Machine learning and portfolio selections. I.

László (Laci) Györfi[1]

[1]Department of Computer Science and Information Theory
Budapest University of Technology and Economics
Budapest, Hungary

May 28, 2008

e-mail: gyorfi@szit.bme.hu
www.szit.bme.hu/˜gyorfi
www.szit.bme.hu/˜oti/portfolio

Growth rate

investment in the stock market

Growth rate

investment in the stock market
d assets

Growth rate

investment in the stock market
d assets
$S_n^{(j)}$ price of asset j at the end of trading period (day) n
initial price $S_0^{(j)} = 1$, $j = 1, \dots, d$

Growth rate

investment in the stock market
d assets
$S_n^{(j)}$ price of asset j at the end of trading period (day) n
initial price $S_0^{(j)} = 1$, $j = 1, \ldots, d$

$$S_n^{(j)} = e^{nW_n^{(j)}} \approx e^{nW^{(j)}}$$

Growth rate

investment in the stock market

d assets

$S_n^{(j)}$ price of asset j at the end of trading period (day) n

initial price $S_0^{(j)} = 1$, $j = 1, \ldots, d$

$$S_n^{(j)} = e^{nW_n^{(j)}} \approx e^{nW^{(j)}}$$

average growth rate

$$W_n^{(j)} = \frac{1}{n} \ln S_n^{(j)}$$

Growth rate

investment in the stock market

d assets

$S_n^{(j)}$ price of asset j at the end of trading period (day) n

initial price $S_0^{(j)} = 1$, $j = 1, \ldots, d$

$$S_n^{(j)} = e^{nW_n^{(j)}} \approx e^{nW^{(j)}}$$

average growth rate

$$W_n^{(j)} = \frac{1}{n} \ln S_n^{(j)}$$

asymptotic average growth rate

$$W^{(j)} = \lim_{n \to \infty} \frac{1}{n} \ln S_n^{(j)}$$

Static portfolio selection: single period investment

the aim is to achieve $\max_j W^{(j)}$

the aim is to achieve $\max_j W^{(j)}$
static portfolio selection

the aim is to achieve $\max_j W^{(j)}$
static portfolio selection
a portfolio vector $\mathbf{b} = (b^{(1)}, \ldots b^{(d)})$
$b^{(j)} \geq 0$, $\sum_j b^{(j)} = 1$

Static portfolio selection: single period investment

the aim is to achieve $\max_j W^{(j)}$
static portfolio selection
a portfolio vector $\mathbf{b} = (b^{(1)}, \ldots b^{(d)})$
$b^{(j)} \geq 0$, $\sum_j b^{(j)} = 1$
$b^{(j)}$ gives the proportion of the investor's capital invested in stock j

Static portfolio selection: single period investment

the aim is to achieve $\max_j W^{(j)}$
static portfolio selection
a portfolio vector $\mathbf{b} = (b^{(1)}, \dots b^{(d)})$
$b^{(j)} \geq 0$, $\sum_j b^{(j)} = 1$
$b^{(j)}$ gives the proportion of the investor's capital invested in stock j
initial capital S_0

Static portfolio selection: single period investment

the aim is to achieve $\max_j W^{(j)}$
static portfolio selection
a portfolio vector $\mathbf{b} = (b^{(1)}, \dots b^{(d)})$
$b^{(j)} \geq 0$, $\sum_j b^{(j)} = 1$
$b^{(j)}$ gives the proportion of the investor's capital invested in stock j
initial capital S_0

$$S_n = S_0 \sum_j b^{(j)} S_n^{(j)}$$

Static portfolio selection: single period investment

the aim is to achieve $\max_j W^{(j)}$
static portfolio selection
a portfolio vector $\mathbf{b} = (b^{(1)}, \dots b^{(d)})$
$b^{(j)} \geq 0$, $\sum_j b^{(j)} = 1$
$b^{(j)}$ gives the proportion of the investor's capital invested in stock j
initial capital S_0

$$S_n = S_0 \sum_j b^{(j)} S_n^{(j)}$$

$$S_0 \max_j b^{(j)} S_n^{(j)} \leq S_n \leq d S_0 \max_j b^{(j)} S_n^{(j)}$$

assume that $b^{(j)} > 0$

$$\frac{1}{n}\ln\left(\max_j S_0 b^{(j)} S_n^{(j)}\right) \le \frac{1}{n}\ln S_n \le \frac{1}{n}\ln\left(\max_j S_0 d b^{(j)} S_n^{(j)}\right)$$

assume that $b^{(j)} > 0$

$$\frac{1}{n} \ln \left(\max_j S_0 b^{(j)} S_n^{(j)} \right) \le \frac{1}{n} \ln S_n \le \frac{1}{n} \ln \left(\max_j S_0 d b^{(j)} S_n^{(j)} \right)$$

$$\begin{aligned} \max_j \left(\frac{1}{n} \ln(S_0 b^{(j)}) + \frac{1}{n} \ln S_n^{(j)} \right) &\le \frac{1}{n} \ln S_n \\ &\le \max_j \left(\frac{1}{n} \ln(S_0 d b^{(j)}) + \frac{1}{n} \ln S_n^{(j)} \right) \end{aligned}$$

assume that $b^{(j)} > 0$

$$\frac{1}{n}\ln\left(\max_j S_0 b^{(j)} S_n^{(j)}\right) \le \frac{1}{n}\ln S_n \le \frac{1}{n}\ln\left(\max_j S_0 d b^{(j)} S_n^{(j)}\right)$$

$$\begin{aligned}
\max_j\left(\frac{1}{n}\ln(S_0 b^{(j)}) + \frac{1}{n}\ln S_n^{(j)}\right) &\le \frac{1}{n}\ln S_n \\
&\le \max_j\left(\frac{1}{n}\ln(S_0 d b^{(j)}) + \frac{1}{n}\ln S_n^{(j)}\right)
\end{aligned}$$

$$\lim_{n\to\infty}\frac{1}{n}\ln S_n = \lim_{n\to\infty}\max_j\frac{1}{n}\ln S_n^{(j)} = \max_j W^{(j)}$$

assume that $b^{(j)} > 0$

$$\frac{1}{n}\ln\left(\max_j S_0 b^{(j)} S_n^{(j)}\right) \leq \frac{1}{n}\ln S_n \leq \frac{1}{n}\ln\left(\max_j S_0 d b^{(j)} S_n^{(j)}\right)$$

$$\begin{aligned} \max_j\left(\frac{1}{n}\ln(S_0 b^{(j)}) + \frac{1}{n}\ln S_n^{(j)}\right) &\leq \frac{1}{n}\ln S_n \\ &\leq \max_j\left(\frac{1}{n}\ln(S_0 d b^{(j)}) + \frac{1}{n}\ln S_n^{(j)}\right) \end{aligned}$$

$$\lim_{n\to\infty}\frac{1}{n}\ln S_n = \lim_{n\to\infty}\max_j\frac{1}{n}\ln S_n^{(j)} = \max_j W^{(j)}$$

we can do much better using multi-period investment

Dynamic portfolio selection: multi-period investment

relative prices

$$x_i^{(j)} = \frac{S_i^{(j)}}{S_{i-1}^{(j)}}$$

relative prices

$$x_i^{(j)} = \frac{S_i^{(j)}}{S_{i-1}^{(j)}}$$

$\mathbf{x}_i = (x_i^{(1)}, \dots x_i^{(d)})$ the return vector on trading period i

relative prices

$$x_i^{(j)} = \frac{S_i^{(j)}}{S_{i-1}^{(j)}}$$

$\mathbf{x}_i = (x_i^{(1)}, \dots x_i^{(d)})$ the return vector on trading period i
multi-period investment

relative prices

$$x_i^{(j)} = \frac{S_i^{(j)}}{S_{i-1}^{(j)}}$$

$\mathbf{x}_i = (x_i^{(1)}, \dots x_i^{(d)})$ the return vector on trading period i
multi-period investment
$x_i^{(j)}$ is the factor by which capital invested in stock j grows during the market period i

relative prices

$$x_i^{(j)} = \frac{S_i^{(j)}}{S_{i-1}^{(j)}}$$

$\mathbf{x}_i = (x_i^{(1)}, \dots x_i^{(d)})$ the return vector on trading period i
multi-period investment
$x_i^{(j)}$ is the factor by which capital invested in stock j grows during the market period i
Constantly Re-balanced Portfolio (CRP)

Dynamic portfolio selection: multi-period investment

relative prices

$$x_i^{(j)} = \frac{S_i^{(j)}}{S_{i-1}^{(j)}}$$

$\mathbf{x}_i = (x_i^{(1)}, \dots x_i^{(d)})$ the return vector on trading period i
multi-period investment
$x_i^{(j)}$ is the factor by which capital invested in stock j grows during the market period i
Constantly Re-balanced Portfolio (CRP)

a portfolio vector $\mathbf{b} = (b^{(1)}, \dots b^{(d)})$

relative prices

$$x_i^{(j)} = \frac{S_i^{(j)}}{S_{i-1}^{(j)}}$$

$\mathbf{x}_i = (x_i^{(1)}, \dots x_i^{(d)})$ the return vector on trading period i
multi-period investment
$x_i^{(j)}$ is the factor by which capital invested in stock j grows during the market period i
Constantly Re-balanced Portfolio (CRP)

a portfolio vector $\mathbf{b} = (b^{(1)}, \dots b^{(d)})$
$b^{(j)}$ gives the proportion of the investor's capital invested in stock j

relative prices

$$x_i^{(j)} = \frac{S_i^{(j)}}{S_{i-1}^{(j)}}$$

$\mathbf{x}_i = (x_i^{(1)}, \dots x_i^{(d)})$ the return vector on trading period i
multi-period investment
$x_i^{(j)}$ is the factor by which capital invested in stock j grows during the market period i
Constantly Re-balanced Portfolio (CRP)

a portfolio vector $\mathbf{b} = (b^{(1)}, \dots b^{(d)})$
$b^{(j)}$ gives the proportion of the investor's capital invested in stock j
$\mathbf{b}$ is the portfolio vector for each trading day

for the first trading period S_0 denotes the initial capital

$$S_1 = S_0 \sum_{j=1}^{d} b^{(j)} x_1^{(j)} = S_0 \langle \mathbf{b} \, , \, \mathbf{x}_1 \rangle$$

for the first trading period S_0 denotes the initial capital

$$S_1 = S_0 \sum_{j=1}^{d} b^{(j)} x_1^{(j)} = S_0 \langle \mathbf{b} \, , \, \mathbf{x}_1 \rangle$$

for the second trading period, S_1 new initial capital

$$S_2 = S_1 \cdot \langle \mathbf{b} \, , \, \mathbf{x}_2 \rangle = S_0 \cdot \langle \mathbf{b} \, , \, \mathbf{x}_1 \rangle \cdot \langle \mathbf{b} \, , \, \mathbf{x}_2 \rangle \, .$$

for the first trading period S_0 denotes the initial capital

$$S_1 = S_0 \sum_{j=1}^{d} b^{(j)} x_1^{(j)} = S_0 \langle \mathbf{b} \, , \, \mathbf{x}_1 \rangle$$

for the second trading period, S_1 new initial capital

$$S_2 = S_1 \cdot \langle \mathbf{b} \, , \, \mathbf{x}_2 \rangle = S_0 \cdot \langle \mathbf{b} \, , \, \mathbf{x}_1 \rangle \cdot \langle \mathbf{b} \, , \, \mathbf{x}_2 \rangle \, .$$

for the nth trading period:

$$S_n = S_{n-1} \langle \mathbf{b} \, , \, \mathbf{x}_n \rangle = S_0 \prod_{i=1}^{n} \langle \mathbf{b} \, , \, \mathbf{x}_i \rangle = S_0 e^{n W_n(\mathbf{b})}$$

with the average growth rate

$$W_n(\mathbf{b}) = \frac{1}{n} \sum_{i=1}^{n} \ln \langle \mathbf{b} \, , \, \mathbf{x}_i \rangle \, .$$

Special market process: $\mathbf{X}_1, \mathbf{X}_2, \ldots$ is independent and identically distributed (i.i.d.)

Special market process: $\mathbf{X}_1, \mathbf{X}_2, \ldots$ is independent and identically distributed (i.i.d.)

log-optimum portfolio $\mathbf{b}^*$

Special market process: $\mathbf{X}_1, \mathbf{X}_2, \ldots$ is independent and identically distributed (i.i.d.)

log-optimum portfolio $\mathbf{b}^*$

$$\mathbf{E}\{\ln \langle \mathbf{b}^* \, , \, \mathbf{X}_1 \rangle\} = \max_{\mathbf{b}} \mathbf{E}\{\ln \langle \mathbf{b} \, , \, \mathbf{X}_1 \rangle\}$$

Special market process: $\mathbf{X}_1, \mathbf{X}_2, \ldots$ is independent and identically distributed (i.i.d.)

log-optimum portfolio $\mathbf{b}^*$

$$\mathbf{E}\{\ln \langle \mathbf{b}^*, \mathbf{X}_1 \rangle\} = \max_{\mathbf{b}} \mathbf{E}\{\ln \langle \mathbf{b}, \mathbf{X}_1 \rangle\}$$

Best Constantly Re-balanced Portfolio (BCRP)

Optimality

If $S_n^* = S_n(\mathbf{b}^*)$ denotes the capital after trading period n achieved by a log-optimum portfolio strategy $\mathbf{b}^*$,

Optimality

If $S_n^* = S_n(\mathbf{b}^*)$ denotes the capital after trading period n achieved by a log-optimum portfolio strategy $\mathbf{b}^*$,
then for any portfolio strategy $\mathbf{b}$ with capital $S_n = S_n(\mathbf{b})$ and for any i.i.d. process $\{\mathbf{X}_n\}_{-\infty}^{\infty}$,

Optimality

If $S_n^* = S_n(\mathbf{b}^*)$ denotes the capital after trading period n achieved by a log-optimum portfolio strategy $\mathbf{b}^*$,
then for any portfolio strategy $\mathbf{b}$ with capital $S_n = S_n(\mathbf{b})$ and for any i.i.d. process $\{\mathbf{X}_n\}_{-\infty}^{\infty}$,

$$\lim_{n\to\infty} \frac{1}{n} \ln S_n \leq \lim_{n\to\infty} \frac{1}{n} \ln S_n^* \quad \text{almost surely}$$

Optimality

If $S_n^* = S_n(\mathbf{b}^*)$ denotes the capital after trading period n achieved by a log-optimum portfolio strategy $\mathbf{b}^*$,
then for any portfolio strategy $\mathbf{b}$ with capital $S_n = S_n(\mathbf{b})$ and for any i.i.d. process $\{\mathbf{X}_n\}_{-\infty}^{\infty}$,

$$\lim_{n\to\infty} \frac{1}{n} \ln S_n \le \lim_{n\to\infty} \frac{1}{n} \ln S_n^* \quad \text{almost surely}$$

and

$$\lim_{n\to\infty} \frac{1}{n} \ln S_n^* = W^* \quad \text{almost surely,}$$

where

$$W^* = \mathbf{E}\{\ln \langle \mathbf{b}^*, \mathbf{X}_1 \rangle\}$$

is the maximal growth rate of any portfolio.

Proof

$$\frac{1}{n}\ln S_n = \frac{1}{n}\sum_{i=1}^{n}\ln\langle \mathbf{b}\,,\,\mathbf{X}_i\rangle$$

Proof

$$
\begin{aligned}
\frac{1}{n}\ln S_n &= \frac{1}{n}\sum_{i=1}^{n}\ln\langle \mathbf{b}\,,\,\mathbf{X}_i\rangle \\
&= \frac{1}{n}\sum_{i=1}^{n}\mathbf{E}\{\ln\langle \mathbf{b}\,,\,\mathbf{X}_i\rangle\} \\
&+ \frac{1}{n}\sum_{i=1}^{n}\left(\ln\langle \mathbf{b}\,,\,\mathbf{X}_i\rangle - \mathbf{E}\{\ln\langle \mathbf{b}\,,\,\mathbf{X}_i\rangle\}\right)
\end{aligned}
$$

Proof

$$\begin{aligned}\frac{1}{n}\ln S_n &= \frac{1}{n}\sum_{i=1}^{n}\ln\langle \mathbf{b}\,,\,\mathbf{X}_i\rangle\\ &= \frac{1}{n}\sum_{i=1}^{n}\mathbf{E}\{\ln\langle \mathbf{b}\,,\,\mathbf{X}_i\rangle\}\\ &+ \frac{1}{n}\sum_{i=1}^{n}\left(\ln\langle \mathbf{b}\,,\,\mathbf{X}_i\rangle - \mathbf{E}\{\ln\langle \mathbf{b}\,,\,\mathbf{X}_i\rangle\}\right)\end{aligned}$$

and

$$\begin{aligned}\frac{1}{n}\ln S_n^* &= \frac{1}{n}\sum_{i=1}^{n}\mathbf{E}\{\ln\langle \mathbf{b}^*\,,\,\mathbf{X}_i\rangle\}\\ &+ \frac{1}{n}\sum_{i=1}^{n}\left(\ln\langle \mathbf{b}^*\,,\,\mathbf{X}_i\rangle - \mathbf{E}\{\ln\langle \mathbf{b}^*\,,\,\mathbf{X}_i\rangle\}\right)\end{aligned}$$

History

gambling, horse racing, information theory

Kelly (1956)
Latané (1959)
Breiman (1961)
Finkelstein and Whitley (1981)
Barron and Cover (1988)

History

gambling, horse racing, information theory

Kelly (1956)
Latané (1959)
Breiman (1961)
Finkelstein and Whitley (1981)
Barron and Cover (1988)

Chapter 15 of D. G. Luenberger, *Investment Science*. Oxford University Press, 1998.

Example 1: 1 stock + cash

$d = 2, \qquad \mathbf{X} = (X^{(1)}, X^{(2)})$

Stock:

$$X^{(1)} = \begin{cases} 2 & \text{with probability } 1/2, \\ 1/2 & \text{with probability } 1/2. \end{cases}$$

Example 1: 1 stock + cash

$d = 2$, $\quad \mathbf{X} = (X^{(1)}, X^{(2)})$

Stock:

$$X^{(1)} = \begin{cases} 2 & \text{with probability } 1/2, \\ 1/2 & \text{with probability } 1/2. \end{cases}$$

$$\mathbf{E}\{X^{(1)}\} = 1/2 \cdot (2 + 1/2) = 5/4 > 1$$

Example 1: 1 stock + cash

$d = 2, \qquad \mathbf{X} = (X^{(1)}, X^{(2)})$

Stock:

$$X^{(1)} = \begin{cases} 2 & \text{with probability } 1/2, \\ 1/2 & \text{with probability } 1/2. \end{cases}$$

$$\mathbf{E}\{X^{(1)}\} = 1/2 \cdot (2 + 1/2) = 5/4 > 1$$

$$\mathbf{E}\{S_n^{(1)}\} = \mathbf{E}\left\{\prod_{i=1}^{n} X_i^{(1)}\right\} = (5/4)^n$$

Example 1: 1 stock + cash

$d = 2, \qquad \mathbf{X} = (X^{(1)}, X^{(2)})$

Stock:

$$X^{(1)} = \begin{cases} 2 & \text{with probability } 1/2, \\ 1/2 & \text{with probability } 1/2. \end{cases}$$

$$\mathbf{E}\{X^{(1)}\} = 1/2 \cdot (2 + 1/2) = 5/4 > 1$$

$$\mathbf{E}\{S_n^{(1)}\} = \mathbf{E}\left\{\prod_{i=1}^{n} X_i^{(1)}\right\} = (5/4)^n$$

What about $S_n^{(1)}$ or $W^{(1)}$?

Example 1: 1 stock + cash

$d = 2, \qquad \mathbf{X} = (X^{(1)}, X^{(2)})$

Stock:

$$X^{(1)} = \begin{cases} 2 & \text{with probability } 1/2, \\ 1/2 & \text{with probability } 1/2. \end{cases}$$

$$\mathbf{E}\{X^{(1)}\} = 1/2 \cdot (2 + 1/2) = 5/4 > 1$$

$$\mathbf{E}\{S_n^{(1)}\} = \mathbf{E}\left\{\prod_{i=1}^{n} X_i^{(1)}\right\} = (5/4)^n$$

What about $S_n^{(1)}$ or $W^{(1)}$?

$$\begin{aligned} W^{(1)} = \lim_{n\to\infty} \frac{1}{n} \ln S_n^{(1)} &= \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \ln X_i^{(1)} = \mathbf{E}\{\ln X^{(1)}\} \\ &= 1/2 \ln 2 + 1/2 \ln(1/2) = 0 \end{aligned}$$

zero growth rate

Cash:

$$X^{(2)} = 1$$

zero growth rate

Cash:

$$X^{(2)} = 1$$

zero growth rate

portfolio

$$\mathbf{b} = (b, 1 - b)$$

Cash:

$$X^{(2)} = 1$$

zero growth rate
portfolio

$$\mathbf{b} = (b, 1-b)$$

$$\begin{aligned} \mathbf{E}\{\ln \langle \mathbf{b}\,,\, \mathbf{X}\rangle\} &= 1/2\,(\ln(2b + (1-b)) + \ln(b/2 + (1-b)) \\ &= 1/2 \ln[(1+b)(1-b/2)] \end{aligned}$$

Cash:

$$X^{(2)} = 1$$

zero growth rate
portfolio

$$\mathbf{b} = (b, 1-b)$$

$$\begin{aligned} \mathbf{E}\{\ln \langle \mathbf{b}\,,\, \mathbf{X} \rangle\} &= 1/2\,(\ln(2b + (1-b)) + \ln(b/2 + (1-b)) \\ &= 1/2 \ln[(1+b)(1-b/2)] \end{aligned}$$

log-optimal portfolio

$$\mathbf{b}^* = (1/2, 1/2)$$

Cash:

$$X^{(2)} = 1$$

zero growth rate
portfolio

$$\mathbf{b} = (b, 1 - b)$$

$$\begin{aligned} \mathbf{E}\{\ln \langle \mathbf{b}\, , \, \mathbf{X} \rangle\} &= 1/2\, (\ln(2b + (1-b)) + \ln(b/2 + (1-b)) \\ &= 1/2 \ln[(1+b)(1-b/2)] \end{aligned}$$

log-optimal portfolio

$$\mathbf{b}^* = (1/2, 1/2)$$

asymptotic average growth rate

$$\mathbf{E}\{\ln \langle \mathbf{b}^*\, , \, \mathbf{X} \rangle\} = 1/2 \ln(9/8) = 0.059 = W^*$$

positive growth rate

Example 2: 2 stocks + cash

$$d = 3, \qquad \mathbf{X} = (X^{(1)}, X^{(2)}, X^{(3)})$$

Example 2: 2 stocks + cash

$d = 3, \qquad \mathbf{X} = (X^{(1)}, X^{(2)}, X^{(3)})$

Stocks:

$$X^{(1)} = \begin{cases} 2 & \text{with probability } 1/2, \\ 1/2 & \text{with probability } 1/2. \end{cases}$$

Example 2: 2 stocks + cash

$d = 3$, $\mathbf{X} = (X^{(1)}, X^{(2)}, X^{(3)})$

Stocks:

$$X^{(1)} = \begin{cases} 2 & \text{with probability } 1/2, \\ 1/2 & \text{with probability } 1/2. \end{cases}$$

$$X^{(2)} = \begin{cases} 2 & \text{with probability } 1/2, \\ 1/2 & \text{with probability } 1/2. \end{cases}$$

Example 2: 2 stocks + cash

$d = 3, \qquad \mathbf{X} = (X^{(1)}, X^{(2)}, X^{(3)})$

Stocks:

$$X^{(1)} = \begin{cases} 2 & \text{with probability } 1/2, \\ 1/2 & \text{with probability } 1/2. \end{cases}$$

$$X^{(2)} = \begin{cases} 2 & \text{with probability } 1/2, \\ 1/2 & \text{with probability } 1/2. \end{cases}$$

Cash:

$$X^{(3)} = 1$$

Example 2: 2 stocks + cash

$d = 3, \qquad \mathbf{X} = (X^{(1)}, X^{(2)}, X^{(3)})$

Stocks:

$$X^{(1)} = \begin{cases} 2 & \text{with probability } 1/2, \\ 1/2 & \text{with probability } 1/2. \end{cases}$$

$$X^{(2)} = \begin{cases} 2 & \text{with probability } 1/2, \\ 1/2 & \text{with probability } 1/2. \end{cases}$$

Cash:

$$X^{(3)} = 1$$

log-optimal portfolio

$$\mathbf{b}^* = (0.46, 0.46, 0.08)$$

Example 2: 2 stocks + cash

$d = 3$, $\quad \mathbf{X} = (X^{(1)}, X^{(2)}, X^{(3)})$

Stocks:

$$X^{(1)} = \begin{cases} 2 & \text{with probability } 1/2, \\ 1/2 & \text{with probability } 1/2. \end{cases}$$

$$X^{(2)} = \begin{cases} 2 & \text{with probability } 1/2, \\ 1/2 & \text{with probability } 1/2. \end{cases}$$

Cash:

$$X^{(3)} = 1$$

log-optimal portfolio

$$\mathbf{b}^* = (0.46, 0.46, 0.08)$$

asymptotic average growth rate

$$\mathbf{E}\{\ln \langle \mathbf{b}^*, \mathbf{X} \rangle\} = 0.112 = W^*$$

Example 3: 3 stocks + cash

$$d = 4, \qquad \mathbf{X} = (X^{(1)}, X^{(2)}, X^{(3)}, X^{(4)})$$

Example 3: 3 stocks + cash

$d = 4, \qquad \mathbf{X} = (X^{(1)}, X^{(2)}, X^{(3)}, X^{(4)})$

log-optimal portfolio

$$\mathbf{b}^* = (1/3, 1/3, 1/3, 0)$$

Example 3: 3 stocks + cash

$d = 4, \qquad \mathbf{X} = (X^{(1)}, X^{(2)}, X^{(3)}, X^{(4)})$
log-optimal portfolio

$$\mathbf{b}^* = (1/3, 1/3, 1/3, 0)$$

the cash has zero weight

$d = 4, \qquad \mathbf{X} = (X^{(1)}, X^{(2)}, X^{(3)}, X^{(4)})$
log-optimal portfolio

$$\mathbf{b}^* = (1/3, 1/3, 1/3, 0)$$

the cash has zero weight
asymptotic average growth rate

$$\mathbf{E}\{\ln \langle \mathbf{b}^*, \mathbf{X} \rangle\} = 0.152 = W^*$$

Example 4: many stocks

d is large

Example 4: many stocks

d is large
log-optimal portfolio

$$\mathbf{b}^* = (1/d, \ldots, 1/d)$$

Example 4: many stocks

d is large

log-optimal portfolio

$$\mathbf{b}^* = (1/d, \dots, 1/d)$$

asymptotic average growth rate

$$\mathbf{E}\{\ln \langle \mathbf{b}^*, \mathbf{X} \rangle\} = 0.223 = W^*$$

Example 5: horse racing

d horses in a race

Example 5: horse racing

d horses in a race

horse j wins with probability p_j

Example 5: horse racing

d horses in a race
horse j wins with probability p_j
payoff o_j: investing 1\$ on horse j results in o_j if it wins, otherwise 0\$

Example 5: horse racing

d horses in a race
horse j wins with probability p_j
payoff o_j: investing 1\$ on horse j results in o_j if it wins, otherwise 0\$

$$\mathbf{X} = (0, \ldots, 0, o_j, 0, \ldots, 0)$$

if horse j wins

Example 5: horse racing

d horses in a race
horse j wins with probability p_j
payoff o_j: investing 1\$ on horse j results in o_j if it wins, otherwise 0\$

$$\mathbf{X} = (0, \dots, 0, o_j, 0, \dots, 0)$$

if horse j wins
repeated races

$$\mathbf{E}\{\ln \langle \mathbf{b}\,,\, \mathbf{X} \rangle\} = \sum_{j=1}^{d} p_j \ln(b^{(j)} o_j) = \sum_{j=1}^{d} p_j \ln b^{(j)} + \sum_{j=1}^{d} p_j \ln o_j$$

Example 5: horse racing

d horses in a race
horse j wins with probability p_j
payoff o_j: investing 1$ on horse j results in o_j if it wins, otherwise 0$

$$\mathbf{X} = (0, \dots, 0, o_j, 0, \dots, 0)$$

if horse j wins
repeated races

$$\mathbf{E}\{\ln \langle \mathbf{b}\, ,\, \mathbf{X} \rangle\} = \sum_{j=1}^{d} p_j \ln(b^{(j)} o_j) = \sum_{j=1}^{d} p_j \ln b^{(j)} + \sum_{j=1}^{d} p_j \ln o_j$$

therefore

$$\arg\max_{\mathbf{b}} \mathbf{E}\{\ln \langle \mathbf{b}\, ,\, \mathbf{X} \rangle\} = \arg\max_{\mathbf{b}} \sum_{j=1}^{d} p_j \ln b^{(j)}$$

$$\arg\max_{\mathbf{b}} \sum_{j=1}^{d} p_j \ln b^{(j)}$$

$$\arg\max_{\mathbf{b}} \sum_{j=1}^{d} p_j \ln b^{(j)}$$

Kullback-Leibler divergence:

$$KL(\mathbf{p}, \mathbf{b}) = \sum_{j=1}^{d} p_j \ln \frac{p_j}{b^{(j)}}$$

$$\arg\max_{\mathbf{b}} \sum_{j=1}^{d} p_j \ln b^{(j)}$$

Kullback-Leibler divergence:

$$KL(\mathbf{p}, \mathbf{b}) = \sum_{j=1}^{d} p_j \ln \frac{p_j}{b^{(j)}}$$

basic property:

$$KL(\mathbf{p}, \mathbf{b}) \geq 0$$

$$\arg\max_{\mathbf{b}} \sum_{j=1}^{d} p_j \ln b^{(j)}$$

Kullback-Leibler divergence:

$$KL(\mathbf{p}, \mathbf{b}) = \sum_{j=1}^{d} p_j \ln \frac{p_j}{b^{(j)}}$$

basic property:

$$KL(\mathbf{p}, \mathbf{b}) \geq 0$$

Proof:

$$KL(\mathbf{p}, \mathbf{b}) = -\sum_{j=1}^{d} p_j \ln \frac{b^{(j)}}{p_j}$$

$$\arg\max_{\mathbf{b}} \sum_{j=1}^{d} p_j \ln b^{(j)}$$

Kullback-Leibler divergence:

$$KL(\mathbf{p}, \mathbf{b}) = \sum_{j=1}^{d} p_j \ln \frac{p_j}{b^{(j)}}$$

basic property:

$$KL(\mathbf{p}, \mathbf{b}) \geq 0$$

Proof:

$$\begin{aligned} KL(\mathbf{p}, \mathbf{b}) = -\sum_{j=1}^{d} p_j \ln \frac{b^{(j)}}{p_j} &\geq -\sum_{j=1}^{d} p_j \left(\frac{b^{(j)}}{p_j} - 1 \right) \\ &= -\sum_{j=1}^{d} b^{(j)} + \sum_{j=1}^{d} p_j = 0 \end{aligned}$$

$$\arg\max_{\mathbf{b}} \sum_{j=1}^{d} p_j \ln b^{(j)} = \mathbf{p}$$

$$\arg\max_{\mathbf{b}} \sum_{j=1}^{d} p_j \ln b^{(j)} = \mathbf{p}$$

independent of the payoffs

$$W^* = \sum_{j=1}^{d} p_j \ln(p_j o_j)$$

$$\arg\max_{\mathbf{b}} \sum_{j=1}^{d} p_j \ln b^{(j)} = \mathbf{p}$$

independent of the payoffs

$$W^* = \sum_{j=1}^{d} p_j \ln(p_j o_j)$$

usual choice of payoffs:

$$o_j = \frac{1}{p_j}$$

$$\arg\max_{\mathbf{b}} \sum_{j=1}^{d} p_j \ln b^{(j)} = \mathbf{p}$$

independent of the payoffs

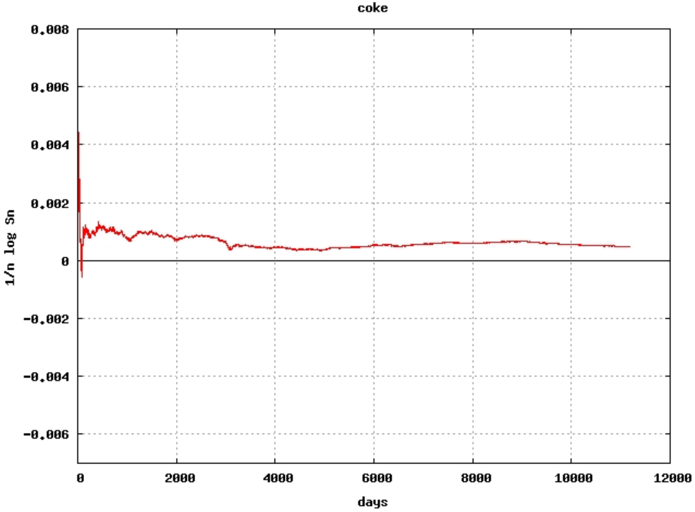
$$W^* = \sum_{j=1}^{d} p_j \ln(p_j o_j)$$

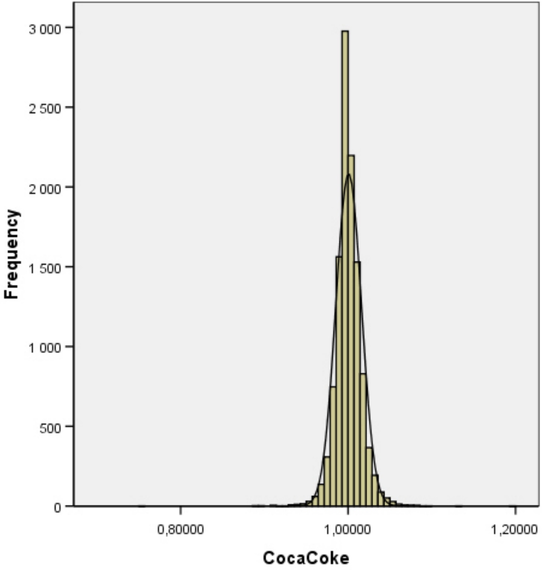
usual choice of payoffs:

$$o_j = \frac{1}{p_j}$$

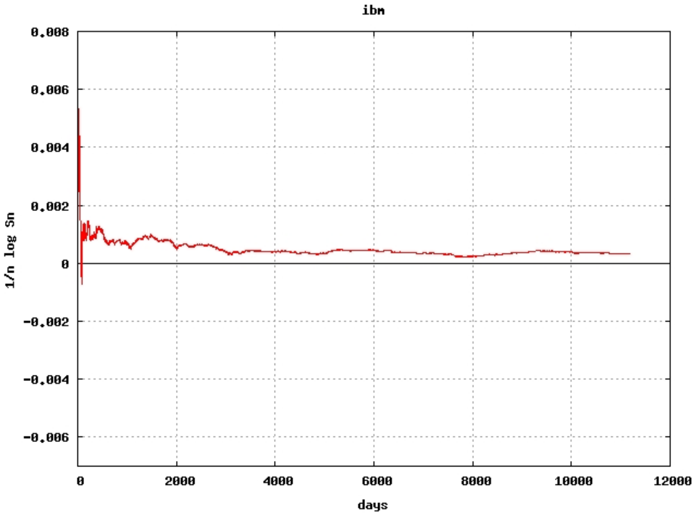
$$W^* = 0$$

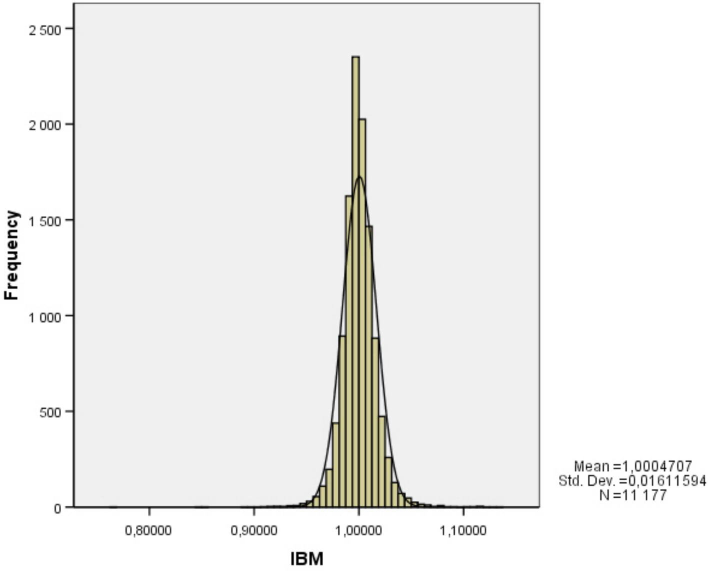
any gambling strategy has negative growth rate

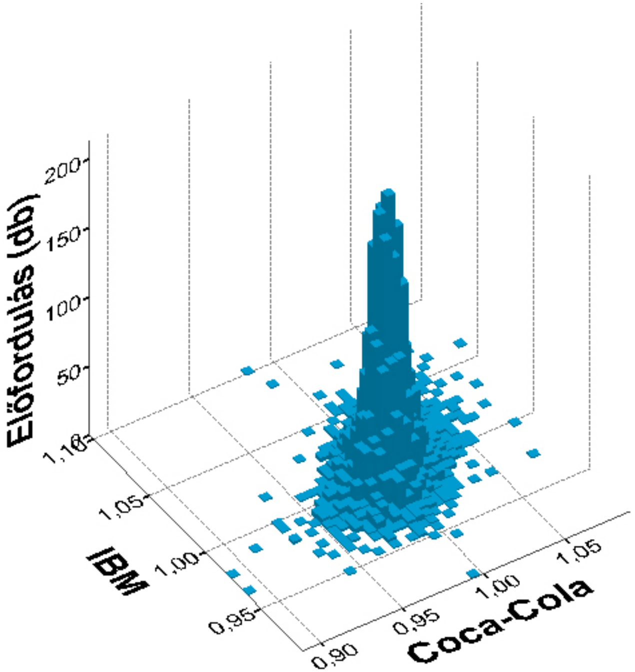

coke
0.008
0.006
0.004
0.002
0
-0.002
-0.004
-0.006
1/n log Sn
0
2000
4000
6000
8000
10000
12000
days



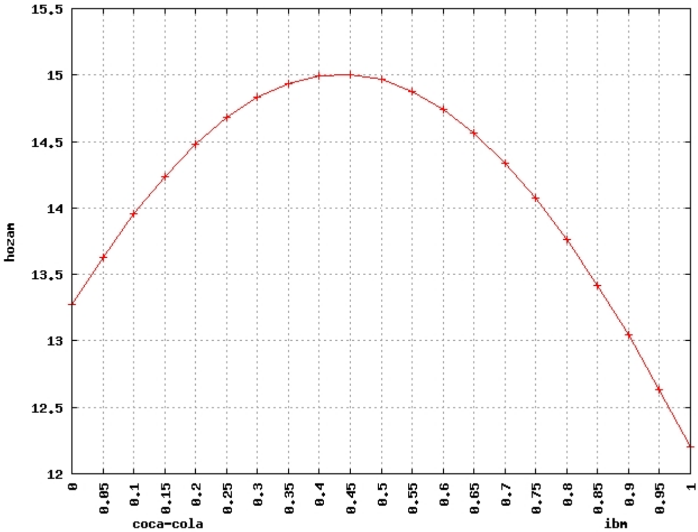
Mean =1,0006105
Std. Dev. =0,01529634
N =11 177


ibm
0.008
0.006
0.004
0.002
0
-0.002
-0.004
-0.006
1/n log Sn
0
2000
4000
6000
8000
10000
12000
days


2 500
2 000
1 500
1 000
500
0
Frequency
0,80000
0,90000
1,00000
1,10000
IBM
Mean =1,0004707
Std. Dev. =0,01611594
N =11 177



Előfordulás (db)
200
150
100
50
1,10
1,05
1,00
0,95
IBM
0,90
0,95
1,00
1,05
Coca-Cola


15.5
15
14.5
14
13.5
13
12.5
12
hozam
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4
0.45
0.5
0.55
0.6
0.65
0.7
0.75
0.8
0.85
0.9
0.95
1
coca-cola
ibm

Consequences

Corollary: with large probability

$$S_n(\mathbf{b}) \quad \text{is not close to} \quad \mathbf{E}\{S_n(\mathbf{b})\}$$

Consequences

Corollary: with large probability

$$S_n(\mathbf{b}) \quad \text{is not close to} \quad \mathbf{E}\{S_n(\mathbf{b})\}$$

Proof:

$$\left\{-\delta < \frac{1}{n}\ln S_n(\mathbf{b}) - \mathbf{E}\{\ln\langle \mathbf{b}\,,\,\mathbf{X}_1\rangle\} < \delta\right\}$$

Corollary: with large probability

$$S_n(\mathbf{b}) \quad \text{is not close to} \quad \mathbf{E}\{S_n(\mathbf{b})\}$$

Proof:

$$\left\{-\delta < \frac{1}{n}\ln S_n(\mathbf{b}) - \mathbf{E}\{\ln \langle \mathbf{b}\,,\, \mathbf{X}_1\rangle\} < \delta\right\}$$

$$\left\{-\delta + \mathbf{E}\{\ln \langle \mathbf{b}\,,\, \mathbf{X}_1\rangle\} < \frac{1}{n}\ln S_n(\mathbf{b}) < \delta + \mathbf{E}\{\ln \langle \mathbf{b}\,,\, \mathbf{X}_1\rangle\}\right\}$$

Consequences

Corollary: with large probability

$$S_n(\mathbf{b}) \quad \text{is not close to} \quad \mathbf{E}\{S_n(\mathbf{b})\}$$

Proof:

$$\left\{ -\delta < \frac{1}{n} \ln S_n(\mathbf{b}) - \mathbf{E}\{\ln \langle \mathbf{b}\,,\, \mathbf{X}_1 \rangle\} < \delta \right\}$$

$$\left\{ -\delta + \mathbf{E}\{\ln \langle \mathbf{b}\,,\, \mathbf{X}_1 \rangle\} < \frac{1}{n} \ln S_n(\mathbf{b}) < \delta + \mathbf{E}\{\ln \langle \mathbf{b}\,,\, \mathbf{X}_1 \rangle\} \right\}$$

$$\left\{ e^{n(-\delta + \mathbf{E}\{\ln \langle \mathbf{b}\,,\, \mathbf{X}_1 \rangle\})} < S_n(\mathbf{b}) < e^{n(\delta + \mathbf{E}\{\ln \langle \mathbf{b}\,,\, \mathbf{X}_1 \rangle\})} \right\}$$

$$S_n(\mathbf{b}) \quad \text{is close to} \quad e^{n\mathbf{E}\{\ln\langle \mathbf{b}\,,\,\mathbf{X}_1\rangle\}}$$

$$S_n(\mathbf{b}) \quad \text{is close to} \quad e^{n\mathbf{E}\{\ln\langle \mathbf{b}\,,\,\mathbf{X}_1\rangle\}}$$

$$\mathbf{E}\{S_n(\mathbf{b})\} = \mathbf{E}\{\prod_{i=1}^{n}\langle \mathbf{b}\,,\,\mathbf{X}_i\rangle\} = \prod_{i=1}^{n}\langle \mathbf{b}\,,\,\mathbf{E}\{\mathbf{X}_i\}\rangle = e^{n\ln\langle \mathbf{b}\,,\,\mathbf{E}\{\mathbf{X}_1\}\rangle}$$

$$S_n(\mathbf{b}) \quad \text{is close to} \quad e^{n\mathbf{E}\{\ln\langle \mathbf{b}\,,\,\mathbf{X}_1\rangle\}}$$

$$\mathbf{E}\{S_n(\mathbf{b})\} = \mathbf{E}\{\prod_{i=1}^{n} \langle \mathbf{b}\,,\,\mathbf{X}_i\rangle\} = \prod_{i=1}^{n} \langle \mathbf{b}\,,\,\mathbf{E}\{\mathbf{X}_i\}\rangle = e^{n\ln\langle \mathbf{b}\,,\,\mathbf{E}\{\mathbf{X}_1\}\rangle}$$

by Jensen inequality

$$\ln\langle \mathbf{b}\,,\,\mathbf{E}\{\mathbf{X}_1\}\rangle > \mathbf{E}\{\ln\langle \mathbf{b}\,,\,\mathbf{X}_1\rangle\}$$

$$S_n(\mathbf{b}) \quad \text{is close to} \quad e^{n\mathbf{E}\{\ln\langle \mathbf{b}\,,\, \mathbf{X}_1\rangle\}}$$

$$\mathbf{E}\{S_n(\mathbf{b})\} = \mathbf{E}\{\prod_{i=1}^{n} \langle \mathbf{b}\,,\, \mathbf{X}_i\rangle\} = \prod_{i=1}^{n} \langle \mathbf{b}\,,\, \mathbf{E}\{\mathbf{X}_i\}\rangle = e^{n \ln\langle \mathbf{b}\,,\, \mathbf{E}\{\mathbf{X}_1\}\rangle}$$

by Jensen inequality

$$\ln \langle \mathbf{b}\,,\, \mathbf{E}\{\mathbf{X}_1\}\rangle > \mathbf{E}\{\ln \langle \mathbf{b}\,,\, \mathbf{X}_1\rangle\}$$

therefore

$$S_n(\mathbf{b}) \quad \text{is much less than} \quad \mathbf{E}\{S_n(\mathbf{b})\}$$

Naive approach

$$\arg\max_{\mathbf{b}} \mathbf{E}\{S_n(\mathbf{b})\}$$

Naive approach

$$\arg\max_{\mathbf{b}} \mathbf{E}\{S_n(\mathbf{b})\}$$

because of

$$\mathbf{E}\{S_n(\mathbf{b})\} = \langle \mathbf{b}\,,\, \mathbf{E}\{\mathbf{X}_1\}\rangle^n$$

Naive approach

$$\arg\max_{\mathbf{b}} \mathbf{E}\{S_n(\mathbf{b})\}$$

because of

$$\mathbf{E}\{S_n(\mathbf{b})\} = \langle \mathbf{b}\,,\, \mathbf{E}\{\mathbf{X}_1\}\rangle^n$$

$$\arg\max_{\mathbf{b}} \mathbf{E}\{S_n(\mathbf{b})\} = \arg\max_{\mathbf{b}} \langle \mathbf{b}\,,\, \mathbf{E}\{\mathbf{X}_1\}\rangle$$

Naive approach

$$\arg\max_{\mathbf{b}} \mathbf{E}\{S_n(\mathbf{b})\}$$

because of

$$\mathbf{E}\{S_n(\mathbf{b})\} = \langle \mathbf{b} \, , \, \mathbf{E}\{\mathbf{X}_1\}\rangle^n$$

$$\arg\max_{\mathbf{b}} \mathbf{E}\{S_n(\mathbf{b})\} = \arg\max_{\mathbf{b}} \langle \mathbf{b} \, , \, \mathbf{E}\{\mathbf{X}_1\}\rangle$$

$\arg\max_{\mathbf{b}} \langle \mathbf{b} \, , \, \mathbf{E}\{\mathbf{X}_1\}\rangle$ is a portfolio vector having 1 at the position, where $\mathbf{E}\{\mathbf{X}_1\}$ has the largest component

Naive approach

$$\arg\max_{\mathbf{b}} \mathbf{E}\{S_n(\mathbf{b})\}$$

because of

$$\mathbf{E}\{S_n(\mathbf{b})\} = \langle \mathbf{b}\,,\, \mathbf{E}\{\mathbf{X}_1\}\rangle^n$$

$$\arg\max_{\mathbf{b}} \mathbf{E}\{S_n(\mathbf{b})\} = \arg\max_{\mathbf{b}} \langle \mathbf{b}\,,\, \mathbf{E}\{\mathbf{X}_1\}\rangle$$

$\arg\max_{\mathbf{b}} \langle \mathbf{b}\,,\, \mathbf{E}\{\mathbf{X}_1\}\rangle$ is a portfolio vector having 1 at the position, where $\mathbf{E}\{\mathbf{X}_1\}$ has the largest component
it is a dangerous portfolio

$$\arg\max_{\mathbf{b}} \mathbf{E}\{S_n(\mathbf{b})\}$$

because of

$$\mathbf{E}\{S_n(\mathbf{b})\} = \langle \mathbf{b}\,,\, \mathbf{E}\{\mathbf{X}_1\}\rangle^n$$

$$\arg\max_{\mathbf{b}} \mathbf{E}\{S_n(\mathbf{b})\} = \arg\max_{\mathbf{b}} \langle \mathbf{b}\,,\, \mathbf{E}\{\mathbf{X}_1\}\rangle$$

$\arg\max_{\mathbf{b}} \langle \mathbf{b}\,,\, \mathbf{E}\{\mathbf{X}_1\}\rangle$ is a portfolio vector having 1 at the position, where $\mathbf{E}\{\mathbf{X}_1\}$ has the largest component
it is a dangerous portfolio
Markowitz:

$$\arg\max_{\mathbf{b}:\mathbf{Var}(\langle \mathbf{b}\,,\,\mathbf{X}_1\rangle)\le\lambda} \langle \mathbf{b}\,,\, \mathbf{E}\{\mathbf{X}_1\}\rangle$$

Semi-log-optimal portfolio

log-optimal:

$$\arg\max_{\mathbf{b}} \mathbf{E}\{\ln \langle \mathbf{b}\,,\ \mathbf{X}_1 \rangle\}$$

Semi-log-optimal portfolio

log-optimal:

$$\arg\max_{\mathbf{b}} \mathbf{E}\{\ln \langle \mathbf{b}\,,\ \mathbf{X}_1\rangle\}$$

Taylor expansion: $\ln z \approx h(z) = z - 1 - \frac{1}{2}(z-1)^2$

Semi-log-optimal portfolio

log-optimal:

$$\arg\max_{\mathbf{b}} \mathbf{E}\{\ln \langle \mathbf{b}\,,\ \mathbf{X}_1\rangle\}$$

Taylor expansion: $\ln z \approx h(z) = z - 1 - \frac{1}{2}(z-1)^2$
semi-log-optimal:

$$\arg\max_{\mathbf{b}} \mathbf{E}\{h(\langle \mathbf{b}\,,\ \mathbf{X}_1\rangle)\}$$

Semi-log-optimal portfolio

log-optimal:

$$\arg\max_{\mathbf{b}} \mathbf{E}\{\ln \langle \mathbf{b}\,,\ \mathbf{X}_1\rangle\}$$

Taylor expansion: $\ln z \approx h(z) = z - 1 - \frac{1}{2}(z-1)^2$
semi-log-optimal:

$$\arg\max_{\mathbf{b}} \mathbf{E}\{h(\langle \mathbf{b}\,,\ \mathbf{X}_1\rangle)\} = \arg\max_{\mathbf{b}}\{\langle \mathbf{b}\,,\ \mathbf{m}\rangle - \langle \mathbf{b}\,,\ \mathbf{C}\mathbf{b}\rangle\}$$

Semi-log-optimal portfolio

log-optimal:

$$\arg\max_{\mathbf{b}} \mathbf{E}\{\ln \langle \mathbf{b}\,,\, \mathbf{X}_1 \rangle\}$$

Taylor expansion: $\ln z \approx h(z) = z - 1 - \frac{1}{2}(z-1)^2$
semi-log-optimal:

$$\arg\max_{\mathbf{b}} \mathbf{E}\{h(\langle \mathbf{b}\,,\, \mathbf{X}_1 \rangle)\} = \arg\max_{\mathbf{b}}\{\langle \mathbf{b}\,,\, \mathbf{m} \rangle - \langle \mathbf{b}\,,\, \mathbf{C}\mathbf{b} \rangle\}$$

Connection to the Markowitz theory.

Semi-log-optimal portfolio

log-optimal:

$$\arg\max_{\mathbf{b}} \mathbf{E}\{\ln \langle \mathbf{b}\,,\ \mathbf{X}_1 \rangle\}$$

Taylor expansion: $\ln z \approx h(z) = z - 1 - \frac{1}{2}(z-1)^2$
semi-log-optimal:

$$\arg\max_{\mathbf{b}} \mathbf{E}\{h(\langle \mathbf{b}\,,\ \mathbf{X}_1 \rangle)\} = \arg\max_{\mathbf{b}} \{\langle \mathbf{b}\,,\ \mathbf{m} \rangle - \langle \mathbf{b}\,,\ \mathbf{C}\mathbf{b} \rangle\}$$

Connection to the Markowitz theory.
Gy. Ottucsák and I. Vajda, "An Asymptotic Analysis of the Mean-Variance portfolio selection", *Statistics and Decisions*, 25, pp. 63-88, 2007.
http://www.szit.bme.hu/˜oti/portfolio/articles/marko.pdf