$ with bounded derivatives up to the third order. Then for $y \in \mathfrak{A},\ z \in \mathfrak{A}$ we have

$$\begin{aligned} R(y) = R(z) + \sum (y_i - z_i)\frac{\partial}{\partial z_i}R(z) + \\ + \tfrac{1}{2}\sum (y_i - z_i)(y_j - z_j)\frac{\partial^2}{\partial z_i \partial z_j}R(z) + \\ + \Theta' \rho^3(y,z), \quad |\Theta'| \le C', \end{aligned} \tag{31}$$

whereas for $y \in \mathfrak{R} - \mathfrak{A}$ and $z \in \mathfrak{A}$,

$$R(y) = R(z) + \Theta'' \rho^3(y,z), \quad |\Theta''| \le C''. \tag{32}$$

Finally, for $y \in \mathfrak{R} - \mathfrak{A},\ z \in \mathfrak{R} - \mathfrak{A}$

$$R(y) = 0. \tag{33}$$

[8] See A.M., §13, formula (126).

If in the corresponding regions $R(y)$ is replaced by (31)–(33), we obtain

$$\int_{\mathfrak{A}} R(y)\frac{\partial}{\partial t}f(s,x,t,y)dV_y =$$

$$= \frac{\partial}{\partial t}\int_{\mathfrak{A}} R(y)f(s,x,t,y)dV_y = \frac{\partial}{\partial t}\int_{\mathfrak{R}} R(y)f(s,x,t,y)dV_y =$$

$$= \lim \frac{1}{\Delta}\int_{\mathfrak{R}} R(y)[f(s,x,t+\Delta,y) - f(s,x,t,y)]dV_y =$$

$$= \lim \frac{1}{\Delta}\Big\{\int_{\mathfrak{R}} R(y)\int_{\mathfrak{R}} f(s,x,t,z)f(t,z,t+\Delta,y)dV_z dV_y -$$

$$- \int_{\mathfrak{R}} R(y)f(s,x,t,y)dV_y\Big\} =$$

$$= \lim \frac{1}{\Delta}\Big\{\int_{\mathfrak{R}} f(s,x,t,z)\int_{\mathfrak{R}} R(y)f(t,z,t+\Delta,y)dV_y dV_z -$$

$$- \int_{\mathfrak{R}} R(z)f(s,x,t,z)dV_z\Big\} =$$

$$= \lim \frac{1}{\Delta}\Big\{\int_{\mathfrak{A}} f(s,x,t,z)\int_{\mathfrak{R}} R(z)f(t,z,t+\Delta,y)dV_y dV_z +$$

$$+ \int_{\mathfrak{A}} f(s,x,t,z)\int_{\mathfrak{A}}\Big[\sum (y_i - z_i)\frac{\partial}{\partial z_i}R(z) +$$

$$+\tfrac{1}{2}\sum (y_i - z_i)(y_j - z_j)\frac{\partial^2}{\partial z_i \partial z_j}R(z)\Big] f(t,z,t+\Delta,y)dV_y dV_z +$$

$$+ \int_{\mathfrak{A}} f(s,x,t,z)\int_{\mathfrak{R}} \Theta''' \rho^3(y,z)f(t,z,t+\Delta,y)dV_y dV_z -$$

$$- \int_{\mathfrak{A}} R(z)f(s,x,t,z)dV_z\Big\} = \lim \frac{1}{\Delta}\Big\{\int_{\mathfrak{A}} f(s,x,t,z)R(z)dV_z +$$

$$+ \int_{\mathfrak{A}} f(s,x,t,z)\Big[\sum a_i(t,z,\Delta)\frac{\partial}{\partial z_i}R(z) +$$

$$+\tfrac{1}{2}\sum b_{ij}(t,z,\Delta)\frac{\partial^2}{\partial z_i \partial z_j}R(z)\Big] dV_z +$$

$$+\Theta \int_{\mathfrak{A}} f(s,x,t,z)\nu(t,z,\Delta)dV_z - \int_{\mathfrak{A}} f(s,x,t,z)R(z)dV_z\Big\} =$$

$$= \int_{\mathfrak{A}} f(s,x,t,z)\Big[\sum A_i(t,z)\frac{\partial}{\partial z_i}R(z)+$$

$$+\sum B_{ij}(t,z)\frac{\partial^2}{\partial z_i \partial z_j}R(z)\Big]dV_z.$$

Replacing z by y in the right-hand side of the equation we obtain

$$\int_{\mathfrak{R}} R(y)\frac{\partial}{\partial t}f(s,x,t,y)dV_y = \int_{\mathfrak{A}} f(s,x,t,y)\Big[\sum A_i(t,y)\frac{\partial}{\partial y_i}R(y)+$$

$$+\sum B_{ij}(t,y)\frac{\partial^2}{\partial y_i \partial y_j}R(y)\Big]dV_y. \qquad (34)$$

Now assume that $A_i(t,z)$ and $B_{ij}(t,z)$ are twice continuously differentiable in $\mathfrak{A}$. Then we set

$$Q(t,y) = |g_{ij}(t,y)|$$

and after integration by parts, we obtain

$$\int_{\mathfrak{A}} f(s,x,t,y)A_i(t,y)\frac{\partial}{\partial y_i}R(y)dV_y =$$

$$= \int_{\mathfrak{A}} f(s,x,t,y)A_i(t,y)Q(t,y)\frac{\partial}{\partial y_i}R(y)dy_1 dy_2 \dots dy_n =$$

$$= -\int_{\mathfrak{A}} \frac{\partial}{\partial y_i}[f(s,x,t,y)A_i(t,y)Q(t,y)]R(y)dy_1 dy_2 \dots dy_n. \qquad (35)$$

Double integration by parts (since all the derivatives vanish on the boundary of $\mathfrak{A}$) yields

$$\int_{\mathfrak{A}} f(s,x,t,y)B_{ij}(t,y)\frac{\partial^2}{\partial y_i \partial y_j}R(y)dV_y =$$

$$= \int_{\mathfrak{A}} \frac{\partial^2}{\partial y_i \partial y_j}[f(s,x,t,y)B_{ij}(t,y)Q(t,y)]R(y)dy_1 dy_2 \dots dy_n. \qquad (36)$$

Formulas (34)–(36) immediately imply that

$$\int_{\mathfrak{A}} R(y)Q(t,y)\frac{\partial}{\partial t}f(s,x,t,y)dy_1 dy_2 \dots dy_n =$$

$$= \int_{\mathfrak{A}} R(y)\Big\{-\sum \frac{\partial}{\partial y_i}[A_i(t,y)Q(t,y)f(s,x,t,y)]+$$

$$+\sum \frac{\partial^2}{\partial y_i \partial y_j}[B_{ij}(t,y)Q(t,y)f(s,x,t,y)]\Big\}dy_1 dy_2 \dots dy_n.$$

Since $R(y)$ is arbitrary, apart from the above conditions, it is easy to conclude that at interior points of $\mathfrak{A}$ the *second differential equation*

$$Q(t,y)\frac{\partial}{\partial t}f(s,x,t,y) = -\sum \frac{\partial}{\partial y_i}[A_i(t,y)Q(t,y)f(s,x,t,y)] + \\ + \sum \frac{\partial^2}{\partial y_i \partial y_j}[B_{ij}(t,y)Q(t,y)f(s,x,t,y)] \quad (37)$$

also holds.

If at time t_0 the differential function of the probability distribution is given, that is, a non-negative function $g(t_0, y)$ of y satisfying the condition

$$\int_{\mathfrak{R}} g(t_0, y)dV_y = 1, \quad (38)$$

then for arbitrary $t > t_0$ the distribution function $g(t, y)$ is given by the formula

$$g(t,y) = \int_{\mathfrak{R}} g(t_0, x)f(t_0, x, t, y)dV_x. \quad (39)$$

The function $g(t, y)$ satisfies the equation [9]

$$Q\frac{\partial g}{\partial t} = -\sum \frac{\partial}{\partial y_i}(A_i Q g) + \sum \frac{\partial^2}{\partial y_i \partial y_j}(B_{ij} Q g). \quad (40)$$

§3. Uniqueness

Under a change of the coordinate system the coefficients $A_i(s, x)$ and $B_{ij}(s, x)$ are transformed in the following way:

$$A'_i = \sum \frac{\partial x'_i}{\partial x_k} A_k + \sum \frac{\partial^2 x'_i}{\partial x_k \partial x_l} B_{kl}, \quad (41)$$

$$B'_{ij} = \sum \frac{\partial x'_i}{\partial x_k}\frac{\partial x'_j}{\partial x_l} B_{kl}. \quad (42)$$

Here we always have

$$B_{ii} = \lim \frac{b_{ii}(s,x,\Delta)}{2\Delta} = \lim \frac{1}{2\Delta}\int_{\mathfrak{A}} f(s,x,s+\Delta,z)(z_i - x_i)^2 dV_z \geq 0. \quad (43)$$

Hence the *quadratic form*

$$\sum B_{ij}\xi_i\xi_j \quad (44)$$

[9] See A.M., §18, formulas (169) and (170).

is non-negative. This is crucial in the proof of the following theorem.[10]

Uniqueness Theorem 1. *If $\mathfrak{R}$ is closed, then* (40) *has at most one solution $g(t, y)$ with given continuous initial condition $g(t_0, y) = g(y)$.*

Proof. Clearly it suffices to consider the initial condition $g(t_0, y) = 0$ and prove that $g(t, y) = 0$ also for $t > t_0$. We can transform (40) into the form

$$\frac{\partial g}{\partial t} = \sum B_{ij} \frac{\partial^2 g}{\partial y_i \partial y_j} + \sum S_i \frac{\partial g}{\partial y_i} + Tg. \tag{45}$$

Now set

$$v(t, y) = g(t, y)e^{-ct}.$$

The function $v(t, y)$ satisfies the equation

$$\frac{\partial v}{\partial t} = \sum B_{ij} \frac{\partial^2 v}{\partial y_i \partial y_j} + \sum S_i \frac{\partial v}{\partial y_i} + Tv - cv. \tag{46}$$

For fixed t_0 and t_1 the constant c can be chosen so large that

$$T(t, y) - c < 0$$

for all y and t, $t_0 \leq t \leq t_1$. Under these conditions $v(t, y)$ cannot have a positive maximum at any point (t, y), $t_0 < t < t_1$, since at such a maximum

$$\frac{\partial v}{\partial t} = 0, \quad \frac{\partial v}{\partial y_i} = 0, \quad \sum B_{ij} \frac{\partial^2 v}{\partial y_i \partial y_j} \leq 0, \quad (T - c)v < 0,$$

which contradicts (46). Neither can there be a negative minimum of $v(t, y)$ within these limits. Since $v(t_0, y) = 0$ at $t = t_0$, we obtain for $t_0 < t < t_1$,

$$v(t, y) < \max v(t_1, y) = e^{-ct_1} \max g(t_1, y)$$

$$g(t, y) < e^{-c(t_1 - t)} \max g(t_1, y).$$

Since c was arbitrarily large, it follows that

$$g(t, y) = 0.$$

[10] See: E. Rothe, 'Über die Wärmeleitungsgleichung', *Math. Ann.* **104** (1931), 353–354 (uniqueness proof).

Uniqueness Theorem 2. *Let $\mathfrak{R}$ be closed. Then there is at most one non-negative continuous solution $f(s,x,t,y)$ for* (2) *and* (3) *that satisfies* (29) *with given twice continuously differentiable coefficients $A_i(t,y)$ and $B_{ij}(t,y)$, and the continuity condition* (4).

The continuity condition (4) *can be replaced by the following, weaker one:*

$$\int_{\mathfrak{R}} f(s,x,t,y)\rho^2(x,y)dV_y \to 0 \quad \text{as } t \to s. \tag{47}$$

Proof. Assume that two different functions $f_1(s,x,t,y)$ and $f_2(s,x,t,y)$ satisfy all our conditions. Then we can choose s and a continuous function $g(x)$ such that

$$g_1(t,y) = \int_{\mathfrak{R}} g(x)f_1(s,x,t,y)dV_x,$$

$$g_2(t,y) = \int_{\mathfrak{R}} g(x)f_2(s,x,t,y)dV_x$$

are also different. By (2) and (47), $g_1(t,y)$ and $g_2(t,y)$ tend to $g(y)$ as $t \to s$. Since the functions $g_1(t,y)$ and $g_2(t,y)$ satisfy (40), this contradicts Uniqueness Theorem 1.

§4. An example

The following example, which is interesting also for applications, demonstrates that the quadratic form (44) need not be positive definite: let $\mathfrak{R}$ be the usual Euclidean plane and let

$$f(s,x_1,x_2,t,y_1,y_2) = \frac{2\sqrt{3}}{\pi(t-s)^2}\exp\Big\{-\frac{(y_1-x_1)^2}{4(t-s)} - \frac{3[y_2-x_2-(t-s)(y_1+x_2)/2]^2}{(t-s)^3}\Big\}. \tag{48}$$

A simple computation shows that

$$B_{11}=1, \quad B_{12}=0, \quad B_{22}=0, \quad A_1=0, \quad A_2(s,x)=x_1.$$

§5. The limit solution

Let $\mathfrak{R}$ be closed and $f(s,x,t,y)$ everywhere positive and dependent only on the difference $t-s$:

$$f(s,x,t,y) = \phi(t-s,x,y). \tag{49}$$

Then general ergodic theorems[11] imply the existence of the limit probability distribution. In other words, for any distribution $g(t,y)$ determined by (38) and (39) and any region $\mathfrak{E}$ the relation

$$\int_{\mathfrak{E}} g(t,y)dV_y \to P(\mathfrak{E}) \quad \text{as } t \to +\infty, \tag{50}$$

holds, where $P(\mathfrak{E})$ does not depend on $g(t_0,y)$. It can easily be proved that $g(t,y)$ is uniformly continuous for large t. From this we deduce that[12]

$$P(\mathfrak{E}) = -\int_{\mathfrak{E}} g(y)dV_y, \tag{51}$$

$$g(t,y) \to g(y) \quad \text{as } t \to +\infty. \tag{52}$$

Clearly, $g(y)$ and $P(\mathfrak{E})$ do not depend on $g(t_0,y)$.

Now, let $g(y)$ be the solution of the equations

$$-\sum \frac{\partial}{\partial y_i}[A_i(y)Q(y)g(y)] + \sum \frac{\partial^2}{\partial y_i \partial y_j}[B_{ij}(y)Q(y)g(y)] = 0, \tag{53}$$

$$\int_{\mathfrak{R}} g(y)dV_y = 1. \tag{53a}$$

Setting $g(t_0,y) = g(y)$ it can easily be seen that $g(t,y) = g(y)$ also for $t > t_0$ (see (40) and Uniqueness Theorem 1). From this we deduce that the *solution of* (53) *and* (53a) (*if it exists*) *is uniquely determined and coincides with the limit function* $g(y)$.

As a particular case, (52) implies

$$f(s,x,t,y) \to g(y) \quad \text{as } t \to +\infty. \tag{54}$$

Klyazma, near Moscow, 12 April 1932

[11] See A.M., §4, Theorem IV.

[12] See footnote 1.