Analysis of a Combinatorial Optimization Problem in Wireless Communications

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joint work with
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The Problem in a Nutshell

Let

$$E \equiv \frac{1}{K} \min_{\boldsymbol{x} \in \mathcal{X}} \boldsymbol{x}^\dagger \boldsymbol{J} \boldsymbol{x}$$

with $\boldsymbol{x} \in \mathbb{C}^K$ and $\boldsymbol{J} \in \mathbb{C}^{K \times K}$.

The Problem in a Nutshell

Let

$$\color{red}{E} \equiv \frac{1}{K} \min_{\boldsymbol{x} \in \color{blue}{\mathcal{X}}} \boldsymbol{x}^\dagger \boldsymbol{J} \boldsymbol{x}$$

with $\boldsymbol{x} \in \mathbb{C}^K$ and $\boldsymbol{J} \in \mathbb{C}^{K \times K}$.

- Sphere:

$$\color{blue}{\mathcal{X}} = \{\boldsymbol{x} : \boldsymbol{x}^\dagger \boldsymbol{x} = K\} \quad \Longrightarrow \quad \color{red}{E = \min \lambda(\boldsymbol{J})}$$

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$$\mathcal{X} = \{+1, -1\}^K \quad \Longrightarrow \quad ???$$

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$$\mathcal{X} = \{+1, -1\}^K \quad \Longrightarrow \quad ???$$

- Vector precoding:

$$\mathcal{X} = (4\mathbb{Z} + 1)^K \quad \Longrightarrow \quad ???$$

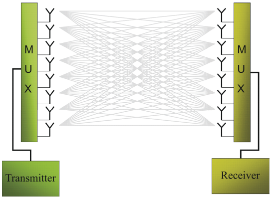
The Gaussian Vector Channel

Let the received vector be given by

$$\boldsymbol{y} = \boldsymbol{H}\boldsymbol{t} + \boldsymbol{n}$$

where

- $\boldsymbol{t}$ is the transmitted vector
- $\boldsymbol{n}$ is uncorrelated (white) Gaussian noise
- $\boldsymbol{H}$ is a coupling matrix accounting for crosstalk



Data rate scales linearly with the minimum of the number of antenna elements.

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Crosstalk can be processed either at receiver or transmitter

Processing at Transmitter

- If the transmitter is a base-station and the receiver is a hand-held device, processing at the transmitter is preferred.
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E.g. let the transmitted vector be

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where $\boldsymbol{x} = \boldsymbol{s}$ is the data to be sent.

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Then,

$$\begin{aligned} \boldsymbol{y} &= \boldsymbol{H}\boldsymbol{t} + \boldsymbol{n}. \\ &= \boldsymbol{H}\boldsymbol{H}^{\dagger}(\boldsymbol{H}\boldsymbol{H}^{\dagger})^{-1}\boldsymbol{x} + \boldsymbol{n}. \\ &= \boldsymbol{s} + \boldsymbol{n}. \end{aligned}$$

No crosstalk anymore due to channel inversion.

Problems of Simple Channel Inversion

Channel inversion implies a significant power amplification, i.e.

$$\mathrm{E}\left\{\boldsymbol{t}^\dagger \boldsymbol{t}\right\} = \mathrm{E}\left\{\boldsymbol{x}^\dagger \left(\boldsymbol{H}\boldsymbol{H}^\dagger\right)^{-1} \boldsymbol{x}\right\} > \mathrm{E}\left\{\boldsymbol{x}^\dagger \boldsymbol{x}\right\} \qquad \text{even for} \quad \mathrm{E}\left\{\boldsymbol{H}\boldsymbol{H}^\dagger\right\} = \mathbf{I}$$

In particular, let

- the entries of $\boldsymbol{H} \in \mathbb{C}^{K \times N}$ be i.i.d. with variance $1/N$.
- $\alpha = \frac{K}{N} \leq 1$;

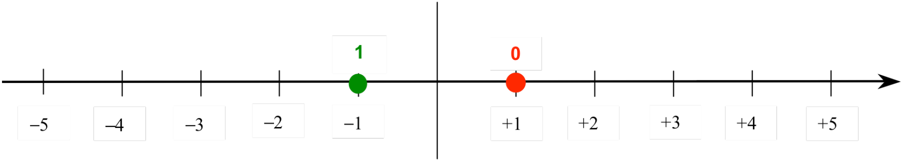
Then, for fixed aspect ratio α

$$\lim_{K \to \infty} \frac{\boldsymbol{x}^\dagger \left(\boldsymbol{H}\boldsymbol{H}^\dagger\right)^{-1} \boldsymbol{x}}{\boldsymbol{x}^\dagger \boldsymbol{x}} = \frac{1}{1-\alpha}$$

(with probability 1).

Vector Precoding

Lattice-based vector precoding



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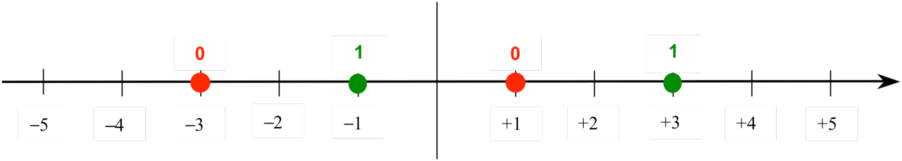

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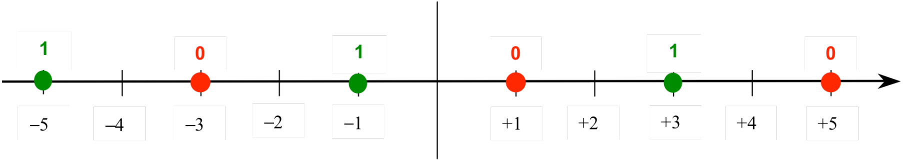
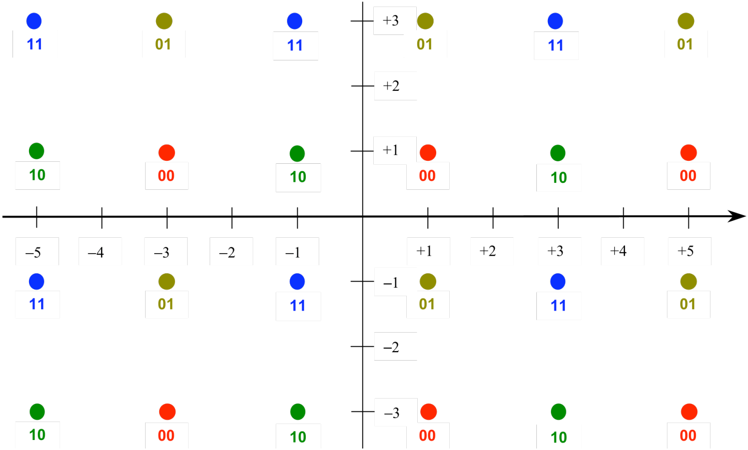


Instead of representing the logical "0" by $+1$, represent it by any element of the set $\{\dots, -7, -3, +1, +5, \dots\} = 4\mathbb{Z} + 1$. Correspondingly, the logical "1" is represented by any element of the set $4\mathbb{Z} - 1$.

Choose that representation that gives the smallest transmit power.

Lattice Relaxation of QPSK



General Vector Precoding

- Let $\mathcal{B}_0$ and $\mathcal{B}_1$ denote the sets presenting 0 and 1, resp.
- Let $(s_1, s_2, \ldots, s_K) \in \{0, 1\}^K$ denote the data to be transmitted.

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Then, the transmitted energy per data symbol is given by

$$E = \frac{1}{K} \min_{\boldsymbol{x} \in \mathcal{X}} \boldsymbol{x}^\dagger \boldsymbol{J} \boldsymbol{x}$$

with

$$\mathcal{X} = \mathcal{B}_{s_1} \times \mathcal{B}_{s_2} \times \cdots \times \mathcal{B}_{s_K}$$

and

$$\boldsymbol{J} = \left(\boldsymbol{H}\boldsymbol{H}^\dagger \right)^{-1}.$$

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What is a smart choice for $\mathcal{B}_0$ and $\mathcal{B}_1$?

Zero Temperature Formulation

Vector precoding is the problem of finding the zero temperature limit of a quadratic energy potential.

The transmitted power is written as a zero temperature limit

$$E = -\lim_{\beta\to\infty} \frac{1}{\beta K} \log \sum_{\boldsymbol{x}\in\mathcal{X}} \mathrm{e}^{-\beta\,\mathrm{tr}(\boldsymbol{x}^\dagger \boldsymbol{J}\boldsymbol{x})}$$

with $\frac{1}{\beta}$ denoting temperature.

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with $\frac{1}{\beta}$ denoting temperature.

The Harish-Chandra Integral (Itzykson-Zuber Integral)

Let $\boldsymbol{P}$ be any positive semi-definite matrix of bounded rank n and $\boldsymbol{J}$ be unitarily invariant and free of $\boldsymbol{P}$, then

$$\lim_{K\to\infty} \frac{1}{K} \log \mathop{\mathrm{E}}_{\boldsymbol{J}} \mathrm{e}^{-K \operatorname{tr} \boldsymbol{JP}} = -\sum_{a=1}^{n} \int_0^{\lambda_a} R_{\boldsymbol{J}}(-w)dw$$

with

- λ_a denoting the positive eigenvalues of $\boldsymbol{P}$ and
- $R_{\boldsymbol{J}}(w)$ denoting the R-transform of the spectral measure of $\boldsymbol{J}$.

(Marinari et al. '94; Guionnet, Maïda '05)

Examples of R-Transforms

For $\boldsymbol{H}$ with i.i.d. entries:

$$
\begin{aligned}
R_{\mathbf{I}}(w) &= 1 \\
R_{\boldsymbol{H}\boldsymbol{H}^\dagger}(w) &= \frac{1}{1-\alpha w} \\
R_{(\boldsymbol{H}\boldsymbol{H}^\dagger)^{-1}}(w) &= \frac{1-\alpha-\sqrt{(1-\alpha)^2-4\alpha w}}{2\alpha w}
\end{aligned}
$$

The Replica Method

We want

$$E = -\lim_{\beta\to\infty}\frac{1}{\beta}\lim_{K\to\infty}\frac{1}{K}\mathop{\mathrm{E}}_{\boldsymbol{J}}\log\sum_{\boldsymbol{x}\in\mathcal{X}}\mathrm{e}^{-\beta\,\mathrm{tr}(\boldsymbol{J}\boldsymbol{x}\boldsymbol{x}^\dagger)}.$$

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We know (Harish-Chandra integral)

$$\lim_{K\to\infty} \frac{1}{K} \log \mathop{\mathrm{E}}_{\boldsymbol{J}} \mathrm{e}^{-K\,\mathrm{tr}\,\boldsymbol{J}\boldsymbol{P}} = -\sum_{a=1}^{n} \int_0^{\lambda_a(\boldsymbol{P})} R_{\boldsymbol{J}}(-w)dw.$$

We would like to exchange expectation and logarithm.

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We would like to exchange expectation and logarithm.

$$\mathop{\mathrm{E}}_{X} \log X = \lim_{n\to 0} \frac{1}{n} \log \mathop{\mathrm{E}}_{X} X^n$$

The Replica Method cont'd

We want

$$\lim_{K\to\infty} \frac{1}{K} \mathop{\mathsf{E}}_{\boldsymbol{J}} \log \sum_{\boldsymbol{x}\in\mathcal{X}} \mathrm{e}^{-\beta \operatorname{tr}(\boldsymbol{J}\boldsymbol{x}\boldsymbol{x}^\dagger)} = \lim_{K\to\infty} \lim_{n\to 0} \frac{1}{nK} \log \mathop{\mathsf{E}}_{\boldsymbol{J}} \left(\sum_{\boldsymbol{x}\in\mathcal{X}} \mathrm{e}^{-\beta \operatorname{tr}(\boldsymbol{J}\boldsymbol{x}\boldsymbol{x}^\dagger)} \right)^n$$

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$$= \lim_{K\to\infty} \lim_{n\to 0} \frac{1}{nK} \log \mathop{\mathsf{E}}_{\boldsymbol{J}} \prod_{a=1}^{n} \sum_{\boldsymbol{x}_a\in\mathcal{X}} \mathrm{e}^{-\beta\,\mathrm{tr}\left(\boldsymbol{J}\boldsymbol{x}_a\boldsymbol{x}_a^\dagger\right)}$$

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$$= \lim_{K\to\infty}\lim_{n\to 0}\frac{1}{nK}\log\mathop{\mathrm{E}}_{\boldsymbol{Q}}\exp\left[-K\sum_{a=1}^{n}\int_0^{\beta\lambda_a(\boldsymbol{Q})}R_{\boldsymbol{J}}(-w)dw\right]$$

with

$$Q_{ab}\equiv\frac{1}{K}\boldsymbol{x}_a^\dagger\boldsymbol{x}_b.$$

Laplace Integration

We find

$$\lim_{K\to\infty}\frac{1}{K}\mathop{\mathrm{E}}_{\boldsymbol{J}}\log\sum_{\boldsymbol{x}\in\mathcal{X}}\mathrm{e}^{-\beta\,\mathrm{tr}(\boldsymbol{J}\boldsymbol{x}\boldsymbol{x}^\dagger)}=\lim_{K\to\infty}\lim_{n\to0}\frac{1}{nK}\log\mathop{\mathrm{E}}_{\boldsymbol{Q}}\exp\left[-K\sum_{a=1}^{n}\int_0^{\beta\lambda_a(\boldsymbol{Q})}R_{\boldsymbol{J}}(-w)dw\right]$$

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$$\rightsquigarrow \lim_{n\to 0} \frac{1}{n} \min_{\boldsymbol{Q}} \mathrm{tr}\left[\boldsymbol{Q} R_{\boldsymbol{J}}(-\beta\boldsymbol{Q})\right].$$

How to optimize over the $n \times n$ matrix $\boldsymbol{Q}$ for $n \to 0$?

Replica Symmetry (RS)

We try the following ansatz

$$\boldsymbol{Q} \equiv \begin{bmatrix} q+\frac{\chi}{\beta} & q & \cdots & q & q \\ q & q+\frac{\chi}{\beta} & \ddots & q & q \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ q & q & \ddots & q+\frac{\chi}{\beta} & q \\ q & q & \cdots & q & q+\frac{\chi}{\beta} \end{bmatrix}$$

with some macroscopic parameters q and χ.

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This is a critical step. In some cases, the structure of $\boldsymbol{Q}$ is more complicated. We try this structure first.

RS Solution

Let q and χ be the simultaneous solutions to

$$q = \mathop{\mathsf{E}}_{s,z} \mathop{\mathrm{argmin}}_{x \in \mathcal{B}_s}{}^2 \left| z\sqrt{2qR_{\boldsymbol{J}}'(-\chi)} - 2xR_{\boldsymbol{J}}(-\chi) \right|$$

$$\chi = \frac{1}{\sqrt{2qR_{\boldsymbol{J}}'(-\chi)}} \mathop{\mathsf{E}}_{s,z} \Re \left\{ z^* \mathop{\mathrm{argmin}}_{x \in \mathcal{B}_s} \left| z\sqrt{2qR_{\boldsymbol{J}}'(-\chi)} - 2xR_{\boldsymbol{J}}(-\chi) \right| \right\}$$

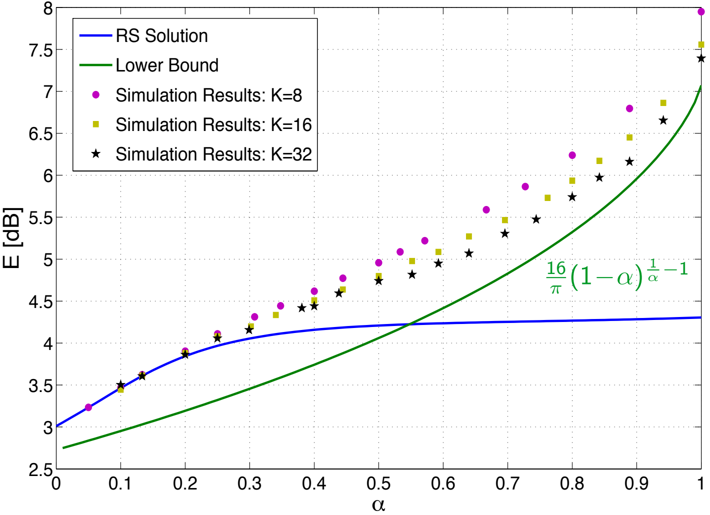
where z is a unit variance zero-mean Gaussian random variable.

Then, replica symmetry (RS) implies

$$E = \frac{1}{K} \min_{\boldsymbol{x} \in \mathcal{X}} \boldsymbol{x}^\dagger \boldsymbol{J} \boldsymbol{x} \to q \frac{\partial}{\partial \chi} \chi R_{\boldsymbol{J}}(-\chi)$$

as $K \to \infty$.

Complex Lattice Precoding



1-Step Replica Symmetry Breaking

$$\boldsymbol{Q} \equiv \left[\begin{array}{ccccccc} \multicolumn{2}{c}{\overbrace{\begin{array}{cc} q+p+\frac{\chi}{\beta} & q+p \end{array}}^{\frac{\mu}{\beta}\text{ columns}}} & q & q & \cdots & q & q \\ q+p & q+p+\frac{\chi}{\beta} & q & q & \cdots & q & q \\ q & q & q+p+\frac{\chi}{\beta} & q+p & \ddots & q & q \\ q & q & q+p & q+p+\frac{\chi}{\beta} & & \vdots & \vdots \\ \vdots & \vdots & \ddots & & \ddots & q & q \\ q & q & q & \cdots & q & q+p+\frac{\chi}{\beta} & q+p \\ q & q & q & \cdots & q & q+p & q+p+\frac{\chi}{\beta} \end{array} \right]$$

with the macroscopic parameters q, p and χ and the blocksize $\frac{\mu}{\beta}$.

1-Step Replica Symmetry Breaking

$$E = \frac{1}{K} \min_{\boldsymbol{x} \in \mathcal{X}} \boldsymbol{x}^\dagger \boldsymbol{J} \boldsymbol{x}$$
$$\to \left(q + p + \frac{\chi}{\mu} \right) R_{\boldsymbol{J}}(-\chi - \mu p) - \frac{\chi}{\mu} R_{\boldsymbol{J}}(-\chi) - q(\mu p + \chi) R'_{\boldsymbol{J}}(-\chi - \mu p)$$

The macroscopic parameters q, p, χ and μ are given by 4 coupled non-linear equations (omited here).

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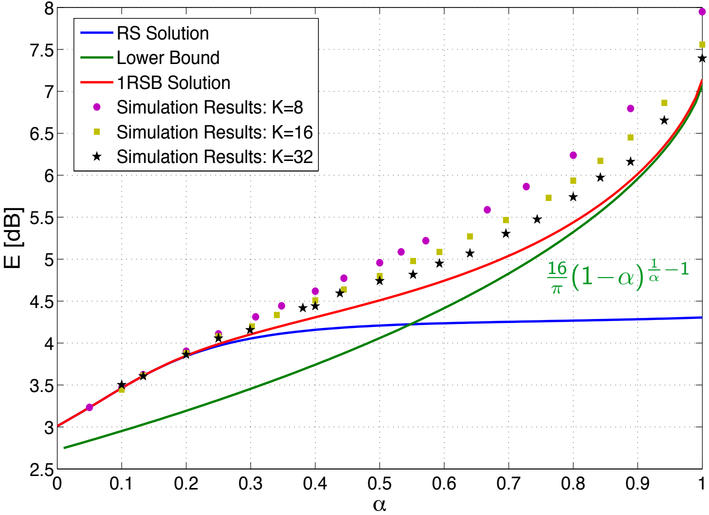

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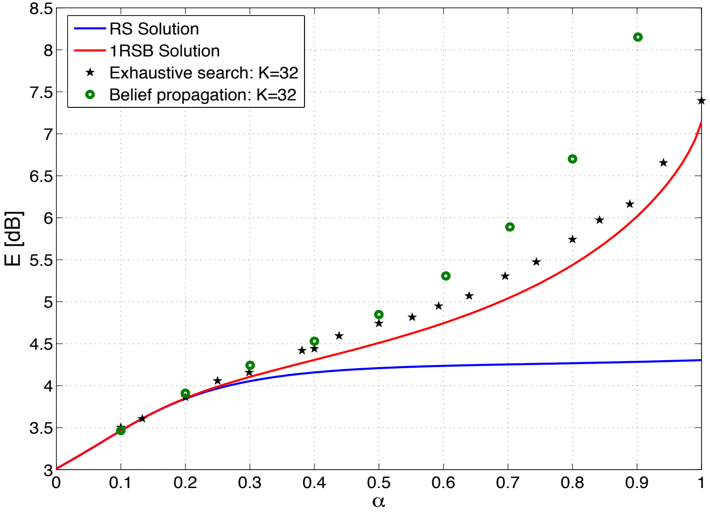
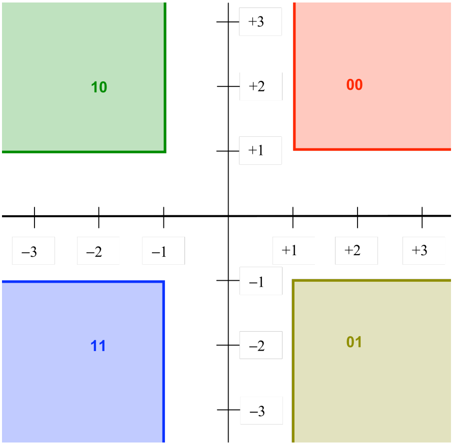
RS (breaking) ranks the difficulty of approximating an NP-hard problem, in practice.

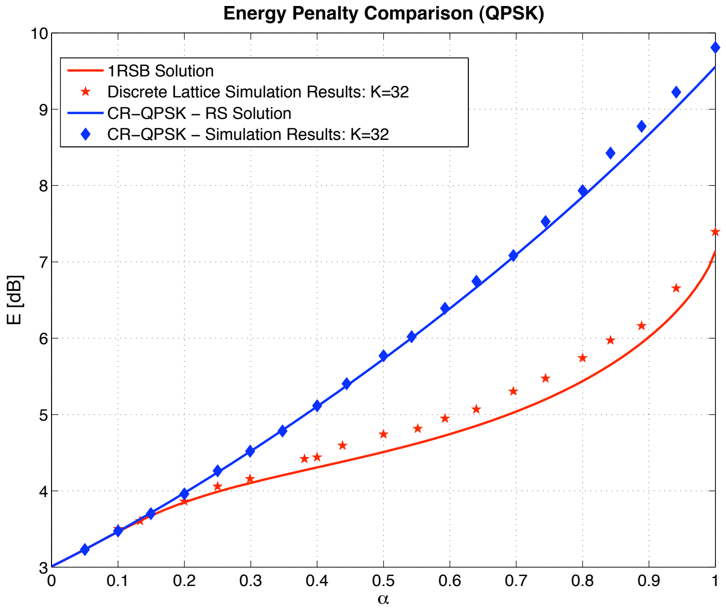
Belief Propagation vs. Exhaustive Search



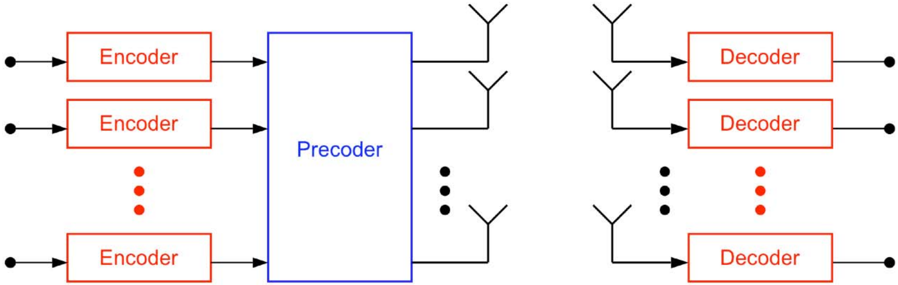
Convex Relaxation



. . . allows for convex programming.


Energy Penalty Comparison (QPSK)
1RSB Solution
Discrete Lattice Simulation Results: K=32
CR–QPSK – RS Solution
CR–QPSK – Simulation Results: K=32
E [dB]
α
10
9
8
7
6
5
4
3
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

Outer Single User Coding



$$I(s_k, y_k) = h(y_k) - h(y_k | s_k)$$

Spectral efficiency (per transmit antenna) is given by $C = \frac{1}{N} \sum_{k=1}^{K} I(s_k, y_k)$

Conditional Channel Output Distribution $P(y|s)$

We characterize the joint distribution of all relevant variables in the channel.

$$p(s,x,y) = p(s)p(x|s)p(y|x,s)$$

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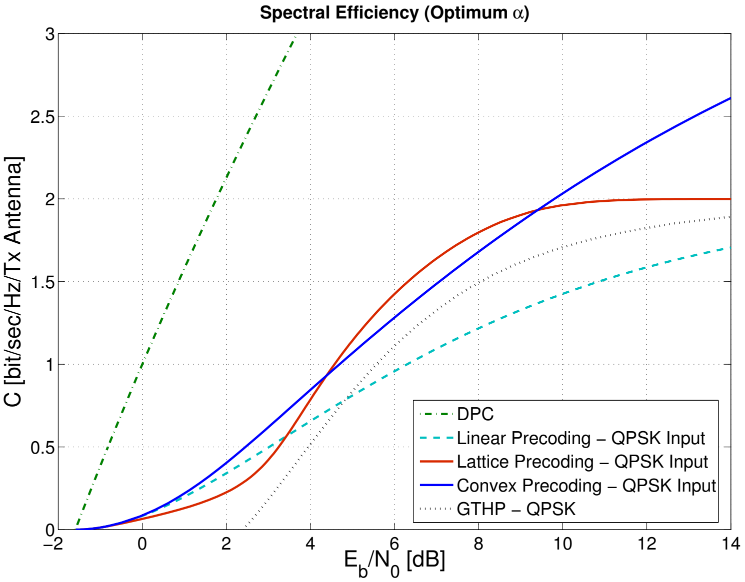
- The channel $p(x|s)$ is found by the replica method via

$$P(x,s) = \lim_{h\to 0} \frac{\partial}{\partial h} \lim_{K\to\infty} \lim_{\beta\to\infty} \mathop{\mathrm{E}}_{\boldsymbol{J}} \frac{1}{\beta K} \log \sum_{\boldsymbol{x}\in\mathcal{X}} \mathrm{e}^{-\beta \boldsymbol{x}^\dagger \boldsymbol{J}\boldsymbol{x} + \beta h \sum\limits_{k=1}^{K} 1\{(x_k,s_k)=(x,s)\}}$$


Spectral Efficiency (Optimum α)
C [bit/sec/Hz/Tx Antenna]
E_b/N_0 [dB]
DPC
Linear Precoding – QPSK Input
Lattice Precoding – QPSK Input
Convex Precoding – QPSK Input
GTHP – QPSK
0
0.5
1
1.5
2
2.5
3
–2
0
2
4
6
8
10
12
14

Inverting Singular Channels

What happens if the channel is rank-deficient, e.g. $K > N$?

Can we precode without interference?

Inverting Singular Channels

What happens if the channel is rank-deficient, e.g. $K > N$?

Can we precode without interference?

- The precoder produces

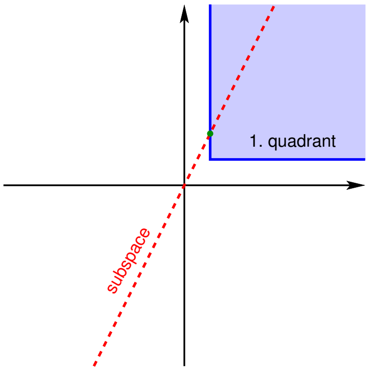
$$\lim_{\epsilon \to 0} \operatorname*{argmin}_{\boldsymbol{x} \in \mathcal{X}} \frac{\boldsymbol{x}^\dagger (\boldsymbol{H}\boldsymbol{H}^\dagger + \epsilon \mathbf{I})^{-1} \boldsymbol{x}}{K}$$

- The received signal becomes

$$\boldsymbol{y} = \lim_{\epsilon \to 0} \boldsymbol{H}\boldsymbol{H}^\dagger (\boldsymbol{H}\boldsymbol{H}^\dagger + \epsilon \mathbf{I})^{-1} \boldsymbol{x} + \boldsymbol{n}.$$

If the energy is finite, there is no interference.

Overloaded Convex Precoding



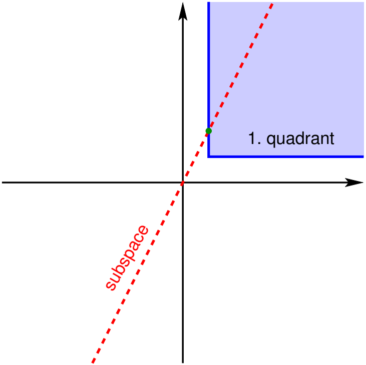
The probability that a random N dimensional subspace in K real dimensions intersects the 1. K-tant is (Wendel '62)

$$P(K, N) = 2^{1-K} \sum_{\ell=0}^{N-1} \binom{K-1}{\ell}$$

As K, N to infinity, we get

$$P(K, N) = \begin{cases} 1 & K < 2N \\ 1/2 & K = 2N \\ 0 & K > 2N \end{cases}$$

Overloaded Convex Precoding



The probability that a random N dimensional subspace in K complex dimensions intersects the 1. K-tant is

$$P(K,N) = 2^{1-2K} \sum_{\ell=0}^{2N-1} \binom{2K-1}{\ell}$$

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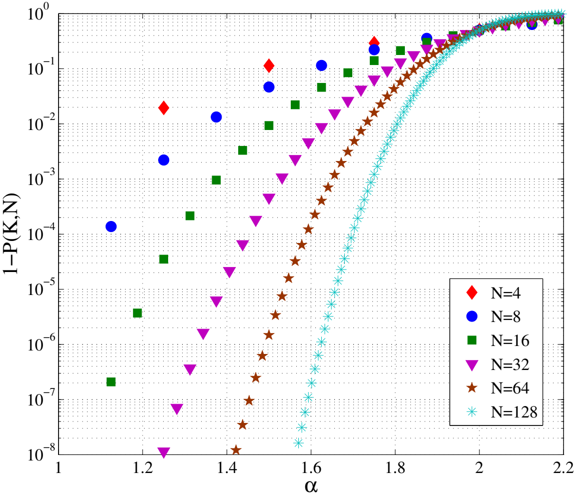
Overloaded Convex Precoding



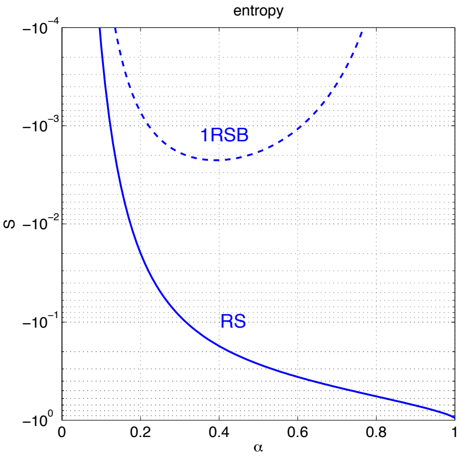
Wanted

More general versions of the Harish-Chandra integral, e.g.

- no unitary invariance, only freeness required
- $$\lim_{K\to\infty} \frac{1}{K} \log \operatorname*{E}_{\boldsymbol{A},\boldsymbol{B}} \mathrm{e}^{-K\mathrm{tr}\boldsymbol{APBP}} = f\{R_{\boldsymbol{A}}(\cdot), R_{\boldsymbol{B}}(\cdot), \dots\}$$
- or other more complicated exponents

Negative Entropy

$$S = \chi R_{\mathbf{J}}(-\chi) - \int_0^{\chi} R_{\mathbf{J}}(-w)dw$$



The closer the entropy is to zero, the better the RSB approximation.