# From Archimedes to statistics: the area of the sphere.

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## I The area of a sphere : some history.

Archimedes (287-212) has shown that when you put a sphere in a cylinder in the tightest manner, the lateral area of the cylinder is the area of the sphere itself. This is not an easy result, and Archimedes was so proud of it that he asked that the corresponding picture should be engraved on his tomb in Agrigente in Sicily : 150 years after his murder during the siege of Syracuse by an ignorant Roman soldier (imagine Werner von Braun killed by a GI in 1945), this detail of the sphere inside the cylinder enabled Cicero to discover the grave and to restore it in the year 75.

Actually, the Archimedes' result is even more precise. Indeed if you cut off the whole by a plane perpendicular to the axis of the cylinder, the remainders of the cylinder and of the sphere have still the same area. I learnt of this result when I was a I5 years old schoolboy (with the original proof that I shall sketch in a few seconds). In these far away times, the proof was actually belonging to the syllabus of the "classe de Première" of the lycées, the 11th grade of the French system. Although these syllabus have changed, I have observed recently that the statement is still fascinating for my grand children.

The principle of the proof given by Archimedes can be easily rediscovered by somebody who likes mathematics, even elementary ones. However, it uses a clever trick which simplifies the calculation : you are stalled if you do not have it. Finally, it still has two minor defects : slight lack of rigor and complication. There are some other proofs, which use classical calculus learnt during first years at the university. I will also give you a completely different proof which uses a weapon borrowed to statistics : the normal distribution.

## II Archimedes' proof.

Consider a sphere of radius $R$ in the three dimensional space and fix a number $a$ in $[0, 2R]$. The lateral area of the cylinder of diameter $2R$ and height $a$ is $2\pi Ra$ : just incise and unfold to get a rectangle of dimensions $2\pi R$ and $a$. The corresponding part of the sphere is called a *calotte* if $a \leq R$ from the Italian calotta, a brimmed hat. We thus have to show that the area of the calotte is $2\pi Ra$. For this, Archimedes approximates the calotte by a finite union of truncated cones, computes the lateral area of each of these truncated cones sums up and passes to the limit to get the desired result. Let us now work on the details :

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## II.1 The area of a cone of revolution.

Imagine a clown's hat of height $h$ and with $r$ as the radius of the basic circle (Well, rather a Chinese hat if $h/r$ is small). What is its area ? Answer : $\pi r\sqrt{r^2+h^2}$. To see this, incise and unfold the hat into a sector. The radius of the sector is $\sqrt{r^2+h^2}$, but finding the value of the angle of the sector $2\pi\alpha$, with $0 < \alpha < 1$ is not so easy. What is known is the length of the circular arc of the sector, namely $2\pi r$. Well, this gives $\alpha$ since this length is also $2\pi\alpha\sqrt{r^2+h^2}$ and we get

$$\alpha = \frac{r}{\sqrt{r^2+h^2}}.$$

Now the area of the sector is exactly $\pi(\sqrt{r^2+h^2})^2$ multiplied by $\alpha$ and we get the result.

## II.2 The area of a truncated cone.

Now we take a clown's hat associated to $(h, r)$, we fix $0 < h' < h$ and we truncate at height $h'$ (that means that we create a new clown's hat associated to $(h', r')$ with $\frac{h}{r} = \frac{h'}{r'}$. Removing the new from the old gives a truncated cone $T$ with area

$$\mathcal{A} = \pi r\sqrt{r^2+h^2} - \pi r'\sqrt{r'^2+h'^2}.$$

Another presentation of $\mathcal{A}$ is obtained by observing

$$\sqrt{1+\frac{h^2}{r^2}} = \sqrt{1+\frac{h'^2}{r'^2}} \quad\Rightarrow\quad r'\sqrt{r^2+h^2} = r\sqrt{r'^2+h'^2}$$

(multiply both sides of the first equality by $rr'$ to get the second) which leads to

$$\mathcal{A} = \pi(r+r')(\sqrt{r^2+h^2} - \sqrt{r'^2+h'^2}) = 2\pi\frac{r+r'}{2} \times (\sqrt{r^2+h^2} - \sqrt{r'^2+h'^2}) \tag{2.1}$$

"The area of the truncated cone is the product of the half sum of the perimeter of the external circles by the length of the lateral edge".

## II.3 Archimedes' trick.

Consider the two circles limiting the truncated cone : there exists one (and only one) sphere $\mathbb{S}$ containing these two circles. Take a plane $P$ through the axis $D$ of symmetry of $T$. The plane $P$ cuts $\mathbb{S}$ under a circle whose center is the center $O$ of the sphere and cuts $T$ under two segments $AA'$ and $BB'$. The points $A$ and $B$ are symmetric with respect to $D$ and we denote by $C$ the intersection of $AB$ and $D$. Similarly, $A'$ and $B'$ are symmetric with respect to $D$ and we denote by $C'$ the intersection of $A'B'$ and $D$. Finally, denote by $H$ the midpoint of $AA'$ and by $M$ the midpoint of $CC'$. A reformulation of 2.1 is

$$\mathcal{A} = 2\pi \times MH \times AA'.$$

The basic remark, Archimedes' trick, is that

$$\mathcal{A} = 2\pi \times OH \times CC'. \tag{2.2}$$

There are many ways to see this but the closest to Archimedes' method is to consider the perpendicular projection $A''$ from $A'$ onto $AC$ to observe that the two triangles $AA'A''$ and $HOM$ are similar since their sides are perpendicular and to conclude

$$\frac{HM}{HO} \overset{(a)}{=} \frac{A''A'}{AA'} \overset{(b)}{=} \frac{CC'}{AA'}$$

(here (a) comes from similarity and (b) comes from the obvious equality $A''A' = CC'$).

### II.4 End of the proof.

Now this is too easy. We split the calotte governed by the parameters $R$ and $a \leq R$ with parallel planes $P_1, \ldots, P_n$ perpendicular to the axis of the calotte, with distances of each other equal to $a/n$. These planes cut the calotte under circles $C_1, \ldots C_n$. We call $C_0$ the pole of the calotte. The pair $C_{k-1}, C_k$ creates a truncated cone $T_k$ whose area $\mathcal{A}_k$ is given by the formula 2.2. Adding the whole the approximation of the area of the calotte is, with obvious notations

$$\begin{aligned} \mathcal{A}_1 + \ldots + \mathcal{A}_n &= 2\pi(OH_1 \times C_0C_1 + OH_2 \times C_1C_2 + \cdots + OH_n \times C_{n-1}C_n) \\ &= 2\pi a \times \frac{1}{n}(OH_1 + \cdots + OH_n) \end{aligned}$$

(we have used $C_{k-1}C_k = a/n$). From modern standards, the only weak point of the Archimedes' proof is that he considers the fact that $\lim_{n\to\infty} \frac{1}{n}(OH_1 + \cdots + OH_n) = R$ is obvious. If you want something more rigorous, observe that

$$0 \leq OH_n \leq OH_{n-1} \leq OH_1 \leq R$$

and that $\lim_{n\to\infty} OH_n = R$. Thus

$$OH_n \leq \frac{1}{n}(OH_1 + \cdots + OH_n) \leq R$$

will lead to the result.

## III The normal distribution and the uniform distribution on the sphere.

Let us assume that you have heard some statistics. If you repeat independently the same random experiment $n$ times, with issues $S$ (success) or $F$ (failure) such that the probability $p \in (0,1)$ of $S$ does not change with the experiment, the number $S_n$ of successes in these $n$ experiments is a random variable such that for $k = 0, \ldots, n$ one has

$$\Pr(S_n = k) = \frac{n!}{(n-k)!k!} p^k (1-p)^{n-k}.$$

A great result of the 18th century mathematics is the Moivre Laplace theorem which says that

$$\lim_{n\to\infty} \Pr\left(\frac{S_n - np}{\sqrt{np(1-p)}} \leq x\right) = \int_{-\infty}^{x} e^{-\frac{t^2}{2}} \frac{dt}{\sqrt{2\pi}}.$$

In some sense $X = \lim_{n\to\infty} \frac{S_n - np}{\sqrt{np(1-p)}}$ exists and is a continuous random variable with density $e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}}$. We call it a normal variable. One more hat in this story is the graph of its density : after the calotte, the clown's hat, the Chinese hat, we have the Napoleon's hat.

Now, consider two other normal variables $Y$ and $Z$ such that $X, Y, Z$ are independent (we can construct other random sequences of independent experiments leading to $S'_n$ and $S''_n$ for obtaining such $Y$ and $Z$ if we have some doubts about their existence). A striking fact is that the random vector of $\mathbb{R}^3$ defined by $V = (X, Y, Z)$ has density

$$e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} \frac{dy}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \frac{dz}{\sqrt{2\pi}} = e^{-\frac{x^2+y^2+z^2}{2}} \frac{dxdydz}{(2\pi)^{3/2}} = e^{-\frac{\|v\|^2}{2}} \frac{dv}{(2\pi)^{3/2}}.$$

Denote by $R = \|V\| = (X^2 + Y^2 + Z^2)^{1/2}$ the distance to zero of the random point $V$ of the space $\mathbb{R}^3$ (this explicit formula comes from the Pythagoras' Theorem). The remarkable point is that the density of $V$ is a function of $r = \|v\| = (x^2 + y^2 + z^2)^{1/2}$ only. This implies that the distribution of $U = \frac{V}{R}$, which is by definition concentrated on the sphere $\mathbb{S}$ of $\mathbb{R}^3$ with center 0 and radius 1, will not change if we rotate the space $\mathbb{R}^3$ in an arbitrary manner. That means that $U$ has the *uniform distribution* on the sphere. That means that if $E$ is any subset of $\mathbb{S}$, then the probability such that $U$ falls into $E$ is proportional to the area $\mathcal{A}(E)$ of $E$. More specifically :

$$\Pr(U \in E) = \frac{\mathcal{A}(E)}{\mathcal{A}(\mathbb{S})}.$$

At this point, you may stop me and tell me that Archimedes says $\mathcal{A}(\mathbb{S}) = 4\pi$ but since we are suppose to have an other proof of Archimedes' statements, we are not allowed to use it. Okay : let us keep $A(\mathbb{S})$ unknown for the proof. Actually, the statement of Archimedes that we aim to prove is that the area of a calotte is the area of the surrounding cylinder. Let us call $E_a$ a calotte of $\mathbb{S}$ governed by $a < 1$. We want to prove that

$$\mathcal{A}(E_a) \stackrel{?}{=} \frac{a}{2}\mathcal{A}(\mathbb{S}).$$

In terms of $U$ this can be rewritten as

$$\Pr(U \in E_a) \stackrel{?}{=} \frac{a}{2}.$$

And after all, since we can rotate the space in an arbitrary manner without changing the distribution of $U$ this can be rewritten

$$\Pr(\frac{Z}{R} \in [1-a, 1]) \stackrel{?}{=} \frac{a}{2}.$$

Because of the equality of distribution of $Z/R$ and $-Z/R$ we get

$$\begin{aligned} a &\stackrel{?}{=} \Pr(\frac{Z}{R} \in [1-a, 1] \cup [-1, a-1]) \\ &\stackrel{?}{=} \Pr(\frac{Z^2}{R^2} \in [(1-a)^2, 1]) \\ &\stackrel{?}{=} \Pr(\frac{Z^2}{X^2 + Y^2 + Z^2} \in [(1-a)^2, 1]) \\ &\stackrel{?}{=} \Pr(\frac{1}{\frac{X^2+Y^2}{Z^2} + 1} \in [(1-a)^2, 1]) \end{aligned}$$

For simplicity, let me write $T = \frac{1}{2}(X^2 + Y^2)$ and $S = \frac{1}{2}Z^2$. Let me introduce $k > 0$ such that $\frac{1}{k+1} = (1-a)^2$. We are led to

$$a \stackrel{?}{=} \Pr(\frac{T}{S} \leq k), \quad 1 - a \stackrel{?}{=} \Pr(\frac{T}{S} > k).$$

Classical calculations in statistics about chi square distributions show that

$$\Pr(T > t) = e^{-t}, \ \ \Pr(S > s) = \int_s^\infty e^{-u} u^{-1/2} \frac{du}{\sqrt{\pi}}$$

Thus

$$\Pr(\frac{T}{S} > k) = \Pr(T > kS) = \int_0^\infty e^{-(1+k)s} s^{-1/2} \frac{ds}{\sqrt{\pi}} = \frac{1}{(1+k)^{1/2}} = 1 - a.$$

Have we proved that the area of $E_a$ is the area of the surrounding cylinder ? Not quite, we have only $\mathcal{A}(E_a) = \frac{a}{2}\mathcal{A}(\mathbb{S})$ and the constant $\mathcal{A}(\mathbb{S})$ is still unknown. To find it, we analyze $\lim_{a\to 0} \mathcal{A}(E_a)/a = \frac{1}{2}\mathcal{A}(\mathbb{S})$. When $a$ is very small, $E_a$ is almost a disc, with radius

$$r_a = \sqrt{1 - (1-a)^2} = \sqrt{2a - a^2}$$

(draw a picture and use Pythagoras) : If we draw a small circle around us on the earth, it creates a calotte, but the earth is so locally flat that the area of the calotte is almost the area of the circle. Thus $\mathcal{A}(E_a)$ is approximated by $\pi r_a^2 = \pi(2a - a^2)$. Taking the limit we get $\mathcal{A}(\mathbb{S}) = 4\pi$ as desired.

## IV A more general result

What does Archimedes' result say in terms of probability distributions ? It says that if $U$ is uniform on the sphere, then the orthogonal projection of $U$ on any diameter is uniform on this diameter. Can we extend the idea ? Actually we have done calculations with three normal random variables since we had a definite geometrical application in mind. But consideration of $n$ normal independent variables is possible while $n$ dimensional spaces can create dizziness among the audience. The result which extends Archimedes' result is this :

"Let $U$ be a uniform random variable on the unit sphere $\mathbb{S}_{n-1}$ of the $n$ dimensional space $\mathbb{R}^n$. Select any $n-2$ dimensional subspace $H$ of $\mathbb{R}^n$ and consider the orthogonal projection $W$ of $U$ on $H$. Then $W$ is uniform on the unit *ball* of $H$."

The difference of dimension equal to 2 is quite important. The uniform distribution on $\mathbb{S}_3$ is projected on a two dimensional plane as a uniform distribution on a disc, but this is false if you do this from $\mathbb{S}_2$ to a plane. Details and proof using the normal distribution can be found in G. Letac, *Integration and Probability*, Springer (1995) pages 92-3.