Machine learning and portfolio selections. II.

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with the average growth rate

$$W_n(\mathbf{B}) = \frac{1}{n} \sum_{i=1}^{n} \ln \left\langle \mathbf{b}(\mathbf{x}_1^{i-1}) \, , \, \mathbf{x}_i \right\rangle .$$

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$$\mathbf{E}\{\ln \left\langle \mathbf{b}^*(\mathbf{X}_1^{n-1}), \mathbf{X}_n \right\rangle \mid \mathbf{X}_1^{n-1}\} = \max_{\mathbf{b}(\cdot)} \mathbf{E}\{\ln \left\langle \mathbf{b}(\mathbf{X}_1^{n-1}), \mathbf{X}_n \right\rangle \mid \mathbf{X}_1^{n-1}\}$$

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where

$$W^* = \mathbf{E}\left\{\max_{\mathbf{b}(\cdot)}\mathbf{E}\{\ln\left\langle \mathbf{b}(\mathbf{X}_{-\infty}^{-1})\,,\,\mathbf{X}_0\right\rangle \mid \mathbf{X}_{-\infty}^{-1}\}\right\}$$

is the maximal growth rate of any portfolio.

Martingale difference sequences

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Definition

there are two sequences of random variables:

$$\{Z_n\} \qquad \{X_n\}$$

- Z_n is a function of $X_1, \ldots, X_n$,
- $\mathbf{E}\{Z_n \mid X_1, \ldots, X_{n-1}\} = 0$ almost surely.

Then $\{Z_n\}$ is called martingale difference sequence with respect to $\{X_n\}$.

Chow Theorem:

Chow Theorem: If $\{Z_n\}$ is a martingale difference sequence with respect to $\{X_n\}$ and

$$\sum_{n=1}^{\infty} \frac{\mathbf{E}\{Z_n^2\}}{n^2} < \infty$$

then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} Z_i = 0 \quad \text{a.s.}$$

Lemma:

A weak law of large numbers

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Proof. Put $i < j$.

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Corollary

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&= \frac{1}{n^2}\sum_{i=1}^{n}\mathbf{E}\{Z_i^2\} \\
&\to 0
\end{aligned}$$

if, for example, $\mathbf{E}\{Z_i^2\}$ is a bounded sequence.

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  almost surely.

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log-optimum portfolio $\mathbf{B}^* = \{\mathbf{b}^*(\cdot)\}$

$$\mathbf{E}\{\ln \left\langle \mathbf{b}^*(\mathbf{X}_1^{n-1}), \mathbf{X}_n \right\rangle \mid \mathbf{X}_1^{n-1}\} = \max_{\mathbf{b}(\cdot)} \mathbf{E}\{\ln \left\langle \mathbf{b}(\mathbf{X}_1^{n-1}), \mathbf{X}_n \right\rangle \mid \mathbf{X}_1^{n-1}\}$$

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If $S_n^* = S_n(\mathbf{B}^*)$ denotes the capital after day n achieved by a log-optimum portfolio strategy $\mathbf{B}^*$, then for any portfolio strategy $\mathbf{B}$ with capital $S_n = S_n(\mathbf{B})$ and for any process $\{\mathbf{X}_n\}_{-\infty}^{\infty}$,

$$\limsup_{n\to\infty} \left(\frac{1}{n} \ln S_n - \frac{1}{n} \ln S_n^* \right) \leq 0 \quad \text{almost surely}$$

Proof of optimality

$$\frac{1}{n}\ln S_n = \frac{1}{n}\sum_{i=1}^{n}\ln\left\langle \mathbf{b}(\mathbf{X}_1^{i-1}), \mathbf{X}_i\right\rangle$$

Proof of optimality

$$
\begin{aligned}
\frac{1}{n}\ln S_n &= \frac{1}{n}\sum_{i=1}^{n}\ln\left\langle \mathbf{b}(\mathbf{X}_1^{i-1}),\, \mathbf{X}_i\right\rangle \\
&= \frac{1}{n}\sum_{i=1}^{n}\mathbf{E}\{\ln\left\langle \mathbf{b}(\mathbf{X}_1^{i-1}),\, \mathbf{X}_i\right\rangle \mid \mathbf{X}_1^{i-1}\} \\
&+ \frac{1}{n}\sum_{i=1}^{n}\left(\ln\left\langle \mathbf{b}(\mathbf{X}_1^{i-1}),\, \mathbf{X}_i\right\rangle - \mathbf{E}\{\ln\left\langle \mathbf{b}(\mathbf{X}_1^{i-1}),\, \mathbf{X}_i\right\rangle \mid \mathbf{X}_1^{i-1}\}\right)
\end{aligned}
$$

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$$= \frac{1}{n}\sum_{i=1}^{n}\mathbf{E}\{\ln\left\langle \mathbf{b}(\mathbf{X}_1^{i-1}), \mathbf{X}_i\right\rangle \mid \mathbf{X}_1^{i-1}\}$$
$$+ \frac{1}{n}\sum_{i=1}^{n}\left(\ln\left\langle \mathbf{b}(\mathbf{X}_1^{i-1}), \mathbf{X}_i\right\rangle - \mathbf{E}\{\ln\left\langle \mathbf{b}(\mathbf{X}_1^{i-1}), \mathbf{X}_i\right\rangle \mid \mathbf{X}_1^{i-1}\}\right)$$

and

$$\frac{1}{n}\ln S_n^* = \frac{1}{n}\sum_{i=1}^{n}\mathbf{E}\{\ln\left\langle \mathbf{b}^*(\mathbf{X}_1^{i-1}), \mathbf{X}_i\right\rangle \mid \mathbf{X}_1^{i-1}\}$$
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These limit relations give rise to the following definition:

Definition

An empirical (data driven) portfolio strategy $\mathbf{B}$ is called **universally consistent with respect to a class $\mathcal{C}$ of stationary and ergodic processes $\{\mathbf{X}_n\}_{-\infty}^{\infty}$,** if for each process in the class,

$$\lim_{n\to\infty} \frac{1}{n} \ln S_n(\mathbf{B}) = W^* \quad \text{almost surely.}$$

Empirical portfolio selection

$$\mathbf{E}\{\ln \left\langle \mathbf{b}^*(\mathbf{X}_1^{n-1}), \mathbf{X}_n \right\rangle \mid \mathbf{X}_1^{n-1}\} = \max_{\mathbf{b}(\cdot)} \mathbf{E}\{\ln \left\langle \mathbf{b}(\mathbf{X}_1^{n-1}), \mathbf{X}_n \right\rangle \mid \mathbf{X}_1^{n-1}\}$$

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$$\mathbf{b}^*(\mathbf{x}_1^{n-1}) = \arg\max_{\mathbf{b}(\cdot)} \mathbf{E}\{\ln \left\langle \mathbf{b}(\mathbf{X}_1^{n-1}),\, \mathbf{X}_n \right\rangle \mid \mathbf{X}_1^{n-1} = \mathbf{x}_1^{n-1}\}$$

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&= \arg\max_{\mathbf{b}(\cdot)} \mathbf{E}\{\ln \left\langle \mathbf{b}(\mathbf{x}_1^{n-1}), \mathbf{X}_n \right\rangle \mid \mathbf{X}_1^{n-1} = \mathbf{x}_1^{n-1}\} \\
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\end{aligned}$$

fixed integer $k > 0$

$$\mathbf{E}\{\ln \left\langle \mathbf{b}(\mathbf{X}_1^{n-1}) \, , \, \mathbf{X}_n \right\rangle \mid \mathbf{X}_1^{n-1}\} \approx \mathbf{E}\{\ln \left\langle \mathbf{b}(\mathbf{X}_{n-k}^{n-1}) \, , \, \mathbf{X}_n \right\rangle \mid \mathbf{X}_{n-k}^{n-1}\}$$

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fixed integer $k > 0$

$$\mathbf{E}\{\ln \left\langle \mathbf{b}(\mathbf{X}_1^{n-1}) \,,\, \mathbf{X}_n \right\rangle \mid \mathbf{X}_1^{n-1}\} \approx \mathbf{E}\{\ln \left\langle \mathbf{b}(\mathbf{X}_{n-k}^{n-1}) \,,\, \mathbf{X}_n \right\rangle \mid \mathbf{X}_{n-k}^{n-1}\}$$

and

$$\mathbf{b}^*(\mathbf{X}_1^{n-1}) \approx \mathbf{b}_k(\mathbf{X}_{n-k}^{n-1}) = \arg\max_{\mathbf{b}(\cdot)} \mathbf{E}\{\ln \left\langle \mathbf{b}(\mathbf{X}_{n-k}^{n-1}) \,,\, \mathbf{X}_n \right\rangle \mid \mathbf{X}_{n-k}^{n-1}\}$$

because of stationarity

$$\mathbf{b}_k(\mathbf{x}_1^k) = \arg\max_{\mathbf{b}(\cdot)} \mathbf{E}\{\ln \left\langle \mathbf{b}(\mathbf{X}_{n-k}^{n-1}), \mathbf{X}_n \right\rangle \mid \mathbf{X}_{n-k}^{n-1} = \mathbf{x}_1^k\}$$

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\end{aligned}$$

which is the maximization of the regression function

$$m_{\mathbf{b}}(\mathbf{x}_1^k) = \mathbf{E}\{\ln \langle \mathbf{b}, \mathbf{X}_{k+1} \rangle \mid \mathbf{X}_1^k = \mathbf{x}_1^k\}$$

Regression function

Y real valued

X observation vector

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$$m(x) = \mathbf{E}\{Y \mid X = x\}$$

i.i.d. data: $D_n = \{(X_1, Y_1), \dots, (X_n, Y_n)\}$

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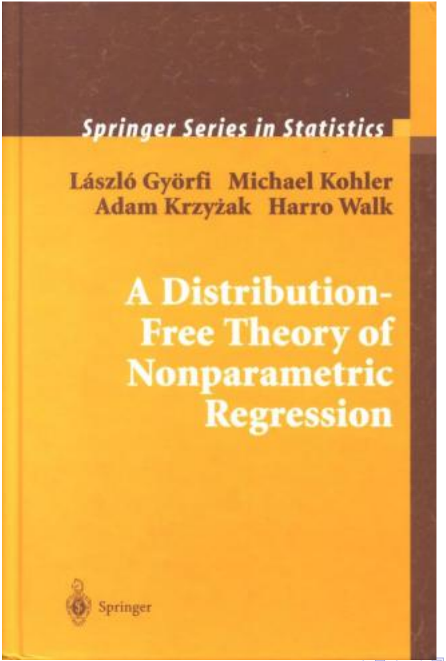
L. Györfi, M. Kohler, A. Krzyzak, H. Walk (2002) *A Distribution-Free Theory of Nonparametric Regression*, Springer-Verlag, New York.


Springer Series in Statistics
László Györfi Michael Kohler
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A Distribution-Free Theory of Nonparametric Regression
Springer

Correspondence

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Kernel-based portfolio selection

choose the radius $r_{k,\ell} > 0$ such that for any fixed k,

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for $n > k + 1$, define the expert $\mathbf{b}^{(k,\ell)}$ by

$$\mathbf{b}^{(k,\ell)}(\mathbf{x}_1^{n-1}) = \arg\max_{\mathbf{b}} \sum_{\left\{k<i<n: \|\mathbf{x}_{i-k}^{i-1} - \mathbf{x}_{n-k}^{n-1}\| \le r_{k,\ell}\right\}} \ln \langle \mathbf{b} \, , \, \mathbf{x}_i \rangle \, ,$$

if the sum is non-void, and $\mathbf{b}_0 = (1/d, \ldots, 1/d)$ otherwise.

Combining elementary portfolios

for fixed $k, \ell = 1, 2, \ldots,$
$\mathbf{B}^{(k,\ell)} = \{\mathbf{b}^{(k,\ell)}(\cdot)\}$, are called elementary portfolios

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Machine learning: combination of experts

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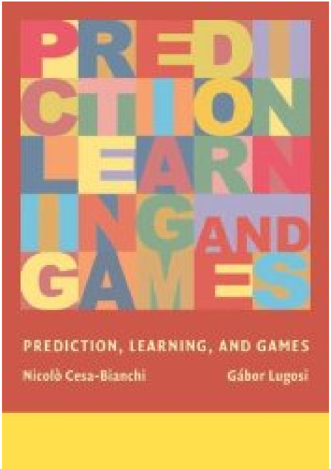
Machine learning: combination of experts

N. Cesa-Bianchi and G. Lugosi, *Prediction, Learning, and Games.* Cambridge University Press, 2006.


PREDICTION LEARNING AND GAMES
PREDICTION, LEARNING, AND GAMES
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the combined portfolio $\mathbf{b}$:

$$\mathbf{b}_n(\mathbf{x}_1^{n-1}) = \sum_{k,\ell} v_{n,k,\ell} \mathbf{b}_n^{(k,\ell)}(\mathbf{x}_1^{n-1}).$$

$$S_n(\mathbf{B}) \quad = \quad \prod_{i=1}^{n} \left\langle \mathbf{b}_i(\mathbf{x}_1^{i-1}) \, , \, \mathbf{x}_i \right\rangle$$

$$
\begin{aligned}
S_n(\mathbf{B}) &= \prod_{i=1}^{n} \left\langle \mathbf{b}_i(\mathbf{x}_1^{i-1}) \,,\, \mathbf{x}_i \right\rangle \\
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&= \sum_{k,\ell} q_{k,\ell} S_n(\mathbf{B}^{(k,\ell)}),
\end{aligned}
$$

The strategy $\mathbf{B}$ then arises from weighing the elementary portfolio strategies $\mathbf{B}^{(k,\ell)} = \{\mathbf{b}_n^{(k,\ell)}\}$ such that the investor's capital becomes

$$S_n(\mathbf{B}) = \sum_{k,\ell} q_{k,\ell} S_n(\mathbf{B}^{(k,\ell)}).$$

Theorem

The kernel-based portfolio scheme is universally consistent with respect to the class of all ergodic processes such that $\mathbf{E}\{|\ln X^{(j)}|\} < \infty$, for $j = 1, 2, \ldots, d$.

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L. Györfi, G. Lugosi, F. Udina (2006) "Nonparametric kernel-based sequential investment strategies", *Mathematical Finance*, 16, pp. 337-357
www.szit.bme.hu/~gyorfi/kernel.pdf

Proof

We have to prove that

$$\liminf_{n\to\infty} W_n(\mathbf{B}) = \liminf_{n\to\infty} \frac{1}{n} \ln S_n(\mathbf{B}) \geq W^* \quad \text{a.s.}$$

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&= \sup_{k,\ell} \left(W_n(\mathbf{B}^{(k,\ell)}) + \frac{\ln q_{k,\ell}}{n} \right).
\end{aligned}$$

Thus

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Because of $\lim_{\ell\to\infty} r_{k,\ell} = 0$, we have that

$$
\sup_{k,\ell} \epsilon_{k,\ell} = \lim_{k\to\infty} \lim_{l\to\infty} \epsilon_{k,\ell} = W^*.
$$

Semi-log-optimal portfolio

empirical log-optimal:

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L. Györfi, A. Urbán, I. Vajda (2007) "Kernel-based semi-log-optimal portfolio selection strategies", *International Journal of Theoretical and Applied Finance*, 10, pp. 505-516.
www.szit.bme.hu/∼gyorfi/semi.pdf

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Our experiment is on the second data set.

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the BCRP had average AAY 24%

The average annual yields of the individual experts.

k / ℓ	1	2	3	4	5
1	68%	54%	23%	21%	16%
2	87%	73%	46%	29%	17%
3	94%	77%	40%	39%	19%
4	94%	90%	46%	42%	32%
5	108%	91%	63%	58%	29%
6	118%	99%	75%	53%	38%
7	122%	100%	81%	71%	54%
8	128%	95%	89%	75%	55%
9	131%	102%	94%	87%	53%
10	131%	108%	107%	97%	65%

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left out
for the remaining 19 large assets, AAY of kernel based semi-log-optimal portfolio is 31%

The average annual yields of the individual experts, for the 19 large assets.

k / ℓ	1	2	3	4	5
1	31%	30%	24%	21%	26%
2	34%	31%	27%	25%	22%
3	35%	29%	26%	24%	23%
4	35%	30%	30%	32%	27%
5	34%	29%	33%	24%	24%
6	35%	29%	28%	24%	27%
7	33%	29%	32%	23%	23%
8	34%	33%	30%	21%	24%
9	37%	33%	28%	19%	21%
10	34%	29%	26%	20%	24%