Limit Theorems for Products of Large Random Matrices

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Topics

- Universality of singular value distribution
- Universality of eigenvalue distribution
- Asymptotic freeness of random matrices
- The main equations for the density of the limit spectral distribution
- Examples

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The basic idea of our approach

We shall investigate the limit spectral distribution of some $n \times n$ matrix $\mathbf{F}$. First we formulate conditions of universality of singular value distribution of matrix $\mathbf{F} - \alpha\mathbf{I}$ and eigenvalue distribution of matrix $\mathbf{F}$. Here $\alpha = x + iy$ and $\mathbf{I}$ denote unit matrix of order n. Furthermore, assume that we know the S-transform of singular value distribution of matrix $\mathbf{F}$.Assume that matrix $\mathbf{V_F} = \begin{bmatrix} \mathbf{O} & \mathbf{F} \\ \mathbf{F}^* & \mathbf{O} \end{bmatrix}$ and matrix $\mathbf{J}(\alpha) = \begin{bmatrix} \mathbf{O} & \alpha\mathbf{I} \\ \overline{\alpha}\mathbf{I} & \mathbf{O} \end{bmatrix}$ are asymptotic free.

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Then we may find the R-transform of matrix $\mathbf{V}_{\mathbf{F}}(\alpha) = \mathbf{V}_{\mathbf{F}} - \mathbf{J}(\alpha)$ as sum of R-transform of matrix $\mathbf{V}_{\mathbf{F}}$ and R-transform of matrix $\mathbf{J}(\alpha)$. The first we find via S-transform, the second we calculate direct. Furthermore, note that the limit measure for the eigenvalues of matrix well defined by its logarithmic potential. Logarithmic potential we may reconstruct by the family of singular distribution of shifted matrices.

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The Notation

- Let $m \geq 1$. For any $n \geq 1$ we shall consider m-tuple of integer $(n_0, n_1, \ldots, n_m)$ with $n_q = n_q(n)$ and $n_0(n) = n$ and there exist $y_1, \ldots, y_m \in (0, 1]$ such that

$$\lim_{n\to\infty} \frac{n}{n_q} = y_q, \text{ for any } q = 1, \ldots, m. \qquad (1)$$

- Let $X_{jk}^{(q)}$ be independent r.v.'s, for $q = 1, \ldots, m$, $1 \leq j \leq n_{q-1}$, $1 \leq k \leq n_q$, with $\mathrm{E}\, X_{jk} = 0$ and $\mathrm{E}\,|X_{jk}|^2 = 1$.

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- For the complex r.v.'s we shall assume that

$$\begin{bmatrix} \mathbf{E}\,\mathrm{Re}^{\,2} X_{jk}^{(q)} & \mathbf{E}\,\mathrm{Re}\, X_{jk}^{(q)}\,\mathrm{Im}\, X_{jk}^{(q)} \\ \mathbf{E}\,\mathrm{Re}\, X_{jk}^{(q)}\,\mathrm{Im}\, X_{jk}^{(q)} & \mathbf{E}\,\mathrm{Im}^{\,2} X_{jk}^{(q)} \end{bmatrix} = \begin{bmatrix} \sigma_{q1}^2 & \rho_q\sigma_{q1}\sigma_{q2} \\ \rho_q\sigma_{q1}\sigma_{q2} & \sigma_{q2}^2 \end{bmatrix}$$

- For any $q = 1, \ldots, m$ we consider the $n_{q-1} \times n_q$ matrix

$$\mathbf{X}^{(q)} := \frac{1}{\sqrt{n_{q-1}}}(X_{jk}^{(q)}). \qquad (2)$$

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The Notation

- Denote by $\mathcal{M}_{p,q}$ the space of $p \times q$ matrices.
- Let $\mathcal{M} = \mathcal{M}_{n_o,n_1} \otimes \mathcal{M}_{n_1,n_2} \otimes \cdots \otimes \mathcal{M}_{n_{m-1},n_m}$.
- Let $\mathbb{F}$ denote a matrix-value map

$$\mathbb{F} : \mathcal{M} \to \mathcal{M}_{n,p} \tag{3}$$

with some $p = p(n) \geq n$.
- We define matrix $\mathbf{F_X} = (f_{jk}) = \mathbb{F}(\mathbf{X}^{(1)}, \ldots, \mathbf{X}^{(m)})$.
- We shall interesting for spectra of matrices

$$\mathbf{W_X}(\alpha) = (\mathbf{F_X} - \alpha\mathbf{I})(\mathbf{F_X} - \alpha\mathbf{I})^* \tag{4}$$

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- Let $Y_{jk}^{(q)}$ be independent Gaussian random variables with covariance

$$\mathrm{cov}(\text{Re } Y_{jk}, \text{Im } Y_{jk}^{(q)}) = \mathrm{cov}(\text{Re } X_{jk}^{(q)}, \text{Im } X_{jk}^{(q)}).$$

- We shall assume that $Y_{jk}^{(q)}$ and $X_{jk}^{(q)}$, for $q = 1, \ldots, m$, are defined on the same probability space and mutually independent.
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- Denote by $\mathbf{F_Y} := \mathbb{F}(\mathbf{Y}^{(1)}, \ldots, \mathbf{Y}^{(m)})$ and $\mathbf{W_Y}(\alpha) = (\mathbf{F_Y} - \alpha\mathbf{I})(\mathbf{F_Y} - \alpha\mathbf{I})^*$. Here and in what follows $\mathbf{I}$ denotes the unit matrix of corresponding dimension.
- To compare asymptotic behaviour of empirical spectral distributions of matrices $\mathbf{W_X}(\alpha)$ and $\mathbf{W_Y}(\alpha)$ we introduce the matrices

$$\mathbf{V_X} = \begin{bmatrix} \mathbf{O} & \mathbf{F_X} \\ \mathbf{F_X^*} & \mathbf{O} \end{bmatrix}, \quad \mathbf{V_Y} = \begin{bmatrix} \mathbf{O} & \mathbf{F_Y} \\ \mathbf{F_Y^*} & \mathbf{O} \end{bmatrix}.$$

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- We define the matrix

$$\mathbf{J}(\alpha) = \begin{bmatrix} \mathbf{O} & \alpha\mathbf{I} \\ \overline{\alpha}\mathbf{I} & \mathbf{O} \end{bmatrix}, \quad \overline{\alpha} = x - iy,$$

and consider shifted matrices

$$\mathbf{V}_{\mathbf{X}}(\alpha) := \mathbf{V}_{\mathbf{X}} - \mathbf{J}(\alpha) \text{ and } \mathbf{V}_{\mathbf{X}}(\alpha) := \mathbf{V}_{\mathbf{X}} - \mathbf{J}(\alpha).$$

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- Furthermore, we denote by $s_1^2(\mathbf{X},\alpha) \geq \ldots \geq s_n^2(\mathbf{X},\alpha)$ the eigenvalues of matrix $\mathbf{W}_{\mathbf{Y}}(\alpha)$ and by $s_1^2(\mathbf{Y},\alpha) \geq \ldots \geq s_n^2(\mathbf{Y},\alpha)$ the eigenvalues of matrix $\mathbf{W}_{\mathbf{Y}}(\alpha)$ correspondingly.

- In these notation the eigenvalues of matrices $\mathbf{V}_{\mathbf{X}}(\alpha)$ and $\mathbf{V}_{\mathbf{Y}}(\alpha)$ are

$$\pm s_1^2(\mathbf{X},\alpha),\ldots \pm s_n^2(\mathbf{X},\alpha) \quad \text{and} \quad \pm s_1^2(\mathbf{Y},\alpha),\ldots \pm s_n^2(\mathbf{Y}\alpha)$$

- Define the empirical spectral distribution of matrices $\mathbf{W}_{\mathbf{X}}(\alpha)$ ($\mathbf{W}_{\mathbf{Y}}(\alpha)$ resp.) and $\mathbf{V}_{\mathbf{X}}(\alpha)$ ($\mathbf{V}_{\mathbf{Y}}(\alpha)$ resp.)

$$\mathcal{G}_n(x,\mathbf{X},\alpha) := \frac{1}{n}\sum_{j=1}^{n}\mathbb{I}\{s_j^2(\mathbf{X},\alpha)\le x\},$$

$$\widetilde{\mathcal{G}}_n(x,\mathbf{X},\alpha) := \frac{1}{2n}\sum_{j=1}^{n}\mathbb{I}\{s_j(\mathbf{X},\alpha)\le x\} + \frac{1}{2n}\sum_{j=1}^{n}\mathbb{I}\{-s_j(\mathbf{X},\alpha)\le x\}.$$

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- Here $\mathbb{I}\{B\}$ denotes indicator of event B.

- The distributions $\mathcal{G}$ and $\widetilde{\mathcal{G}}$ are connected by formula

$$\widetilde{\mathcal{G}}(x) = \frac{1 + \mathrm{sign}(x)\, \mathcal{G}(x^2)}{2}.$$

- We introduce now the resolvent matrices

$$\mathbf{R}_{\mathbf{X}}(\alpha, z) = (\mathbf{V}_{\mathbf{X}}(\alpha) - z\mathbf{I})^{-1}, \quad \mathbf{R}_{\mathbf{Y}}(\alpha, z) = (\mathbf{V}_{\mathbf{Y}}(\alpha) - z\mathbf{I})^{-1}.$$

- We define the following matrices

$$\mathbf{Z}^{(q)} = \mathbf{X}^{(q)} \cos\varphi + \mathbf{Y}^{(q)} \sin\varphi,$$

for any $\varphi \in [0, \frac{\pi}{2}]$ and any $q = 1, \ldots, m$.

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- Let

$$\mathbf{F}(\varphi) = \mathbb{F}(\mathbf{Z}^{(1)}(\varphi), \ldots, \mathbf{Z}^{(m)}(\varphi)), \quad \mathbf{V}(\alpha, \varphi) = \mathbf{V}_{\mathbf{Z}}(\alpha).$$

- We have

$$\mathbf{F}(0) = \mathbf{F}_{\mathbf{X}}, \quad \mathbf{F}(\frac{\pi}{2}) = \mathbf{F}_{\mathbf{Y}}, \quad \mathbf{V}(\alpha, 0) = \mathbf{V}_{\mathbf{X}}(\alpha), \quad \mathbf{V}(\frac{\pi}{2}) = \mathbf{V}_{\mathbf{Y}}(\alpha).$$

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- Furthermore, we define the corresponding resolvent matrices

$$\mathbf{R} := \mathbf{R}(z, \alpha, \varphi) = (\mathbf{V}(\alpha, \varphi) - z\mathbf{I})^{-1}.$$

- Stieltjes transform of singular values distribution of matrix $\mathbf{V}(\alpha, \varphi)$,

$$m_n(z, \alpha, \varphi) := \frac{1}{2n} \mathrm{Tr}\, \mathbf{R}(z, \alpha, \varphi).$$

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Lindeberg condition

We shall assume that r.v.'s $X_{jk}^{(q)}$ satisfy the Lindeberg condition, i.e.

$$L_n(\tau) = \max_{1\le q\le m} \frac{1}{n^2}\sum_{j=1}^{n_{q-1}}\sum_{k=1}^{n_q} \mathrm{E}\,|X_{jk}^{(q)}|^2\mathbb{I}\{|X_{jk}^{(q)}| > \tau\sqrt{n}\} \to 0 \text{ as } n\to\infty,$$

for any $\tau > 0$. (5)

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To formulate the conditions on the function **F** we need some additional notations. In what follows we shall omit argument φ in the notation.

- Define the function

$$g_{jk}^{(q)} := g_{jk}^{(q)}(\mathbf{Z}^{(1)}, \ldots, \mathbf{Z}^{(m)}) := \operatorname{Tr} \frac{\partial \mathbf{V}}{\partial Z_{jk}^{(q)}} \mathbf{R}^2.$$

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- Let θ be random variables distributed in $[0, 1]$ and independent on all $Z_{jk}^{(q)}$.

- We shall assume that there exist constants $A_1 > 0$ and $A_2 > 0$ and $\tau_0 > 0$ such that

$$\sup_{q,n,j,k,\varphi} \left| \mathsf{E}\left\{ \frac{\partial g_{jk}^{(q)}}{\partial Z_{jk}^{(q)}}(\theta) \Big| Z_{jk}^{(q)} \right\} \right| \le A_1 \quad \text{a.s.,} \tag{6}$$

- and, for any $\tau \le \tau_0$,

$$\sup_{q,n,j,k} \mathbb{I}\{|Z_{jk}^{(q)}| \le \tau\sqrt{n}\} \left| \mathsf{E}\left\{ \frac{\partial^2 g_{jk}^{(q)}}{\partial {Z_{jk}^{(q)}}^2}(\theta) \Big| Z_{jk}^{(q)} \right\} \right| \le A_2 \text{ a.s.} \tag{7}$$

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Universality of singular values distribution

Theorem 2.1

Let $X_{jk}^{(q)}$'s and $Y_{jk}^{(q)}$'s be random variables as described above and assume that $X_{jk}^{(q)}$ satisfy the Lindeberg condition (5)*. Assume that function $\mathbb{F}$ is such that the conditions* (6) *and* (7) *hold. Then*

$$|\,\mathrm{E}\, m_n(z,\alpha,\frac{\pi}{2}) - \mathrm{E}\, m_n(z,\alpha,0)| \to 0 \text{ as } n \to \infty.$$

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Remark 2.2

Under conditions of Theorem 2.1 the expected distribution function of singular value of matrix $\mathbf{F}_{\mathbf{X}}(\alpha)$ has the same limit as distribution function of singular values of matrix $\mathbf{F}_{\mathbf{Y}}(\alpha)$.

Example

- For $m = 1$ and $\mathbb{F}(\mathbf{X}) = \mathbf{X}$

$$g_{jk} = \begin{cases} 2[\mathbf{R}^2]_{jk}, & \text{for } j \neq k \\ [\mathbf{R}^2]_{jj}, & \text{otherwise.} \end{cases}$$

- It is straightforward to check that

$$|\frac{\partial g_{jk}}{\partial Z_{jk}}| \leq Cv^{-3}, |\frac{\partial^2 g_{jk}}{\partial Z_{jk}^2}| \leq Cv^{-4},$$

for $z = u + iv$.

Let μ a probability measure on the complex plane. Define the logarithmic potential of measure μ as

$$U_\mu(\alpha) = -\int_{\mathbb{C}} \log|\alpha - \zeta| d\zeta.$$

Let $\mu_{\mathbf{X}}$ (resp. $\mu_{\mathbf{Y}}$) denote the empirical spectral measure of the matrix $\mathbf{F}_{\mathbf{X}}$ (resp. $\mathbf{F}_{\mathbf{Y}}$), i.e. $\mu_{\mathbf{X}}$ (resp. $\mu_{\mathbf{Y}}$) is the uniform distribution on the eigenvalues $\{\lambda_1(\mathbf{X}), \ldots \lambda_n(\mathbf{X})\}$ (resp. $\{\lambda_1(\mathbf{Y}), \ldots \lambda_n(\mathbf{Y})\}$ of the matrix $\mathbf{F}_{\mathbf{X}}$ (resp. $\mathbf{F}_{\mathbf{Y}}$).Then

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$$G_{\mathbf{X}}(x,\alpha)=\mathsf{E}\,\mathcal{G}_{\mathbf{X}}(x,\alpha).$$

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The function $\log|x|$ is uniformly integrated with respect to distribution functions $G_{\mathbf{X}}(x,\alpha)$ if

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Definition 1

Let random matrices $\mathbf{X}^{(1)}, \ldots, \mathbf{X}^{(m)}$ be independent random matrices of order $n_0 \times n_1, \ldots n_{m-1} \times n_m$ respectively. Assume that random variables $X_{jk}^{(q)}$ are mutually independent , for $q = 1, \ldots, m$ and $j = 1, \ldots, n_{q-1}, k = 1 \ldots, n_q$. Let $\mathrm{E}\, X_{jk}^{(q)} = 0$, $\mathrm{E}\,|X_{jk}^{(q)}|^2 = 1$ and random variables $X_{jk}^{(q)}$ have uniformly integrated second moment, i.e.

$$\sup_{q,j,k,n} \mathrm{E}\,|X_{jk}^{(q)}|^2 \mathbb{I}\{|X_{jk}^{(q)}| > M\} \to 0 \quad \text{as} \quad n \to \infty.$$

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Definition 2

Let matrix-valued functions $\mathbf{F_X} = \mathbb{F}(\mathbf{X}^{(1)}, \ldots, \mathbf{X}^{(m)})$ is such that the function $\log|x|$ is uniformly integrated with respect to singular values distribution of matrices $G_{\mathbf{X}}(x, \alpha)$. Then we say that matrices $\mathbf{F_X}$ satisfy the condition $(C1)$.

Theorem 3.1

Let random matrices $\mathbf{X}^{(1)}, \ldots, \mathbf{X}^{(m)}$ *and* $\mathbf{Y}^{(1)}, \ldots, \mathbf{Y}^{(m)}$ *satisfy the conditions* $(C0)$. *Let matrices* $\mathbb{F}_{\mathbf{X}} = \mathbb{F}(\mathbf{X}^{(1)}, \ldots, \mathbf{X}^{(m)}$ *and* $\mathbb{F}_{\mathbf{Y}} = \mathbb{F}(\mathbf{Y}^{(1)}, \ldots, \mathbf{Y}^{(m)}$ *satisfy the condition* $(C1)$. *Assume the functions* $\mathbb{F}$ *satisfy the conditions* (6) *and* (7) *of Theorem 2.1. Then the matrices* $\mathbb{F}_{\mathbf{X}}$ *and* $\mathbb{F}_{\mathbf{Y}}$ *have the same limit distribution of eigenvalues.*

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This Proposition is bounded on the following Lemma from Bordenave and Chafai "Around circular law", Probability surveys, vol. 9(2012).

Lemma 3.1

Let $(\mathbf{X}_n)$ *be a sequence of random matrices. Let* $\nu_n(\cdot, z)$ *be the empirical distribution function of singular values of matrix* $\mathbf{X}_n - z\mathbf{I}$. *Suppose a.a.* $z \in \mathbb{C}$ *there exists a probability measure* $\nu(\cdot, z)$ *on* $[0, \infty)$ *such*

1) $\nu_n(\cdot, z) \to \nu(\cdot, z)$ *weak as* $n \to \infty$ *in probability;*

2) the function $\log x$ *is uniformly integrated in probability with respect to measures* $\nu_n(\cdot, z)$.

Then there exists a probability measure μ on the complex plane $\mathbb{C}$ such that empirical spectral measures μ_n of matrices $\mathbf{X}_n$ weakly convergence to the measure μ in probability. Moreover

$$U_\mu(z) = -\int_{\mathbb{C}} \log|\zeta - z| d\mu(\zeta) = -\int_0^\infty \log x d\nu_n(x, z). \tag{9}$$

We recall the definition of Voiculescu asymptotic freeness. Two sequences of matrices $(\mathbf{A}_n)_{n\in\mathbb{N}}$ and $(\mathbf{B}_n)_{n\in\mathbb{N}}$ are asymptotic free if for all $k \geq 1$ and all $p_1, m_1, \ldots, p_k, m_k$ the following relations

- there exist measures $\mu_{\mathbf{A}}$ and $\mu_{\mathbf{B}}$ such that

$$\lim_{n\to\infty} \frac{1}{n} \mathrm{E}\,\mathrm{Tr}\,\mathbf{A}_n^{p_1} = M_{p_1}(\mathbf{A}) := \int x^{p_1} d\mu_{\mathbf{A}},$$
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$$\begin{aligned}\lim_{n\to\infty} \mathrm{E}\,\frac{1}{n}\mathrm{Tr}\,\big((\mathbf{A}_n^{p_1} - M_{p_1}(\mathbf{A})\mathbf{I})(\mathbf{B}_n^{m_1} - M_{m_1}(\mathbf{B})\mathbf{I})\cdots \\ \times\,(\mathbf{A}_n^{p_k} - M_{p_k}(\mathbf{A})\mathbf{I})(\mathbf{B}_n^{m_k} - M_{m_k}(\mathbf{B})\mathbf{I})\big) = 0.\end{aligned} \tag{11}$$

Consider sequences of $n \times n$ random matrices $\mathbf{X}_n$, and define matrices

$$\mathbf{A}_n = \begin{bmatrix} \mathbf{O} & \mathbf{F}_n \\ \mathbf{F}_n{}^* & \mathbf{O} \end{bmatrix}.$$

$$\lim_{n\to\infty} \mathrm{E}\,\frac{1}{n}\mathrm{Tr}\,\Big((\mathbf{A}_n^{p_1} - M_{p_1}(\mathbf{A})\mathbf{I})(\mathbf{B}_n^{m_1} - M_{m_1}(\mathbf{B})\mathbf{I})\cdots \\ \times (\mathbf{A}_n^{p_k} - M_{p_k}(\mathbf{A})\mathbf{I})(\mathbf{B}_n^{m_k} - M_{m_k}(\mathbf{B})\mathbf{I})\Big) = 0. \tag{11}$$

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For any $z = u + iv$, introduce matrices

$$\mathbf{B}_n = \mathbf{J}(\alpha) = \begin{bmatrix} \mathbf{O} & -\alpha\mathbf{I} \\ -\overline{\alpha}\mathbf{I} & \mathbf{O} \end{bmatrix}.$$

We apply the definition of asymptotic freeness to matrices $(\mathbf{A}_n)_{n\in\mathbb{N}}$ and $(\mathbf{B}_n)_{n\in\mathbb{N}}$ defined in such way. Note that

$$\mathbf{B}_n^m = \begin{cases} |\alpha|^{2p}\,\mathbf{I}_{2p}, & \text{if } m = 2p \\ |\alpha|^{2p}\,\mathbf{J}(\alpha), & \text{if } m = 2p+1 \end{cases}. \tag{12}$$

From here it follows immediately that

$$\mathbf{J}_n^{2p}(\alpha) - (\lim \frac{1}{2m}\mathrm{Tr}\,\mathbf{J}_m^{2p}(\alpha))\mathbf{I}_{2p} = \mathbf{O}.$$

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This implies relation (11) holds if at least one of the $m_1, m_2, \dots, m_k$ is even. We may rewrite relation (11) for our case as follows

$$\lim_{n\to\infty} \mathsf{E}\,\frac{1}{n}\mathrm{Tr}\,\Big((\mathbf{A}_n^{n_1} - (\lim_{m\to\infty}\frac{1}{m}\mathsf{E}\,\mathrm{Tr}\,\mathbf{A}_m^{n_1})\mathbf{I})\mathbf{J}(\alpha)\cdots (\mathbf{A}_n^{n_k} - (\lim_{m\to\infty}\frac{1}{m}\mathsf{E}\,\mathrm{Tr}\,\mathbf{A}_m^{n_k})\mathbf{I})\mathbf{J}(\alpha)\Big) = 0. \quad (13)$$

S-transform

The Voiculescu S-transform was defined for non-negative distribution. By several authors it was extend to symmetric distributions. We define Voiculescu *S-transform* of distribution as follows. Let $M(z)$ denote the generic moment function of random variable X with distribution function $F_X(x)$, $M(z) = \sum_{k=1}^{\infty} \varphi(X^k) z^k$, where $\varphi(X^k) :== \int_{-\infty}^{\infty} x^k dF_X(x)$. Let $M^{-1}(z)$ denote inverse function of $M(z)$ w.r.t. composition of functions.

Define *S-transform* of distribution $F(x)$ with $\varphi(X) \neq 0$, by equality

$$S_X(z) := \frac{z+1}{z} M^{-1}(z).$$

It is well-known that for free random variables ξ and η with $\varphi(\xi) \neq 0$ and $\varphi(\eta) \neq 0$

$$S_{\eta\xi}(z) = S_\eta S_\xi.$$

Consider now the case distribution with vanishing mean.

Definition 3

Let X be random variable with $\varphi(X) = 0$ and $\varphi(X^2) \neq 0$. Then its two S-transform S_X and $\widetilde{S}_X$ are defined as follows. Let χ and $\widetilde{\chi}$ denote two inverses under composition of the series

$$\psi(z) := \sum_{n=1}^{\infty} \varphi(X^n) z^n = \varphi(X^2) z^2 + \varphi(X^3) z^3 + \cdots, \tag{14}$$

then

$$S_X(z) := \chi(z) \frac{1+z}{z} \quad \text{and} \quad \widetilde{S}_X(z) := \widetilde{\chi}(z) \frac{1+z}{z} \quad \text{and} \tag{15}$$

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Theorem 4.1

Let X and Y be free random variables such that $\varphi(X) = 0$, $\varphi(X^2) \neq 0$ and $\varphi(Y) \neq 0$. Then

$$S_{XY}(z) = S_X(z)S_Y(z) \quad \text{and} \quad \widetilde{S}_{XY}(z) = \widetilde{S}_X(z)S_Y(z). \tag{16}$$

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We may interpret this equality for random matrices as follows. Let **X** and **Y** be two asymptotic free random square matrices of order $n \times n$. Denote by μ_n and ν_n the empirical spectral measures of matrices $\mathbf{XX}^*$ and $\mathbf{YY}^*$ respectively. Assume that the measures μ_n and ν_n weakly convergence to some measures μ and ν, $\mu_n \to \mu$ and $\nu_n \to \nu$.Then the spectral measure of matrix $\mathbf{XYY}^*\mathbf{X}^*$ convergence to some measure $\mu \boxtimes \nu$ and

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R-transform of matrix $\mathbf{J}(\alpha)$

Introduce the following $2n \times 2n$ block-matrix

$$\mathbf{J}(\alpha) = \begin{pmatrix} \mathbf{O} & -\alpha\mathbf{I} \\ -\overline{\alpha}\mathbf{I} & \mathbf{O} \end{pmatrix}, \tag{17}$$

where $\mathbf{O}$ is $n \times n$ matrix withe zero entries, and $\mathbf{I}$ denotes $n \times n$ unit matrix. This matrix has a spectral distribution $V(\cdot) = \frac{1}{2}\delta_{|\alpha|} + \frac{1}{2}\delta_{-|\alpha|}$, and δ_a denote the unit atom in the point a. We calculate now the R-transform of distribution $V(x)$.

R-transform of matrix $\mathbf{J}(\alpha)$

It is straightforward to check that generic moments function $M(z)$ of distribution $V(x)$ defined by equality

$$M(z) = \frac{|\alpha|^2 z^2}{1 - |\alpha|^2 z^2}.$$

From here it follows that

$$M^{-1}(z) = \frac{1}{|\alpha|}\sqrt{\frac{z}{1+z}}.$$

and

$$S(z) = \frac{1}{|\alpha|}\sqrt{\frac{1+z}{z}}.$$

R-transform of matrix $\mathbf{J}(\alpha)$

It is straightforward to check that generic moments function $M(z)$ of distribution $V(x)$ defined by equality

$$M(z) = \frac{|\alpha|^2 z^2}{1 - |\alpha|^2 z^2}.$$

From here it follows that

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Here and in the what follows we denote by f^{-1} inverse function with respect to composition. Using relation between S- and R-transforms, we get

$$R^{-1}(z) = zS(z) = \frac{\sqrt{z(1+z)}}{|\alpha|}.$$

From here it follows,

$$R^2(z) + R(z) - |\alpha|^2 z^2 = 0.$$

Solving this equation, we obtain

$$R(z) = \frac{-1 + \sqrt{1 + 4|\alpha|^2 z^2}}{2}.$$

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Equations for the Stieltjes transform of limit spectral of shifted matrices

Theorem 4.2

Assume that spectral measure of matrix $\mathbf{V}$ *has a limit* μ_V *and corresponding R-transform* $R_V(z)$*. Assume also that matrices* $\mathbf{V}$ *and* $\mathbf{J}(\alpha)$ *are asymptotically free. Then Stieltjes transform* $s(z,\alpha)$ *of expected spectral distribution of matrix satisfies the following system of equations*

$$w = z + \frac{R_\alpha(-s(z,\alpha))}{s(z,\alpha)} \tag{18}$$

$$s(z,\alpha) = (1 + ws(z,\alpha))S_V(-(1 + ws(z,\alpha)). \tag{19}$$

Density of probability distribution of eigenvalues

We compute the density of the limit measure of empirical spectral distribution of matrix $\mathbf{V_F}$. Let $\varkappa(x,\alpha) = -\sqrt{-1}s(\sqrt{-1}x,\alpha)$, where $x > 0$. We shall assume that distribution function $G_{\mathbf{F}}(x,\alpha)$ has the density with respect to Lebesgue measure, $g(x,\alpha) = \frac{dG_{\mathbf{F}}(x,\alpha)}{dx}$. Shall assume as well that

$$\lim_{C\to\infty} \frac{\partial}{\partial u} \int_{-\infty}^{\infty} \log\left(1 + \frac{u^2}{C^2}\right) g(u,\alpha)du = 0 \qquad (20)$$

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Theorem 4.3

Under assumption of asymptotic freeness of matrices $\mathbf{V}$ *and* $\mathbf{J}(z)$ *we have*

$$p(u,v) = \frac{1}{2\pi}\Delta V(\alpha) = -\frac{i}{2\pi|\alpha|^2}\left(u\frac{\partial t}{\partial u} + v\frac{\partial t}{\partial v}\right), \qquad (21)$$

where function $t = t(z,\alpha)$ *satisfies the following system of equations*

$$t(1+it) = i|\alpha|^2\varkappa^2,$$

$$t = |\alpha|^2\varkappa S_V(-(1+it)). \qquad (22)$$

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Circular law

In this Section we give several examples of investigation of limit distribution. We start from simplest model of Girko–Ginibre matrix.

Let $\mathbf{X}$ be an $n \times n$ random matrix with independent entries X_{jk} such that $\mathrm{E}\, X_{jk} = 0$ and $\mathrm{E}\, |X_{jk}|^2 = 1$. First we must check the conditions of Theorem 2.1. Note that in this case $\mathbb{F} = \mathbf{I}$ and

$$g_{jk} = \begin{cases} 2[\mathbf{R}^2]_{jk}, & \text{for } j \neq k \\ [\mathbf{R}^2]_{jj}, & \text{otherwise.} \end{cases} \tag{23}$$

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It is straightforward to check that

$$|\frac{\partial g_{jk}}{\partial Z_{jk}}| \leq Cv^{-3}, |\frac{\partial^2 g_{jk}}{\partial Z_{jk}^2}| \leq Cv^{-4}, \tag{24}$$

for $z = u + iv$. Thus the conditions of Theorem 2.1 are hold.

Furthermore, to prove the uniform integration of the function $\log x$ with respect to singular value distribution of matrices $\mathbf{X}$ we may use the following results.

Theorem 5.1

Let X_{jk} be independent random variables with $\mathrm{E}\,X_{jk} = 0$ and $\mathrm{E}\,|X_{jk}|^2 = 1$. Assume that square of random variables X_{jk} are uniformly integrated., i.e.

$$\sup_{j,k,n} \mathrm{E}\,|X_{jk}|^2 \mathbb{I}\{|X_{jk}| > M\} \to 0 \quad as \quad M \to \infty. \tag{25}$$

then there exist positive constant $A > 0$ and $B > 0$ such that

Theorem 5.2

Under conditions of Theorem 5.1 there exist a constant $0<\gamma_0<1$ *and constant* $c>0$ *such that*

$$\Pr\{s_{n-k}(z)\ge c\sqrt{\frac{k}{n}},\quad \textit{for}\quad n-1\ge k\ge n^{\gamma_0}\}\ge 1-c_1\exp\{-c_2n\}. \tag{27}$$

The proof of this Theorem is given in [2] or in [**?**]. Theorem 5.1 and 5.2 allows us to prove the uniform inegration of $\log x$ with respect to singular values distribution of matrices $\mathbf{X}-z\mathbf{I}$.

We may assume now that X_{jk} are Gaussians and all moments are finite, $\mathrm{E}\,|X_{jk}|^p \le C_p < \infty$. Let

$$\mathbf{V} = \begin{bmatrix} \mathbf{O} & \frac{1}{\sqrt{n}}\mathbf{X} \\ \frac{1}{\sqrt{n}}\mathbf{X}^* & \mathbf{O} \end{bmatrix},$$

where **O** denotes matrix with zero entries. First we check that matrices **V** and $\mathbf{J}(\alpha)$ are asymptotic free.

Lemma 5.1

Let $\mathbf{X}$ be random matrices of dimension $n \times n$. Let the entries of these matrices are independent standard complex Gaussian random variables. Then random matrices
$\mathbf{V} = \begin{bmatrix} \mathbf{O} & \mathbf{X} \\ \mathbf{X}^* & \mathbf{O} \end{bmatrix}$ *are asymptotically free.*

The limit distribution for spectral distribution function of matrix $\mathbf{V}$ is semi-circular law. According to definition, we may take

$$S_V(z) = -\frac{1}{\sqrt{z}}.$$

Applying now equations (22), we get

$$t = \begin{cases} i|\alpha|^2 & u^2+v^2 \le 1 \\ 0, & u^2+v^2 > 1 \end{cases} \tag{28}$$

$$\varkappa = \begin{cases} \sqrt{1-|\alpha|^2}, & u^2+v^2 \le 1 \\ 0, & u^2+v^2 > 1 \end{cases} \tag{29}$$

It is straightforward to check that

$$u\frac{\partial t}{\partial u} + v\frac{\partial t}{\partial v} = \begin{cases} 0, & u^2+v^2 > 1 \\ 2i|\alpha|^2, & u^2+v^2 \le 1 \end{cases} \tag{30}$$

Equality (**??**) immediately implies that, for $x^2 + y^2 > 1$

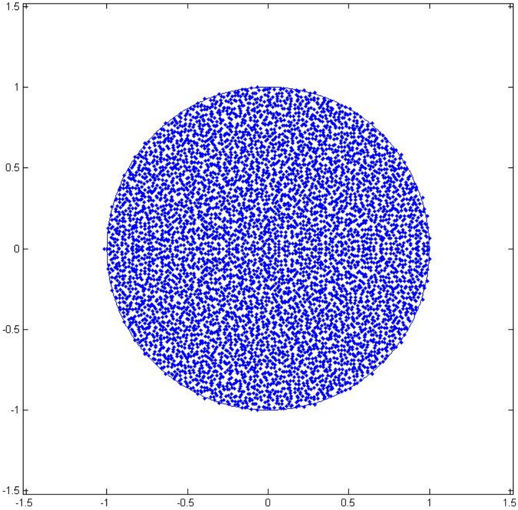
$$\Delta V(\alpha) = 0.$$

If $x^2 + y^2 \le 1$, we have

$$\Delta V(\alpha) = 2 \tag{31}$$

From the last two equalities it follows that spectral density $p(x, y)$ of the limit empirical spectral measure of matrix $\mathbf{X}$ is defined by equality

$$p(x, y) = \begin{cases} \frac{1}{\pi}, & x^2 + y^2 \le 1, \\ 0, & x^2 + y^2 > 1. \end{cases}$$


1.5
1
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-1
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1
1.5

Product of independent random matrices

Let $m \geq 1$. Consider independent random matrices $\mathbf{X}^{(q)}$, $q = 1, \ldots, m$ with independent entries $X_{jk}^{(q)}$, $1 \leq j, k \leq n$, $q = 1, \ldots, m$. Let $\mathbf{W} = n^{-\frac{m}{2}} \prod_{q=1}^{m} (\mathbf{X}^{(q)})^{k_q}$, for $k_1, \ldots, k_q \geq 1$ and $k_1 + \ldots + k_q = k$, and

$$\mathbf{J}(\alpha) = \begin{bmatrix} \mathbf{O} & \alpha\mathbf{I} \\ \overline{\alpha}\mathbf{I} & \mathbf{O} \end{bmatrix}, \quad \mathbf{V} = \begin{bmatrix} \mathbf{W} & \mathbf{O} \\ \mathbf{O} & \mathbf{W}^* \end{bmatrix}, \quad \mathbf{V}(\alpha) = \mathbf{V}\mathbf{J}(1). \tag{32}$$

To define the limit eigenvalue distribution of matrix $\mathbf{V}(\alpha)$ we may consider the Gaussian matrices only. Since Gaussian matrices $\mathbf{X}^{(q)}$ are asymptotic free, for $q = 1, \ldots, m$ we find the S-transform of limit singular values distribution of matrix $\mathbf{W}$. Since all matrices are square matrices S-transform of limit distribution of matrix $\mathbf{W}$ is product of S-transforms of Marchenko–Pastur distribution with parameter $y = 1$. This implies

$$S_{\mathrm{W}}(z) = \frac{1}{(1+z)^k}. \tag{33}$$

Furthermore, the limit spectral distribution of matrix **V** is symmetrization of limit distribution of singular values of matrix **W**. According Theorem 6 in Octavia Arizmendi E. and Victor Perez–Abreu *The S-transfom of Symmetric Probability Measures with unbounded supports.* Communication del CIMAT, 2008, we have

$$S_{\mathbf{V}}^2(z) = \frac{1+z}{z} S_{\mathbf{W}}. \tag{34}$$

This implies immediately that

$$S_v(z) = -\frac{1}{\sqrt{z}(1+z)^{\frac{k-1}{2}}}. \tag{35}$$

We rewrite equations (22) for this cases

$$
\begin{aligned}
t(1+it) &= i|\alpha|^2\varkappa^2, \\
t\sqrt{1+it} &= i|\alpha|^2\varkappa(-it)^{-\frac{k-1}{2}}.
\end{aligned}
\tag{36}
$$

Solving this system we find that

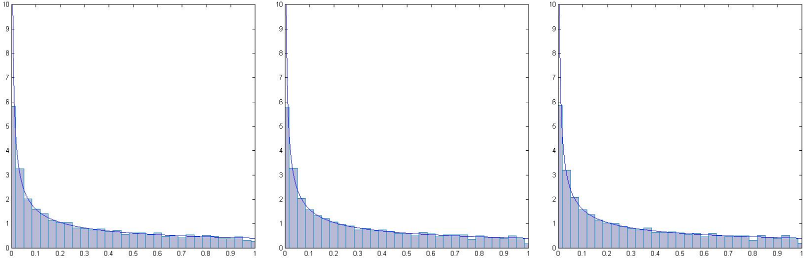
$$(-it)^m = \begin{cases} 0, & u^2+v^2>1 \\ |\alpha|^2 \varkappa S_{\mathbf{V}}(-(1+it)) & u^2+v^2 \le 1 \end{cases} \tag{37}$$

and, for $u^2+v^2 \le 1$,

$$u\frac{\partial t}{\partial u} + v\frac{\partial t}{\partial v} = \frac{2i|\alpha|^2}{k(-it)^{k-1}} = \frac{2i|\alpha|^{\frac{2}{k}}}{k}. \tag{38}$$

These relations immediately imply that

$$p(x,y) = \begin{cases} \frac{1}{\pi k (u^2+v^2)^{\frac{k-1}{k}}}, & x^2+y^2 \le 1, \\ 0, & x^2+y^2>1. \end{cases} \tag{39}$$



(a) X^5 (b) $X^2 Y^3$ (c) $YXYXY$

Figure: Histograms of the eigenvalues radial projection, $n = 5000$.

Product of rectangular matrices

Let $m \geq 1$ be fixed. Let for any $n \geq 1$ are given integer $n_0 = n, n_1 \geq n, \ldots, n_{m-1} \geq n$ and $n_m = n$. Assume that $y_q = \lim_{n\to\infty} \frac{n}{n_q} \in (0, 1]$, $q = 1, \ldots, m$. Note that $p_m = 1$. Consider independent random matrices $\mathbf{X}^{(q)}$ of order $n_{q-1} \times n_q$, $q = 1, \ldots, m$. Put $\mathbf{W} = \prod_{q=1}^{m} \frac{1}{\sqrt{n_{q-1}}}\mathbf{X}^{(q)}$ and let

$$\mathbf{V} = \begin{bmatrix} \mathbf{O} & \mathbf{W} \\ \mathbf{W}^* & \mathbf{O} \end{bmatrix}$$

The proof of universality of singular value distribution of product of rectangular matrices is similar to one for product of square matrices. Moreover, bounds for minimal singular values are similar to bounds of minimal singular values of product square matrices. Using results of Section 1 and relation (34), we may shown that for Gaussian matrtices the Stieltjes transform of limit distribution of singular values distribution is defined by formula

$$S_{\mathbf{V}}(z) = -\frac{1}{\sqrt{z}} \prod_{q=1}^{m-1} \frac{1}{\sqrt{1 + y_q z}}.$$

Putting it in (18), we get

$$t(1+it) = i|\alpha|^2 \varkappa^2$$

$$t\sqrt{1+it} = i|\alpha|^2 \prod_{q=1}^{m-1} \frac{1}{1-y_q-iy_q t}. \tag{40}$$

Solving this system, we obtain

$$-it \prod_{q=1}^{m-1} (1-y_q-y_q it) = |\alpha|^2. \tag{41}$$

For $m=2$ and $u^2+v^2\le 1$, we have

$$-it(1-y_1-ity_1)=|\alpha|^2 \tag{42}$$

and

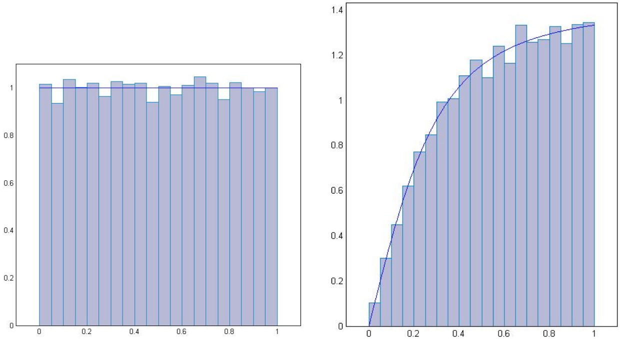
$$t=\frac{-(1-y_1)+\sqrt{(1-y_1)^2+4|\alpha|^2y_1}}{2y_1}.$$

The last relation implies that

$$u\frac{\partial t}{\partial u}+v\frac{\partial t}{\partial v}=\frac{2i}{\sqrt{(1-y_1)^2+4|\alpha|^2y_1}}.$$

Finally, we obtain

$$p(u,v)=\frac{1}{\pi\sqrt{(1-y_1)^2+4(u^2+v^2)y_1}}I\{u^2+v^2\le 1\}.$$



(a) $y = 1$

$n = 5000, p = 5000$

(b) $y = 0.5$

$n = 5000, p = 10000$

Figure: The eigenvalues radial projection histogram of the product of two rectangular matrices of sizes $n \times p$, $p \times n$.

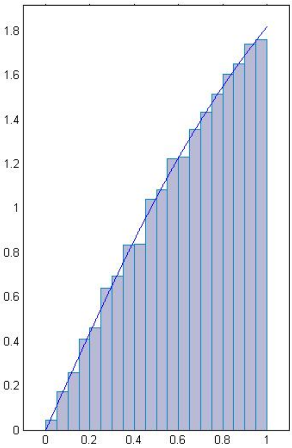


Figure: The eigenvalues radial projection histogram of the product of two rectangular matrices of sizes 4000×40000, 40000×4000. $y = 0.1$.

Eigenvalue distribution of matrix $(\mathbf{XX}^*)^{-\frac{1}{2}}\mathbf{Y}$

Let $\mathbf{X}$ and $\mathbf{Y}$ be independent $n \times n$ random matrices with independent entries. Consider matrix $W = (\mathbf{XX}^*)^{-\frac{1}{2}}\mathbf{Y}$. First, find S-transform $S(z)$ of matrix $(\mathbf{XX}^*)^{-1}$. Note that matrix $\mathbf{XX}^*$ has in the limit Marchenko-Pastur distribution and its S-transform $\widetilde{S}(z) = \frac{1}{z+1}$. Corresponding Stieltjes transform is $g(z) = \frac{-1+\sqrt{\frac{z-4}{z}}}{2}$. Furthermore, we note that formally $M(z) = zg(z)$, where $M(z)$ denotes the generating moment function of spectral distribution of matrix $(\mathbf{XX}^*)^{-1}$.

This implies

$$M(z) = \frac{-z + \sqrt{z(z-4)}}{2}. \tag{43}$$

From this equality it follows that

$$M^{-1}(z) = \frac{-z^2}{1+z} \tag{44}$$

and

$$S_{(XX^*)^{-1}}(z) = -z. \tag{45}$$

By multiplicative property,using that S-transform $S_Y(z)$ of matrix $\mathbf{YY}^*$ is $S_Y(z) = \frac{1}{z+1}$, we obtain that S-transform $S_W(z)$ of matrix $\mathbf{WW}^*$ is

$$S_W(z) = -\frac{z}{z+1} \tag{46}$$

and

$$S_V(z) = i. \tag{47}$$

Solving now the system

$$t(1+it) = i|\alpha|^2\varkappa^2 \tag{48}$$

$$t = i|\alpha|^2\varkappa, \tag{49}$$

we find

$$t = \frac{i|\alpha|^2}{1+|\alpha|^2} \tag{50}$$

and

$$u\frac{\partial t}{\partial u}+v\frac{\partial t}{\partial v}=\frac{2i|\alpha|^2}{(1+|\alpha|^2)^2}. \tag{51}$$

The last equality and equality (refdensity together imply

$$p(u,v)=\frac{1}{\pi(1+(u^2+v^2))^2} \tag{52}$$

We find as well the density of limit distribution of singular values of matrix $\mathbf{X}(\mathbf{YY}^*)^{-1}$.

$$f(u) = \frac{1}{\pi\sqrt{u}(1+\sqrt{u})}, \quad u \geq 0. \tag{53}$$

For radial proection we have

$$p(r) = \frac{2r}{(1+r^2)^2} \tag{54}$$

We find as well the density of limit distribution of singular values of matrix $\mathbf{X}(\mathbf{Y}\mathbf{Y}^*)^{-1}$.

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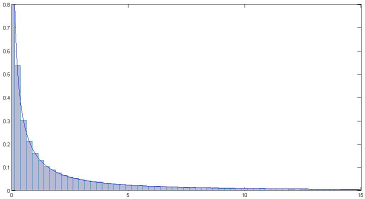
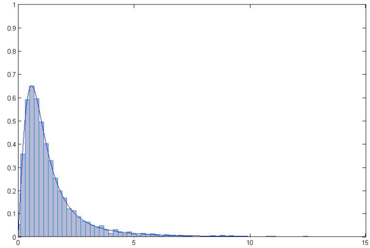


(a) $m = 1$

Figure: The squared singular values histogram of the product $X(YY^*)^{-1/2}$, $n = 5000$.



(a) $m = 1$

Figure: The eigenvalues radial projection histogram of the product $X(YY^*)^{-1/2}$, $n = 5000$.

Eigenvalue distribution of matrix $\prod_{q=1}^m(\mathbf{X}^{(q)}\mathbf{X}^{(q)*})^{-\frac12}\mathbf{Y}^{(q)}$

Let for $m \geq 1$ given n-by-n random matrices $\mathbf{X}^{(q)}$ and $\mathbf{Y}^{(q)}$. Let all matrices be independent and have independent entries. Consider matrix $\mathbf{W} = \prod_{q=1}^m(\mathbf{X}^{(q)}\mathbf{X}^{(q)*})^{-\frac12}\mathbf{Y}^{(q)}$. First, find S-transform $S_{\mathbf{W}}(z)$ of matrix $\mathbf{W}\mathbf{W}^*$. Note that, for any $\nu = 1, \ldots, m$, matrix $\mathbf{X}^{(q)}\mathbf{X}^{(q)*}\mathbf{Y}^{(q)}$ has S-transform $\widetilde{S}(z) = -\frac{z}{z+1}$. By multiplicative property of S-transform, we have

$$S_{\mathbf{W}}(z) = (-\frac{z}{z+1})^m.$$

From here it follows that

Solving now the system

$$t(1+it) = i|\alpha|^2\varkappa^2 \tag{55}$$

$$t = i^{\frac{m+1}{2}}|\alpha|^2\varkappa\left(\frac{1+it}{t}\right)^{\frac{m-1}{2}} \tag{56}$$

we find

$$t = \frac{i|\alpha|^{\frac{2}{m}}}{1+|\alpha|^{\frac{2}{m}}} \tag{57}$$

and

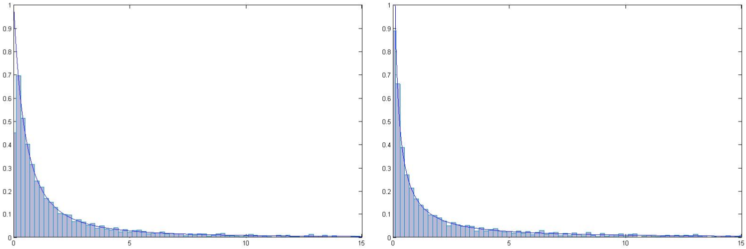
$$u\frac{\partial t}{\partial u} + v\frac{\partial t}{\partial v} = \frac{2i|\alpha|^{\frac{2}{m}}}{m(1+|\alpha|^{\frac{2}{m}})^2}. \tag{58}$$

The last equality and equality (21) together imply

$$p(u,v)=\frac{1}{\pi m(u^2+v^2)^{\frac{m-1}{m}}(1+(u^2+v^2)^{\frac{1}{m}})^2} \tag{59}$$

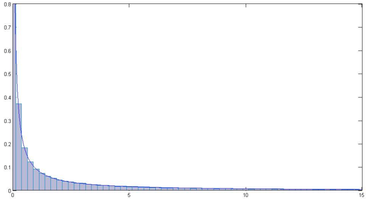
The limit singular value distribution has the density

$$p(u)=\frac{\sin(\frac{\pi}{m+1})}{u^{\frac{m}{m+1}}((u^{\frac{1}{m+1}}+\cos(\frac{\pi}{m+1}))^2+\sin^2(\frac{\pi}{m+1})}. \tag{60}$$

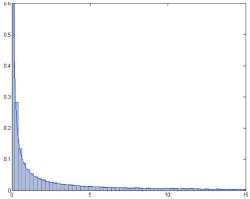


(a) $m = 2$ (b) $m = 3$

Figure: The eigenvalues radial projection histogram of the product $\prod_{k=1}^{m} X_k(Y_k Y_k{}^*)^{-1/2}$, $n = 5000$.



(a) $m = 2$



(b) $m = 3$

Figure: The squared singular values histogram of the product $\prod_{k=1}^{m} X_k(Y_k Y_k{}^*)^{-1/2}$, $n = 5000$.

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KARELIA
Northwest Federal District
ARKHANGEL'SK
KOMI
PECHORSKIY
BASSEYN
VOLOGDA
KIROV
PERM
KOSTROMA
UDMURTIA
TATARSTAN
BASHKORTOSTAN
Arkhangelsk
Petrozavodsk
Helsinki
Tallinn
Riga
Minsk
Vologda
Kotlas
Kirov
Perm'
Kazan'
Nizhniy Novgorod
Yaroslavl'
Ufa
Samara
Novgorod
Pskov
Tver'
Smolensk
Kaluga
Tula
Ryazan'
Vladimir
Ivanovo
Kostroma
Cherepovets
Rybinsk
Izhevsk
Cheboksary
Saransk
Ul'yanovsk
Syzran'
Tampere
Turku
Vyborg
Tikhvin
Berezniki
Glazov
Kungur
Yoshkar-Ola
Dzerzhinsk
Murom
Kolomna
Serpukhov
Novomoskovsk
Bryansk
Vitsyebsk
Sterlitamak
Dimitrovgrad
Tol'yatti
Naberezhnyye Chelny

Thank you for your attention!