

# Sparse estimation in high-dimensional models, II

Alexandre Tsybakov

Laboratoire de Statistique, CREST  
and  
Laboratoire de Probabilités et Modèles Aléatoires,  
Université Paris 6

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# Nonparametric regression model

Assume that we observe the pairs  $(X_1, Y_1), \dots, (X_n, Y_n) \in \mathbb{R}^d \times \mathbb{R}$  where

$$Y_i = f(X_i) + \xi_i, \quad i = 1, \dots, n.$$

- Regression function  $f : \mathbb{R}^d \rightarrow \mathbb{R}$  is unknown
- Errors  $\xi_i$  are independent Gaussian  $\mathcal{N}(0, \sigma^2)$  random variables.
- $X_i \in \mathbb{R}^d$  are arbitrary fixed (non-random) points.

We want to estimate  $f$  based on the data  $(X_1, Y_1), \dots, (X_n, Y_n)$ .

## Dictionary, linear approximation

Let  $f_1, \dots, f_M$  be a finite **dictionary of functions**,  $f_j : \mathbb{R}^d \rightarrow \mathbb{R}$ .  
We approximate the regression function  $f$  by linear combination

$$f_\lambda(x) = \sum_{j=1}^M \lambda_j f_j(x) \quad \text{with weights} \quad \lambda = (\lambda_1, \dots, \lambda_M).$$

We believe that

$$f(x) \approx \sum_{j=1}^M \lambda_j f_j(x)$$

for some  $\lambda = (\lambda_1, \dots, \lambda_M)$ .

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Possibly  $M \gg n$

# Scenarios

(LinReg) Exact equality: there exists  $\lambda^* \in \mathbb{R}^M$  such that

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(NPREg)  $f_1, \dots, f_M$  are the first  $M$  functions of a basis (usually orthonormal) and  $M \leq n$ , there exists  $\lambda^*$  such that  $f - f_{\lambda^*}$  is small: **nonparametric estimation of regression**;

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- (Weak) **learning**:  $f_1, \dots, f_M$  are “weak learners”, i.e., some rough approximations to  $f$ ;  $M$  is extremely large.

# Sparsity of a vector

The number of non-zero coordinates of  $\lambda$ :

$$M(\lambda) = \sum_{j=1}^M \mathbb{I}_{\{\lambda_j \neq 0\}}$$

The value  $M(\lambda)$  characterizes the **sparsity** of vector  $\lambda \in \mathbb{R}^M$ : the smaller  $M(\lambda)$ , the “sparser”  $\lambda$ .

# Sparsity of the model

**Intuitive formulation of sparsity assumption:**

$$f(x) \approx \sum_{j=1}^M \lambda_j f_j(x) \quad ("f \text{ is well approximated by } f_\lambda")$$

where the vector  $\lambda = (\lambda_1, \dots, \lambda_M)$  is sparse:

$$M(\lambda) \ll M.$$

# Strong sparsity

## Strong sparsity:

$f$  admits an exact sparse representation

$$f = f_{\lambda^*}$$

for some  $\lambda^* \in \mathbb{R}^M$ , with

$$M(\lambda^*) \ll M$$

⇒ Scenario (LinReg)

# Sparsity and dimension reduction

The empirical norm:

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Let  $\hat{\lambda}_{\text{OLS}}$  be the ordinary least squares (OLS) estimator.  
Elementary result:

$$\mathbb{E} \|\hat{f}_{\hat{\lambda}_{\text{OLS}}} - f\|_n^2 \leq \|f - f_\lambda\|_n^2 + \frac{\sigma^2 M}{n}$$

for any  $\lambda \in \mathbb{R}^M$ .

# Sparsity and dimension reduction

For any  $\lambda \in \mathbb{R}^M$  the “oracular” OLS that acts only on the relevant  $M(\lambda)$  coordinates satisfies

$$\mathbb{E} \|\hat{f}_{\hat{\lambda}_{\text{OLS}}}^{\text{oracle}} - f\|_n^2 \leq \|f - f_\lambda\|_n^2 + \frac{\sigma^2 M(\lambda)}{n}.$$

This is only an OLS oracle, not an estimator! The set of relevant coordinates should be known.

## Sparsity oracle inequalities

Do there exist estimators with similar behavior? Choose some other data-driven weights  $\hat{\lambda} = (\hat{\lambda}_1, \dots, \hat{\lambda}_M)$  and estimate  $f$  by

$$\hat{f}(x) = f_{\hat{\lambda}}(x) = \sum_{j=1}^M \hat{\lambda}_j f_j(x).$$

Can we find  $\hat{\lambda}$  such that

$$\mathbb{E} \|f_{\hat{\lambda}} - f\|_n^2 \lesssim \|f - f_{\lambda}\|_n^2 + \frac{\sigma^2 M(\lambda)}{n}, \quad \forall \lambda?$$

## Sparsity oracle inequalities (SOI)

Realizable task: look for an estimator  $f_{\hat{\lambda}}$  satisfying a **sparsity oracle inequality (SOI)**

$$\mathbb{E} \|f_{\hat{\lambda}} - f\|_n^2 \leq \inf_{\lambda \in \mathbb{R}^M} \left\{ C \|f - f_{\lambda}\|_n^2 + C' \frac{M(\lambda) \log M}{n} \right\}$$

with some constants  $C \geq 1$ ,  $C' > 0$  and an inevitable extra  $\log M$  in the variance term.  $C = 1 \Rightarrow$  **sharp SOI**.

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“In probability” form of sparsity oracle inequalities:  
with probability close to 1,

$$\|\hat{f}_\lambda - f\|_n^2 \leq \inf_{\lambda \in \mathbb{R}^M} \left\{ C \|f - f_\lambda\|_n^2 + C' \frac{M(\lambda) \log M}{n} \right\}.$$

## Implications of SOI: Scenario (LinReg)

Assume that we have found an estimator  $f_{\hat{\lambda}}$  satisfying SOI. Some consequences for different scenarios:

(LinReg) **linear regression**:  $f = f_{\lambda^*}$  for some  $\lambda^*$ . Using SOI:

$$\begin{aligned} \mathbb{E} \|f_{\hat{\lambda}} - f\|_n^2 &\leq C \left\{ \|f - f_{\lambda^*}\|_n^2 + \frac{M(\lambda^*) \log M}{n} \right\} \\ &= \frac{CM(\lambda^*) \log M}{n} \end{aligned}$$

(the desired result for Scenario (LinReg)).

## Implications of SOI: Scenario (NPREg)

(NPREg) **nonparametric regression.** If  $f$  belongs to standard smoothness classes of functions,  $\min_{\lambda \in \Lambda_m} \|f - f_\lambda\|_n \leq Cm^{-\beta}$  for some  $\beta > 0$  ( $\Lambda_m =$  the set of vectors with only first  $m$  non-zero coefficients,  $m \leq M$ ). Using SOI:

$$\begin{aligned} \mathbb{E} \|\hat{f}_\lambda - f\|_n^2 &\leq C \inf_{m \geq 1} \left\{ \min_{\lambda \in \Lambda_m} \|f - f_\lambda\|_n^2 + \frac{m \log M}{n} \right\} \\ &\leq C \inf_{m \geq 1} \left\{ \frac{1}{m^{2\beta}} + \frac{m \log M}{n} \right\} \\ &= O \left( \left( \frac{\log n}{n} \right)^{2\beta/(2\beta+1)} \right) \quad \text{for } M \leq n \end{aligned}$$

(optimal rate of convergence, up to logs, in Scenario (NPREg)).

## Implications of SOI: Scenario (Agg)

(Agg) **aggregation of arbitrary estimators**: in this case  $f_1, \dots, f_M$  are preliminary estimators of  $f$  based on a pilot (training) sample independent of the observations  $(X_1, Y_1), \dots, (X_n, Y_n)$ . The training sample is considered as frozen. Assume that SOI holds with leading constant 1. Then:

$$\begin{aligned} \mathbb{E} \|f_{\hat{\lambda}} - f\|_n^2 &\leq \inf_{\lambda \in \mathbb{R}^M} \left\{ \|f - f_{\lambda}\|_n^2 + \frac{CM(\lambda) \log M}{n} \right\} \\ &\leq \min_{1 \leq j \leq M} \|f - f_j\|_n^2 + \frac{C \log M}{n} \end{aligned}$$

$\implies f_{\hat{\lambda}}$  attains optimal rate of Model Selection type aggregation  $\frac{\log M}{n}$  (T., 2003).

## Implications of SOI: Scenario (Agg)

Similar conclusion holds for Convex aggregation. We restrict  $\lambda$  to the simplex

$$\Lambda^M = \{\lambda \in \mathbb{R}^M : \lambda_j \geq 0, \sum_{j=1}^M \lambda_j = 1\}.$$

From SOI with leading constant 1 + “Maurey argument”:

$$\begin{aligned} \mathbb{E} \|f_{\hat{\lambda}} - f\|_n^2 &\leq \inf_{\lambda \in \mathbb{R}^M} \left\{ \|f - f_\lambda\|_n^2 + \frac{CM(\lambda) \log M}{n} \right\} \\ &\leq \inf_{\lambda \in \Lambda^M} \|f - f_\lambda\|_n^2 + C' \sqrt{\frac{\log M}{n}}. \end{aligned}$$

$\implies f_{\hat{\lambda}}$  attains optimal rate of Convex aggregation  $\sqrt{\frac{\log M}{n}}$   
 [Nemirovski (2000), Juditsky and Nemirovski (2000)].

## Sparsity oracle inequalities

Conclusion: all these nice properties are simultaneously satisfied for one and the same procedure, whenever it obeys a SOI.

Ultimate target:

- no assumptions on the dictionary  $f_1, \dots, f_M$
- SOI with leading constant 1
- computational feasibility

## Definition of the BIC method

First idea: penalize least squares directly by  $M(\lambda)$  (BIC criterion, Schwarz (1978), Foster and George (1994)).

$$\hat{\lambda}^{BIC} = \arg \min_{\lambda \in \mathbb{R}^M} \left\{ \|\mathbf{y} - \mathbf{f}_\lambda\|_n^2 + \gamma \frac{M(\lambda) \log M}{n} \right\},$$

where  $\gamma > 0$  and

$$\|\mathbf{y} - \mathbf{f}_\lambda\|_n^2 \triangleq \frac{1}{n} \sum_{i=1}^n \left( Y_i - \mathbf{f}_\lambda(X_i) \right)^2, \quad \mathbf{y} = (Y_1, \dots, Y_n).$$

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Remarks:

- If the matrix  $X = (f_j(X_i))_{i,j}$  has orthonormal columns, BIC is equivalent to hard thresholding of the components of  $X^T \mathbf{y} / n$  at the level  $\sqrt{\gamma(\log M)/n}$ .
- In general: non-convex, discontinuous minimization problem!

## Sparsity oracle inequality for BIC

**Theorem.** [Bunea/ T/ Wegkamp (2004)]: if  $\gamma > K_0\sigma^2$  for an absolute constant  $K_0$ , and **with no assumption on the dictionary**  $f_1, \dots, f_M$ , the BIC estimator satisfies, with probability close to 1,

$$\|f_{\hat{\lambda}^{BIC}} - f\|_n^2 \leq (1+\varepsilon) \inf_{\lambda \in \mathbb{R}^M} \left\{ \|f - f_\lambda\|_n^2 + C(\varepsilon) \frac{M(\lambda) \log M}{n} \right\}, \quad \forall \varepsilon > 0.$$

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Remarks:

- the BIC is realizable only for small  $M$  (say,  $M \leq 20$ ),
- the leading constant is **not** 1.

# LASSO

Second popular idea: LASSO [Frank and Friedman (1993, "Bridge" regression), name: Tibshirani (1996), Chen and Donoho (1998, basis pursuit), etc.]: instead of penalizing the residual sum of squares by  $M(\lambda)$ , as in the BIC, penalize by the  $\ell_1$  norm of  $\lambda$ :

$$\hat{\lambda}^L = \arg \min_{\lambda \in \mathbb{R}^M} \{ \|\mathbf{y} - f_\lambda\|_n^2 + 2r|\lambda|_1 \},$$

where  $|\lambda|_1 = \sum_{j=1}^M |\lambda_j|$ ,  $r > 0$  a tuning constant. A sensible choice:

$$r = A \sqrt{\frac{\log M}{n}} \quad \text{for } A > 0 \text{ large enough.}$$

- If the matrix  $X = (f_j(X_i))_{i,j}$  has orthonormal columns, LASSO is equivalent to soft thresholding of the components of  $X^T \mathbf{y} / n$  at the level  $r$ .

## Relaxed positive definiteness

For a vector  $\Delta = (a_j)_{j=1,\dots,M}$  and a subset of indices  $J \subseteq \{1, \dots, M\}$  write

$$\Delta_J = (a_j \mathbf{1}\{a_j \in J\})_{j=1,\dots,M}.$$

The Gram matrix:

$$\Psi_M = (\langle f_j, f_{j'} \rangle_n)_{1 \leq j, j' \leq M}.$$

### Assumption RE( $s, c_0$ ). (Bickel/Ritov/T, 2007)

For an integer  $1 \leq s \leq M$  and  $c_0 > 0$  there exists  $\kappa = \kappa(s, c_0)$ :

$$\Delta^T \Psi_M \Delta \geq \kappa |\Delta_J|_2^2$$

for all  $J \subseteq \{1, \dots, M\}$  such that  $|J| \leq s$  and  $|\Delta_{J^c}|_1 \leq c_0 |\Delta_J|_1$ .

## More specific assumptions

Assumption RE is more general than other assumptions on the Gram matrix in the LASSO/Dantzig literature:

- “Uniform uncertainty principle” (Candes/Tao),
- Mutual coherence assumption (Bunea/T/Wegkamp),
- Incoherent design assumption (Meinshausen/Yu, Zhang/Huang).

Most of the LASSO/Dantzig papers focus on the linear regression scenario (LinReg).

## Sparsity oracle inequality for the LASSO

Theorem [Bickel, Ritov and T., 2007]

Let  $\|f_j\|_n = 1, j = 1, \dots, M$ . Fix some  $\varepsilon > 0$ . Let Assumption  $RE(s, c_0)$  be satisfied with  $c_0 = 3 + 4/\varepsilon$ . Consider the LASSO estimator  $f_{\hat{\lambda}_L}$  with the tuning constant

$$r = A\sigma\sqrt{\frac{\log M}{n}}$$

for some  $A > 2\sqrt{2}$ . Then, for all  $M \geq 3, n \geq 1$  with probability at least  $1 - M^{1-A^2/8}$  we have:  $\forall \lambda \in \mathbb{R}^M : M(\lambda) = s$ ,

$$\|f_{\hat{\lambda}_L} - f\|_n^2 \leq (1 + \varepsilon)\|f_\lambda - f\|_n^2 + C(\varepsilon) \left( \frac{M(\lambda) \log M}{\kappa n} \right).$$

## Disadvantages of the LASSO:

- SOI for the LASSO holds under very restrictive assumptions on the dictionary involving  $\kappa$ . Moreover, the assumptions depend on the (unknown) number  $s$  of non-zero components of the oracle vector, or eventually on the upper bound on this number. Such assumptions are unavoidable: Candès and Plan (2008).
- **Bad behavior when  $\kappa$  is small.**
- The leading constant in SOI is **not** 1.

Same problems with the Dantzig selector: the properties of Dantzig selector are essentially the same as for LASSO, cf. Bickel, Ritov and T. (2007).

## Sparse exponential weighting

Choose  $\widehat{\lambda}^{EW}$  according to:

$$\widehat{\lambda}_j^{EW} = \int_{\mathbb{R}^M} \lambda_j S_n(d\lambda), \quad j = 1, \dots, M,$$

where the probability measure  $S_n$  is given by

$$S_n(d\lambda) = \frac{\exp \left\{ -n \|\mathbf{y} - \mathbf{f}_\lambda\|_n^2 / \beta \right\} \pi(d\lambda)}{\int_{\mathbb{R}^M} \exp \left\{ -n \|\mathbf{y} - \mathbf{f}_w\|_n^2 / \beta \right\} \pi(dw)}$$

with some  $\beta > 0$  and some prior measure  $\pi$ .

- Bayesian estimator if  $\beta = 2\sigma^2$ , but we need a larger  $\beta$ .
- Non-discrete  $\pi$ : is the fast computation possible?

## A PAC-Bayesian bound

Lemma [Dalalyan and T., 2007]

The estimator with exponential weights  $f_{\hat{\lambda}^{EW}}$  defined with  $\beta \geq 4\sigma^2$  and any prior  $\pi$  satisfies:

$$\mathbb{E} \|f_{\hat{\lambda}^{EW}} - f\|_n^2 \leq \inf_P \left\{ \int \|f_\lambda - f\|_n^2 P(d\lambda) + \frac{\beta \mathcal{K}(P, \pi)}{n} \right\}$$

where the infimum is taken over all probability measures  $P$  on  $\mathbb{R}^M$  and  $\mathcal{K}(P, \pi)$  denotes the Kullback-Leibler divergence between  $P$  and  $\pi$ .

## Sparsity prior

Choose a specific prior measure  $\pi$  with Lebesgue density  $q$  defined by

$$q(\lambda) = \prod_{j=1}^M \tau^{-1} q_0(\lambda_j/\tau), \quad \forall \lambda \in \mathbb{R}^M,$$

where  $q_0$  is the Student  $t_3$  density,

$$q_0(t) \sim |t|^{-4}, \quad \text{for large } |t|$$

and  $\tau \sim (Mn)^{-1/2}$ . We will call this prior the **sparsity prior**. The resulting estimator  $f_{\hat{\lambda}^{EW}}$  is called the **Sparse Exponential Weighting (SEW)** estimator.

## SOI for the SEW estimator

### Theorem [Dalalyan and T., 2007]

Let  $\max_{1 \leq j \leq M} \|f_j\|_n \leq c_0 < \infty$ . Then the exponential weighted estimator  $f_{\hat{\lambda}^{EW}}$  defined with  $\beta \geq 4\sigma^2$  and with the **sparsity prior**  $\pi$  satisfies:

$$\mathbb{E} \|f_{\hat{\lambda}^{EW}} - f\|_n^2 \leq \inf_{\lambda \in \mathbb{R}^M} \left\{ \|f_\lambda - f\|_n^2 + \frac{CM(\lambda)}{n} \log \left( 1 + \frac{|\lambda|_1 \sqrt{Mn}}{M(\lambda)} \right) \right\}$$

where  $|\lambda|_1$  is the  $\ell_1$ -norm of  $\lambda$ .

- **No assumption on the dictionary.**
- **Leading constant 1.**
- $\ell_1$ -norm of  $\lambda$ , but under the log.
- Fast computation for at least  $M \sim 10^3$ .

## SEW estimator: discussion

$$\widehat{\lambda}_j^{EW} = \int_{\mathbb{R}^M} \lambda_j S_n(d\lambda) = \int_{\mathbb{R}^M} \lambda_j \theta_n(\lambda) d\lambda, \quad j = 1, \dots, M,$$

with posterior density  $\theta_n(\lambda) = S_n(d\lambda)/d\lambda$ :

$$\begin{aligned} \theta_n(\lambda) &\asymp \exp \left\{ -n \|\mathbf{y} - \mathbf{f}_\lambda\|_n^2 / \beta + \log q(\lambda) \right\} \\ &\asymp \exp \left\{ -n \|\mathbf{y} - \mathbf{f}_\lambda\|_n^2 / \beta - C \sum_{j=1}^M \log(1 + |\lambda_j|/\tau) \right\} \end{aligned}$$

Maximizer of this density (the MAP estimator):

$$\widehat{\lambda} = \arg \min_{\lambda \in \mathbb{R}^M} \left\{ \|\mathbf{y} - \mathbf{f}_\lambda\|_n^2 + \frac{\gamma}{n} \sum_{j=1}^M \log(1 + |\lambda_j|/\tau) \right\}.$$

## Exponential weights: models with i.i.d. data

- An i.i.d. sample  $Z_1, \dots, Z_n$  from the distribution of an abstract random variable  $Z \in \mathcal{Z}$ .
- $Q(Z, f_\lambda)$  a given real-valued loss (prediction loss).

Define the probability measure  $S_n$  on  $\mathbb{R}^M$  by

$$S_n(d\lambda) = \frac{\exp \left\{ - \sum_{i=1}^n Q(Z_i, f_\lambda) / \beta \right\} \pi(d\lambda)}{\int_{\mathbb{R}^M} \exp \left\{ - \sum_{i=1}^n Q(Z_i, f_w) / \beta \right\} \pi(dw)}$$

with some  $\beta > 0$  and some prior measure  $\pi$ . Generalization of the previous definition: we replace

$$n \|\mathbf{y} - f_\lambda\|_n^2 \rightsquigarrow \sum_{i=1}^n Q(Z_i, f_\lambda).$$

## Mirror averaging

Cumulative exponential weights (**mirror averaging**):

$$\widehat{\lambda}_j^{MA} = \int_{\mathbb{R}^M} \lambda_j S(d\lambda), \quad j = 1, \dots, M, \quad \text{with } S = \frac{1}{n} \sum_{i=1}^n S_i$$

cf. Juditsky/Rigollet/T (2005) [even more general method: Juditsky/Nazin/T/Vayatis (2005)]. In a particular case we get the “progressive mixture method” of Catoni and Yang.

Choose a prior measure  $\pi$  supported on a convex compact  $\Lambda \subset \mathbb{R}^M$  (e.g., on an  $\ell_1$  ball).

### Assumption JRT (2005).

The mapping  $\lambda \mapsto Q(Z, f_\lambda)$  is convex for all  $Z$  and there exists  $\beta > 0$  such that the function

$$\lambda \mapsto \mathbb{E} \exp \left( \frac{Q(Z, f_{\lambda'}) - Q(Z, f_\lambda)}{\beta} \right)$$

is concave on a convex compact set  $\Lambda \subset \mathbb{R}^M$  for all  $\lambda' \in \Lambda$ .

## PAC-Bayesian bound for mirror averaging

Define the average risk:  $A(\lambda) = \mathbb{E}Q(Z, f_\lambda)$ .

**Lemma (PAC-Bayesian bound).**

Let  $f_{\hat{\lambda}^{MA}}$  be a mirror averaging estimator defined with  $\beta$  satisfying Assumption JRT and any prior  $\pi$  supported on a convex compact set  $\Lambda$ . Then

$$\mathbb{E} A(\hat{\lambda}^{MA}) \leq \inf_P \left\{ \int A(\lambda) P(d\lambda) + \frac{\beta \mathcal{K}(P, \pi)}{n+1} \right\}$$

where the infimum is taken over all probability measures  $P$  on  $\Lambda$  and  $\mathcal{K}(P, \pi)$  is the Kullback-Leibler divergence between  $P$  and  $\pi$ .

Proof follows the scheme of Juditsky, Rigollet and T. (2005), cf. Rigollet and Zhao (2006), Audibert (2006), Lounici (2007).

## SOI for Mirror Averaging

Theorem [Dalalyan, Rigollet and T., 2007]

Assume that  $\sup_{|\lambda|_1 \leq 2R} \text{Spec}\{\nabla^2 A(\lambda)\} < \infty$  for some  $R > 0$ . Let  $f_{\hat{\lambda}^{MA}}$  be a mirror averaging estimator satisfying assumptions of the PAC lemma, with the **sparsity prior**  $\pi$  truncated to  $\{\lambda : |\lambda|_1 \leq 2R\}$  and  $\tau \sim 1/\sqrt{M(n \vee M)}$ . Then

$$\mathbb{E} A(\hat{\lambda}^{MA}) \leq \inf_{|\lambda|_1 \leq R} \left\{ A(\lambda) + \frac{CR^2 M(\lambda)}{n} \log \left( \frac{C'R\sqrt{M(n \vee M)}}{M(\lambda)} \right) \right\}.$$

- **No restrictive assumption on the dictionary.**
- **Leading constant 1.**

## Comparison with SOI for the LASSO

The LASSO type estimators

$$\hat{\lambda} = \arg \min_{\lambda \in \mathbb{R}^M} \left\{ \frac{1}{n} \sum_{i=1}^n Q(Z_i, f_\lambda) + r \sum_{j=1}^M |\lambda_j|^p \right\}, \quad 1 \leq p \leq 1 + \frac{1}{\log M},$$

van de Geer (2006) [ $p=1$ ,  $\Psi_M > 0$ ] and Koltchinskii (2007):

$$\mathbb{E} A(\hat{\lambda}) \leq \inf_{|\lambda|_1 \leq R} \left( \boxed{3} A(\lambda) + \frac{CR^2 M(\lambda) \log M}{\boxed{\kappa(\lambda)} n} \right)$$

where  $\kappa(\lambda)$  is a quantity analogous to  $\kappa$  in Assumption RE. To get the correct rate, we need to consider only  $\lambda$  such that  $\kappa(\lambda) \geq c$ , which is equivalent to RE.

## Example: Gaussian regression, squared loss

- Gaussian regression with random design :  
 $Z = (X, Y)$ ,  $X \in \mathbb{R}^d$ ,  $Y \in \mathbb{R}$  such that

$$Y = f(X) + \xi,$$

$$\xi|X \sim \mathcal{N}(0, \sigma^2), \quad X \sim P_X, \quad \|f\|_\infty \leq L.$$

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- Assumption on the dictionary:  $\|f_j\|_\infty \leq L$ ,  $j = 1, \dots, M$ .
- The loss function  
 $Q(Z, f_\lambda) = (Y - f_\lambda(X))^2$  where  $f_\lambda = \sum_{j=1}^M \lambda_j f_j$ .
- Then  $A(\lambda) = \mathbb{E} Q(Z, f_\lambda) = \|f_\lambda - f\|_X^2 + \sigma^2$ ,  $\|f\|_X^2 \triangleq \int f^2 dP_X$ .

## SOI for regression with squared loss

### Corollary

Under the conditions of this example, for all  $\beta \geq 2\sigma^2 + 8L^2$ ,

$$\mathbb{E} \|\hat{f}_{\hat{\lambda}^{MA}} - f\|_X^2 \leq \inf_{\lambda \in \Lambda^M} \left\{ \|f_\lambda - f\|_X^2 + \frac{CM(\lambda)}{n} \log \left( \frac{C' \sqrt{M(n \vee M)}}{M(\lambda)} \right) \right\}.$$

Here  $\Lambda^M$  is the simplex:

$$\Lambda^M = \left\{ \lambda \in \mathbb{R}^M : \lambda_j \geq 0, \sum_{j=1}^M \lambda_j = 1 \right\}.$$

## Example: density estimation with $L_2$ loss

- $Z = X \in \mathbb{R}^d$  with density  $f$ , such that  $\|f\|_\infty \leq L$ .
- Assumption on the dictionary:  $f_1, \dots, f_M$  are probability densities such that  $\|f_j\|_\infty \leq L$ .

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- The loss function:

$$Q(X, f_\lambda) = \|f_\lambda\|^2 - 2f_\lambda(X) \quad \text{where} \quad \|f\|^2 = \int f^2(x) dx.$$

- The associated risk:

$$A(\lambda) = \mathbb{E} Q(X, f_\lambda) = \|f - f_\lambda\|^2 - \|f\|^2.$$

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- Assumption JRT holds: if  $\beta > 12L$  the mapping

$$\lambda \mapsto \mathbb{E} \exp \left( \frac{Q(X, f_{\lambda'}) - Q(X, f_\lambda)}{\beta} \right)$$

is concave on the simplex  $\Lambda^M$  for all  $\lambda' \in \Lambda^M$  [JRT (2005)].

## SOI for density estimation with $L_2$ loss

### Corollary

Under the conditions of this example, for all  $\beta > 12L$ ,

$$\mathbb{E} \|\hat{f}_{\hat{\lambda}^{MA}} - f\|^2 \leq \inf_{\lambda \in \Lambda^M} \left\{ \|\mathbf{f}_\lambda - f\|^2 + \frac{CM(\lambda)}{n} \log \left( \frac{C' \sqrt{M(n \vee M)}}{M(\lambda)} \right) \right\}.$$

Here  $\Lambda^M$  is the simplex:

$$\Lambda^M = \left\{ \lambda \in \mathbb{R}^M : \lambda_j \geq 0, \sum_{j=1}^M \lambda_j = 1 \right\}.$$

# Computation of SEW estimators

Consider the linear regression scenario:

$$\mathbf{y} = X\lambda + W.$$

$X$  is a  $n \times M$  deterministic design matrix,  $\lambda \in \mathbb{R}^M$  is an unknown vector and  $W \in \mathbb{R}^M$  is a Gaussian vector with i.i.d. components, with variances  $\sigma^2$ . The SEW estimator

$$\hat{\lambda}^{SEW} \triangleq \int_{\mathbb{R}^M} \mathbf{u} g(\mathbf{u}) d\mathbf{u}$$

where the posterior density

$$g(\mathbf{u}) \propto \exp(-V(\mathbf{u}))$$

$$V(\mathbf{u}) = \beta^{-1} \|\mathbf{y} - X\mathbf{u}\|^2 + 2 \sum_{j=1}^M \log(\tau^2 + u_j^2).$$

## Langevin Monte Carlo

**Remark:** the posterior density  $g(\cdot)$  is the invariant density of the Langevin diffusion

$$\mathbf{L}_t = -\nabla V(\mathbf{L}_t) dt + \sqrt{2} d\mathbf{W}_t, \quad \mathbf{L}_0 = 0, \quad t > 0.$$

Here  $\mathbf{W}_t$  is a  $M$ -dimensional Brownian motion.

Let now  $\eta_1, \eta_2, \dots$  be i.i.d. standard normal random vectors. Set

$$\bar{\mathbf{L}}_0 = 0, \quad \bar{\mathbf{L}}_{k+1} = \bar{\mathbf{L}}_k - h\nabla V(\bar{\mathbf{L}}_k) + \sqrt{2h} \eta_k, \quad k = 0, 1, \dots$$

Then

$$\frac{1}{\lfloor Th^{-1} \rfloor} \sum_{k=1}^{\lfloor Th^{-1} \rfloor} \bar{\mathbf{L}}_k \approx \frac{1}{T} \int_0^T \mathbf{L}_t dt \xrightarrow[T \rightarrow \infty]{a.s.} \int_{\mathbb{R}^M} \mathbf{u} g(\mathbf{u}) d\mathbf{u} = \hat{\lambda}^{SEW}.$$

## Simulations

### Example 1: model selection properties when the coherence is low

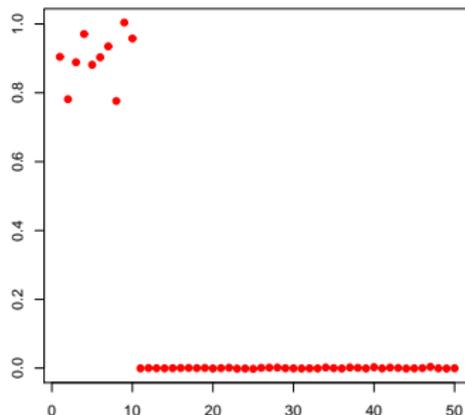
The entries of  $X$  are i.i.d. Rademacher random variables independent of the noise  $W$ .

$$\lambda_j = I(j \leq S) \quad \text{and} \quad \sigma^2 = \frac{S}{9n}.$$

We apply the SEW estimator using Langevin Monte-Carlo with

$$\tau = 4\sigma/\sqrt{M}, \quad \beta = 4\sigma^2, \quad h = 0.0001.$$

## Simulations



**Figure:** Typical result for Example 1 with  $n = 200$ ,  $M = 500$ ,  $S = 10$ ,  $h = 10^{-4}$ ,  $T = 5$ . The estimates of first 50 coefficients are plotted. In this example, we have  $\frac{1}{n} \|X(\hat{\lambda} - \lambda)\|^2 = 0.0021$ . The time of computation of the estimator was about 30 seconds.

## Simulations

### Example 2: Comparison with the LASSO/LARS

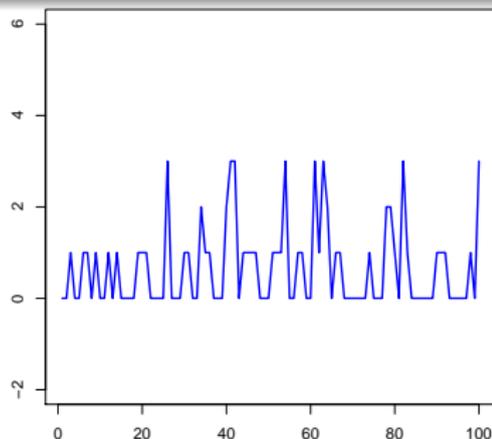
Choose  $X_1, \dots, X_n$  i.i.d. uniformly distributed in  $[0, 1]^2$  and set  $f_j(t) = I\{[0, j_1/k] \times [0, j_2/k]\}(t)$ ,  $j = (j_1, j_2) \in \{1, \dots, k\}^2$ ,  $t \in [0, 1]^2$ .

We get a matrix  $X$  which has  $k^2$  columns some of which are nearly collinear. The number of covariates is  $M = k^2$ . Set  $\sigma = 1$ ,  $k = 15$ ,  $n = 100$ ,  $\lambda_j^* = 0$  for every  $j \in \{1, \dots, k\} \setminus \{87, 110, 200\}$  and  $\lambda_j^* = 1$  for  $j \in \{87, 110, 200\}$ .

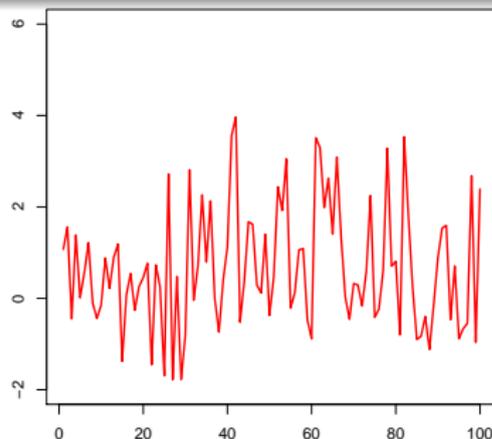
Applying SEW estimator with Langevin Monte-Carlo and

$$\tau = \frac{4\sigma}{\sqrt{\sum_{j,i} f_j^2(X_i)}}, \quad \beta = 4\sigma^2, \quad h = 0.0005.$$

## Simulations



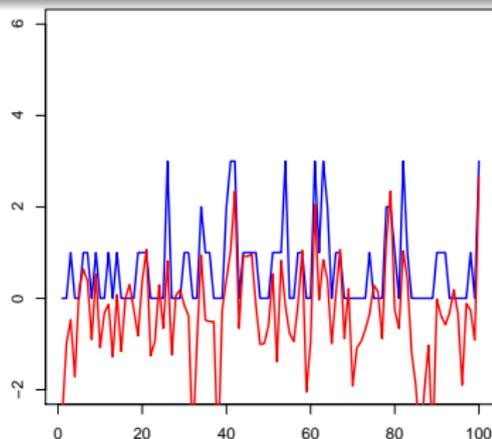
The original signal



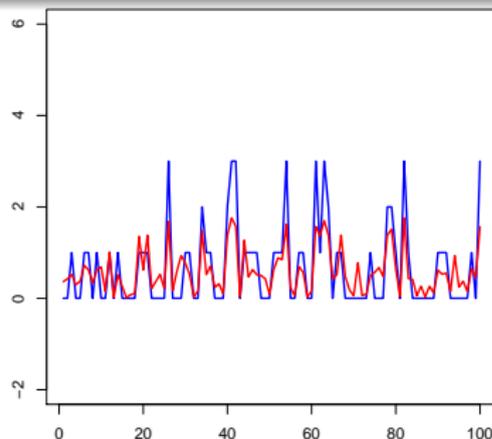
The noisy signal

**Figure:** Typical result for Example 2 with  $n = 100$ ,  $M = 225$ ,  $M(\lambda^*) = 3$ ,  $h = 5 \cdot 10^{-4}$ ,  $T = 2$ . In this example, we have  $\frac{1}{n} \|X(\hat{\lambda} - \lambda^*)\|^2 = 0.28$  for our estimator and  $\frac{1}{n} \|X(\hat{\lambda} - \lambda^*)\|^2 = 1.72$  for the LASSO. The time of computation of the SEW estimator was about 5 seconds.

## Simulations



The original signal and the LASSO-Cp



The original signal and the EW-aggregate

**Figure:** Typical result for Example 2 with  $n = 100$ ,  $M = 225$ ,  $M(\lambda^*) = 3$ ,  $h = 5 \cdot 10^{-4}$ ,  $T = 2$ . In this example, we have  $\frac{1}{n} \|X(\hat{\lambda} - \lambda^*)\|^2 = 0.28$  for our estimator and  $\frac{1}{n} \|X(\hat{\lambda} - \lambda^*)\|^2 = 1.72$  for the LASSO. The time of computation of the SEW estimator was about 5 seconds.

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