Multiple Orthogonal Polynomials and the Normal Matrix Model

Arno Kuijlaars

Department of Mathematics
KU Leuven, Belgium

Joint work with Pavel Bleher
Adv. Math. 2012

Random Matrices and their Applications
Télécom ParisTech, Paris, 8 October 2012

1. Orthogonal and multiple orthogonal polynomials

- **Orthogonal polynomial** $P_n(x) = x^n + \cdots$ **satisfies**

$$\int_{-\infty}^{\infty} P_n(x) x^k w(x)\, dx = 0, \qquad k = 0, 1, \ldots, n-1,$$

- **OPs have many nice properties including a three term recurrence relation**

$$xP_n(x) = P_{n+1}(x) + b_n P_n(x) + a_n P_{n-1}(x)$$

1. Orthogonal and multiple orthogonal polynomials

- **Orthogonal polynomial** $P_n(x) = x^n + \cdots$ **satisfies**

$$\int_{-\infty}^{\infty} P_n(x) x^k w(x)\, dx = 0, \qquad k = 0, 1, \dots, n-1,$$

- **OPs have many nice properties including a three term recurrence relation**

$$xP_n(x) = P_{n+1}(x) + b_n P_n(x) + a_n P_{n-1}(x)$$

and a Riemann-Hilbert problem

Riemann Hilbert problem

- **Fokas-Its-Kitaev (1992) characterized OPs by means of 2×2 matrix valued Riemann-Hilbert problem**

  (1) $Y : \mathbb{C} \setminus \mathbb{R} \to \mathbb{C}^{2\times 2}$ **is analytic,**

  (2) $Y_+ = Y_- \begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix}$ **on** $\mathbb{R}$**,**

  (3) $Y(z) = (I_2 + O(1/z)) \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix}$ **as** $z \to \infty$**.**

Riemann Hilbert problem

- **Fokas-Its-Kitaev (1992) characterized OPs by means of** 2×2 **matrix valued Riemann-Hilbert problem**

  (1) $Y : \mathbb{C} \setminus \mathbb{R} \to \mathbb{C}^{2\times 2}$ **is analytic,**

  (2) $Y_+ = Y_- \begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix}$ **on** $\mathbb{R}$,

  (3) $Y(z) = (I_2 + O(1/z)) \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix}$ **as** $z \to \infty$.

- **Unique solution**

$$Y(z) = \begin{pmatrix} P_n(z) & \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{P_n(s)w(s)}{s-z} ds \\ -2\pi i \gamma_{n-1}^{-1} P_{n-1}(z) & -\gamma_{n-1}^{-1} \int_{-\infty}^{\infty} \frac{P_{n-1}(s)w(s)}{s-z} ds \end{pmatrix}$$

where $\gamma_{n-1} = \int_{-\infty}^{\infty} P_{n-1}(x) x^{n-1} w(x) dx > 0.$

Multiple orthogonal polynomials

- **Multiple orthogonal polynomial (MOP) is a monic polynomial of degree $n_1 + n_2$**

$$P_{n_1,n_2}(x) = x^{n_1+n_2} + \cdots$$

characterized by

$$\int_{-\infty}^{\infty} P_{n_1,n_2}(x) x^k w_1(x)\, dx = 0, \qquad k = 0, 1, \ldots, n_1 - 1,$$
$$\int_{-\infty}^{\infty} P_{n_1,n_2}(x) x^k w_2(x)\, dx = 0, \qquad k = 0, 1, \ldots, n_2 - 1.$$

- **Immediate extension to r weights $w_1, \ldots, w_r$ and $(n_1, \ldots, n_r) \in \mathbb{N}^r$.**

MOP in random matrix theory

- **MOPs appear in random matrix theory and related stochastic processes**

(a) **Random matrices with external source**

(b) **Non-intersecting Brownian motions**

(c) **Non-intersecting squared Bessel paths**

(d) **Coupled random matrices**

  - **two matrix model**
  - **Cauchy matrix model**

Properties of MOPS 1: short recurrence

- **MOPs P_{n_1,n_2} with two weight functions**
- **The polynomials Q_n defined by**

$$Q_{2k} = P_{k,k}, \qquad Q_{2k+1} = P_{k+1,k}$$

have a four term recurrence

$$xQ_n(x) = Q_{n+1}(x) + a_n Q_n(x) + b_n Q_{n-1}(x) + c_n Q_{n-2}(x)$$

Properties of MOPS 1: short recurrence

- **MOPs P_{n_1,n_2} with two weight functions**
- **The polynomials Q_n defined by**

$$Q_{2k} = P_{k,k}, \qquad Q_{2k+1} = P_{k+1,k}$$

have a four term recurrence

$$xQ_n(x) = Q_{n+1}(x) + a_n Q_n(x) + b_n Q_{n-1}(x) + c_n Q_{n-2}(x)$$

- **MOPs with r weight functions and near-diagonal multi-indices satisfy an $r+2$-term recurrence.**

Properties of MOPS 2: RH problem

- **MOPs with two weight functions have a** **Riemann-Hilbert problem** **of size** 3×3

(1) $Y : \mathbb{C} \setminus \mathbb{R} \to \mathbb{C}^{3\times 3}$ **is analytic,**

(2) $Y_+ = Y_- \begin{pmatrix} 1 & w_1 & w_2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ **on** $\mathbb{R}$**,**

(3) $Y(z) = (I_3 + O(1/z)) \begin{pmatrix} z^{n_1+n_2} & 0 & 0 \\ 0 & z^{-n_1} & 0 \\ 0 & 0 & z^{-n_2} \end{pmatrix}$ **as** $z \to \infty$**.**

Van Assche-Geronimo-K (2001)

Properties of MOPS 2: RH problem

- **MOPs with two weight functions have a Riemann-Hilbert problem of size** 3×3

(1) $Y : \mathbb{C} \setminus \mathbb{R} \to \mathbb{C}^{3\times 3}$ **is analytic,**

(2) $Y_+ = Y_- \begin{pmatrix} 1 & w_1 & w_2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ **on** $\mathbb{R}$**,**

(3) $Y(z) = (I_3 + O(1/z)) \begin{pmatrix} z^{n_1+n_2} & 0 & 0 \\ 0 & z^{-n_1} & 0 \\ 0 & 0 & z^{-n_2} \end{pmatrix}$ **as** $z \to \infty$**.**

Van Assche-Geronimo-K (2001)

- **RH problem has a unique solution if and only if the MOP** P_{n_1,n_2} **uniquely exists and in that case**

$$Y_{11}(z) = P_{n_1,n_2}(z)$$

- **MOPs with** r **weight functions have a RH problem of size** $(r+1) \times (r+1)$**.**

2. Normal matrix model

- **Probability measure on $n \times n$ complex matrices**

$$\frac{1}{Z_n} e^{-\frac{n}{t_0} \mathrm{Tr}(MM^* - V(M) - \overline{V}(M^*))} dM, \qquad t_0 > 0,$$

with

$$V(M) = \sum_{k=1}^{\infty} \frac{t_k}{k} M^k$$

2. Normal matrix model

- **Probability measure on $n \times n$ complex matrices**

$$\frac{1}{Z_n} e^{-\frac{n}{t_0} \operatorname{Tr}(MM^* - V(M) - \overline{V}(M^*))} dM, \qquad t_0 > 0,$$

with

$$V(M) = \sum_{k=1}^{\infty} \frac{t_k}{k} M^k$$

- **Model depends on parameters**

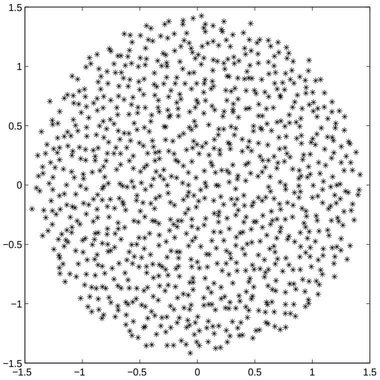
$$t_0 > 0, \qquad t_1, t_2, \ldots, t_k, \ldots.$$

- **For $t_1 = t_2 = \cdots = 0$ this is the Ginibre ensemble.**

Ginibre (1965)

Ginibre ensemble

- **Eigenvalues in the Ginibre ensemble have a limiting distribution as $n \to \infty$ that is uniform in a disk around 0 with radius $\sqrt{t_0}$.**



Laplacian growth

- **For general $t_1, t_2, \ldots$, and t_0 sufficiently small, the eigenvalues of M fill out a two-dimensional domain**

$$\Omega = \Omega(t_0, t_1, \ldots)$$

- **Ω is characterized by**

$$t_0 = \frac{1}{\pi} \operatorname{area}(\Omega), \qquad t_k = -\frac{1}{\pi} \iint_{\mathbb{C} \setminus \Omega} \frac{dA(z)}{z^k}, \quad k \geq 1$$

Laplacian growth

- **For general $t_1, t_2, \ldots$, and t_0 sufficiently small, the eigenvalues of M fill out a two-dimensional domain**

$$\Omega = \Omega(t_0, t_1, \ldots)$$

- **Ω is characterized by**

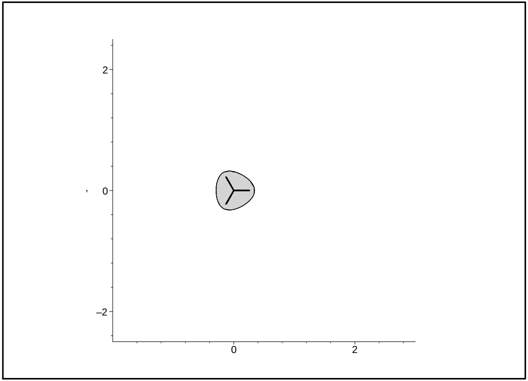
$$t_0 = \frac{1}{\pi}\,\mathrm{area}(\Omega), \qquad t_k = -\frac{1}{\pi}\iint_{\mathbb{C}\setminus\Omega} \frac{dA(z)}{z^k}, \quad k \geq 1$$

- **As a function of t_0, the boundary of Ω evolves according to the model of Laplacian growth.**
- **Laplacian growth is unstable. Singularities develop in finite time.**

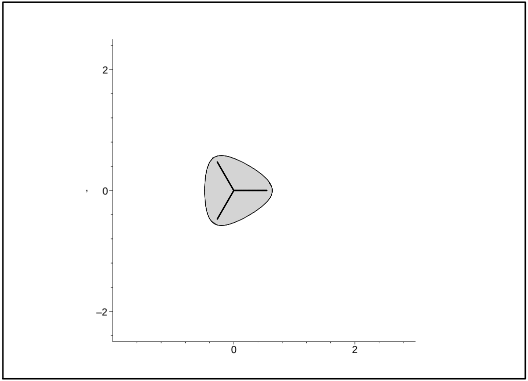
Wiegmann-Zabrodin (2000)

Teoderescu-Bettelheim-Agam-Zabrodin-Wiegmann (2005)

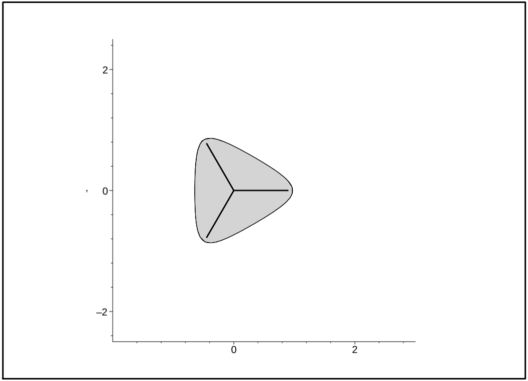
Cubic case $V(z) = \frac{t_3}{3} z^3$



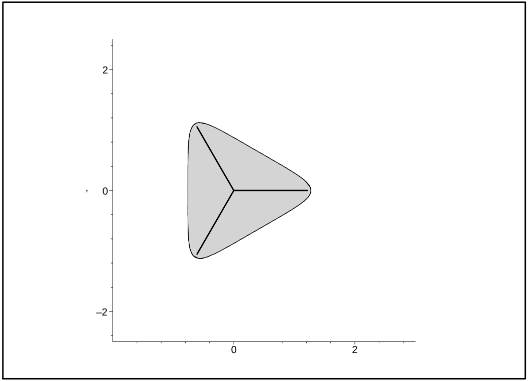
Cubic case



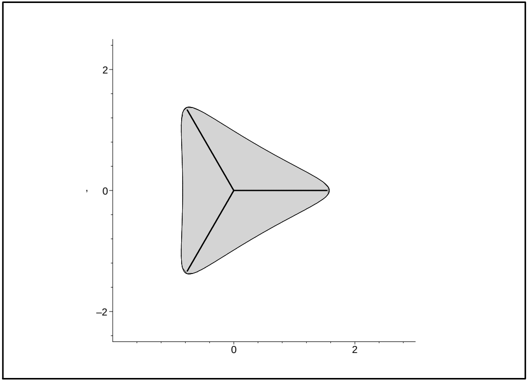
Cubic case



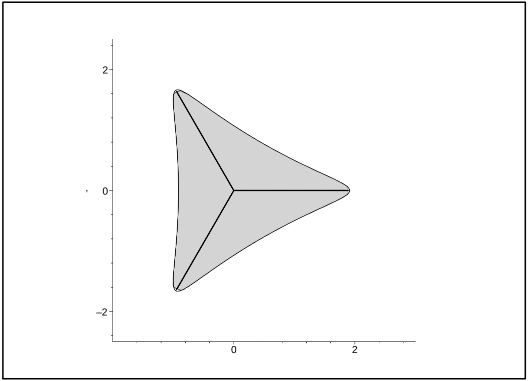
Cubic case



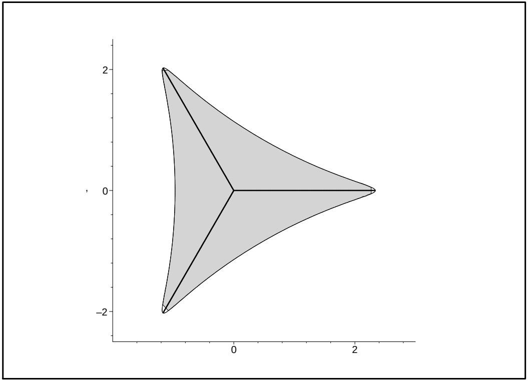
Cubic case



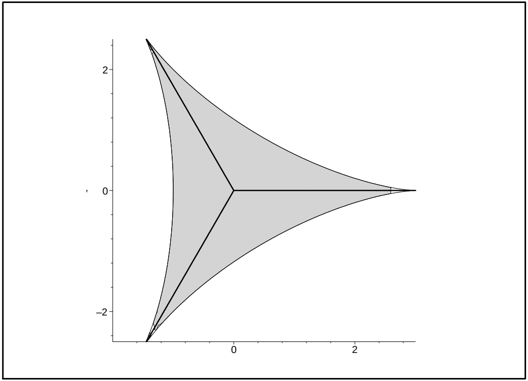
Cubic case



Cubic case



Cubic case



3. Mathematical problem

- **Normal matrix model**

$$\frac{1}{Z_n} e^{-\frac{n}{t_0}\operatorname{Tr}(MM^* - V(M) - \overline{V}(M^*))} dM, \qquad t_0 > 0,$$

is not well-defined if V is a polynomial of degree ≥ 3

3. Mathematical problem

- **Normal matrix model**

$$\frac{1}{Z_n} e^{-\frac{n}{t_0} \operatorname{Tr}(MM^* - V(M) - \overline{V}(M^*))} dM, \qquad t_0 > 0,$$

is not well-defined if V is a polynomial of degree ≥ 3

- **The normalization constant (partition function)**

$$Z_n = \int e^{-\frac{n}{t_0} \operatorname{Tr}(MM^* - V(M) - \overline{V}(M^*))} dM = +\infty.$$

is divergent.

Elbau-Felder approach

- **Elbau and Felder use a cut-off.**
- **They restrict to matrices with eigenvalues in a well-chosen bounded domain D.**

Elbau-Felder approach

- **Elbau and Felder use a cut-off.**
- **They restrict to matrices with eigenvalues in a well-chosen bounded domain D.**
- **Then the induced probability measure on eigenvalues is a determinantal point process on D.**
- **Eigenvalues fill out a domain Ω that evolves according to Laplacian growth provided t_0 is small enough.**

Elbau-Felder (2005)

Orthogonal polynomials

- **Average characteristic polynomial**

$$P_n(z) = \mathbb{E}\left[zI_n - M\right]$$

in the cut-off model is an orthogonal polynomial for scalar product

$$\langle f, g \rangle = \iint_D f(z)\overline{g(z)} e^{-\frac{n}{t_0}(|z|^2 - V(z) - \overline{V(z)})} dA(z)$$

Elbau (ETH thesis, arXiv 2007)

- **Orthogonality does not make sense if $D = \mathbb{C}$, since integrals would diverge if f and g are polynomials**

Recurrence relation

- **OPs in the cut-off model satisfy a recurrence relation**

$$zP_n(z) = P_{n+1}(z) + a_n^{(1)} P_n(z) + \cdots + a_n^{(r)} P_{n-r}(z) + \text{“remainder term”}$$

Recurrence relation

- **OPs in the cut-off model satisfy a recurrence relation**

$$zP_n(z) = P_{n+1}(z) + a_n^{(1)} P_n(z) + \cdots + a_n^{(r)} P_{n-r}(z) + \text{“remainder term”}$$

- **Remainder term comes from boundary integrals that are due to the cut-off.**
- **Remainder term is exponentially small for $t_0 > 0$ sufficiently small.**

Zeros of OPs

- **Conjecture: The zeros of P_n do not fill out the twodimensional domain Ω as $n \to \infty$, but instead accumulate along a contour Σ_1 inside Ω.**
- **Singularities appear when Σ_1 meets the boundary of Ω.**

Zeros of OPs

- **Conjecture: The zeros of P_n do not fill out the twodimensional domain Ω as $n \to \infty$, but instead accumulate along a contour Σ_1 inside Ω.**
- **Singularities appear when Σ_1 meets the boundary of Ω.**
- **In the cubic case**

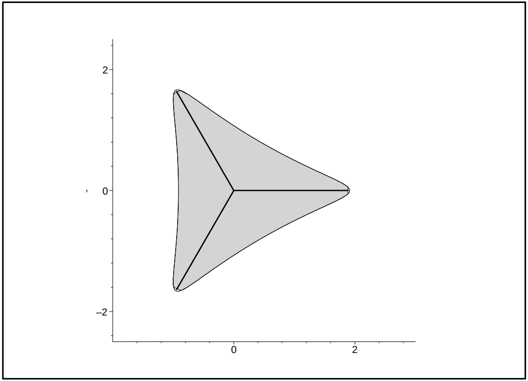
$$V(z) = \frac{t_3}{3} z^3, \qquad t_3 > 0,$$

the contour is a three-star

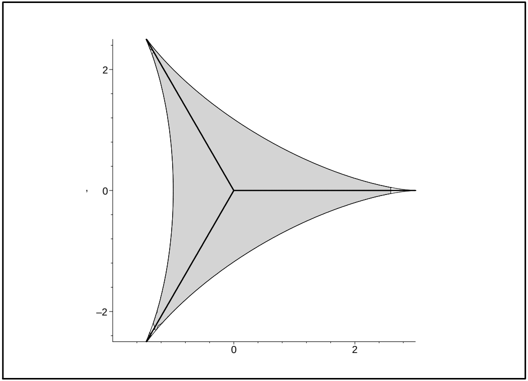
$$\Sigma_1 = [0, x^*] \cup [0, e^{2\pi i/3} x^*] \cup [0, e^{-2\pi i/3} x^*].$$

Elbau (ETH thesis, arXiv 2007)

Cubic case



Cubic case



4. Different approach

- **Scalar product in the cut-off model**

$$\langle f, g\rangle = \iint_D f(z)\overline{g(z)} e^{-\frac{n}{t_0}(|z|^2 - V(z) - \overline{V(z)})} dA(z)$$

satisfies (due to Green's theorem)

$$n\langle zf, g\rangle = t_0\langle f, g'\rangle + n\langle f, V'g\rangle - \frac{t_0}{2i}\oint_{\partial D} f(z)\overline{g(z)} e^{-\frac{n}{t_0}\left(|z|^2 - V(z) - \overline{V(z)}\right)} dz$$

- **Our idea: drop the boundary term**

Hermitian form

- **Consider an a priori abstract sesquilinear form on the space of polynomials satisfying**

$$n\langle zf, g\rangle = t_0\langle f, g'\rangle + n\langle f, V'g\rangle$$

Hermitian form

- **Consider an a priori abstract sesquilinear form on the space of polynomials satisfying**

$$n\langle zf, g\rangle = t_0\langle f, g'\rangle + n\langle f, V'g\rangle$$

- **We also want to keep the Hermitian form condition**

$$\langle g, f\rangle = \overline{\langle f, g\rangle}$$

Double integral representations

Theorem (Bertola 2003, Bleher-K 2012)

(a) **The real vector space of Hermitian forms satisfying**

$$n\langle zf, g\rangle = t_0\langle f, g'\rangle + n\langle f, V'g\rangle$$

is r^2 dimensional, where $r = \deg V - 1$.

Double integral representations

Theorem (Bertola 2003, Bleher-K 2012)

(a) **The real vector space of Hermitian forms satisfying**

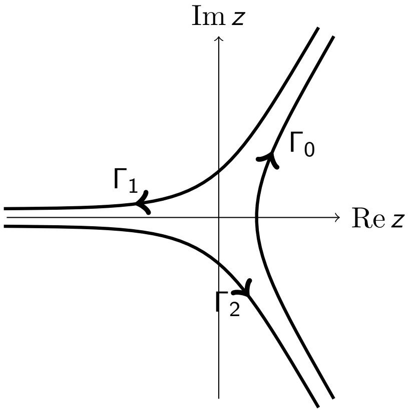
$$n\langle zf, g\rangle = t_0\langle f, g'\rangle + n\langle f, V'g\rangle$$

is r^2 dimensional, where $r = \deg V - 1$.

(b) **Any such Hermitian form can be written as**

$$\langle f, g\rangle = \sum_{j,k=0}^{r} C_{j,k} \int_{\Gamma_j} dz \int_{\overline{\Gamma}_k} ds\, f(z)\overline{g}(s) e^{-\frac{n}{t_0}\left(zs - V(z) - \overline{V}(s)\right)}$$

- $(C_{j,k})_{j,k=0,\ldots r}$ **is a Hermitian matrix with zero row and column sums**
- $\Gamma_0, \ldots, \Gamma_r$ **is a system of unbounded contours along which the integrals converge**



- **Contours Γ_0, Γ_1, Γ_2 for $V(z) = \frac{t_3}{3} z^3$ with $t_3 > 0$**
- **The contours extend to infinity at asymptotic angles $\pm\pi/3$ and π**

Orthogonal polynomials

- **Orthogonal polynomial** $P_n(z) = z^n + \cdots$ **for the Hermitian form**

$$\langle P_n, z^k \rangle = 0, \qquad \textbf{for } k = 0, 1, \ldots, n-1,$$

Orthogonal polynomials

- **Orthogonal polynomial $P_n(z) = z^n + \cdots$ for the Hermitian form**

$$\langle P_n, z^k \rangle = 0, \qquad \textbf{for } k = 0, 1, \ldots, n-1,$$

can also be seen as a multiple orthogonal polynomial with r weights

  - **due to double integral representation, and integration by parts...**

Orthogonal polynomials

- **Orthogonal polynomial $P_n(z) = z^n + \cdots$ for the Hermitian form**

$$\langle P_n, z^k \rangle = 0, \qquad \text{for } k = 0, 1, \ldots, n-1,$$

can also be seen as a multiple orthogonal polynomial with r weights

  - **due to double integral representation, and integration by parts...**

- **Weights are on**

$$\Gamma = \bigcup_{j=0}^{r} \Gamma_j$$

instead of on the real line.

MOP in cubic case

- **For** $V(z) = \frac{t_3}{3} z^3$ **the two weights are**

$$\begin{cases} w_0(z) = e^{\frac{nt_3}{3t_0} z^3} \sum_{k=0}^{2} C_{j,k} \int_{\overline{\Gamma}_k} e^{-\frac{n}{t_0}\left(zs - \frac{t_3}{3} s^3\right)} ds \\ w_1(z) = e^{\frac{nt_3}{3t_0} z^3} \sum_{k=0}^{2} C_{j,k} \int_{\overline{\Gamma}_k} s e^{-\frac{n}{t_0}\left(zs - \frac{t_3}{3} s^3\right)} ds \end{cases} \qquad z \in \Gamma_j,$$

MOP in cubic case

- **For** $V(z) = \frac{t_3}{3} z^3$ **the two weights are**

$$\begin{cases} w_0(z) = e^{\frac{nt_3}{3t_0} z^3} \sum_{k=0}^{2} C_{j,k} \int_{\Gamma_k} e^{-\frac{n}{t_0}\left(zs - \frac{t_3}{3} s^3\right)} ds \\ w_1(z) = e^{\frac{nt_3}{3t_0} z^3} \sum_{k=0}^{2} C_{j,k} \int_{\Gamma_k} s e^{-\frac{n}{t_0}\left(zs - \frac{t_3}{3} s^3\right)} ds \end{cases} \qquad z \in \Gamma_j,$$

- **Multiple orthogonality on** $\Gamma = \Gamma_0 \cup \Gamma_1 \cup \Gamma_2$

$$\int_{\Gamma} P_n(z) z^k w_0(z) dz = 0, \qquad k = 0, \dots, \lceil \tfrac{n}{2} \rceil - 1,$$

$$\int_{\Gamma} P_n(z) z^k w_1(z) dz = 0, \qquad k = 0, \dots, \lfloor \tfrac{n}{2} \rfloor - 1,$$

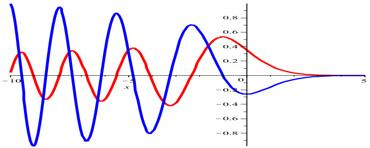
Airy functions

- **Weight w_0 is expressed in terms of the Airy function**

$$\mathrm{Ai}(z) = \frac{1}{2\pi i} \int_{\Gamma_0} e^{\frac{1}{3}s^3 - zs} ds$$

and weight w_1 in terms of the derivative

$$\mathrm{Ai}'(z) = -\frac{1}{2\pi i} \int_{\Gamma_0} s e^{\frac{1}{3}s^3 - zs} ds$$



Riemann-Hilbert problem

- **RH problem of size 3×3 with jumps on Γ that characterizes the orthogonal polynomials**

Riemann-Hilbert problem

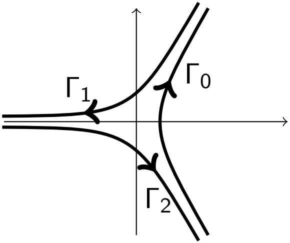
- **RH problem of size 3×3 with jumps on Γ that characterizes the orthogonal polynomials**

(1) $Y : \mathbb{C} \setminus \Gamma \to \mathbb{C}^{3\times 3}$ **is analytic,**

(2) $Y_+ = Y_- \begin{pmatrix} 1 & w_0 & w_1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ **on Γ,**

(3) $Y(z) = (I_3 + O(1/z)) \begin{pmatrix} z^n & 0 & 0 \\ 0 & z^{-n/2} & 0 \\ 0 & 0 & z^{-n/2} \end{pmatrix}$ **as $z \to \infty$.**

(assume n is even)



Riemann-Hilbert problem

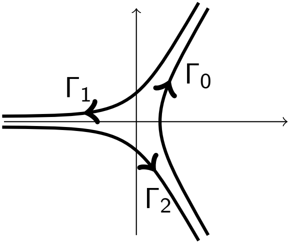
- **RH problem of size 3×3 with jumps on Γ that characterizes the orthogonal polynomials**

(1) $Y : \mathbb{C} \setminus \Gamma \to \mathbb{C}^{3\times 3}$ **is analytic,**

(2) $Y_+ = Y_- \begin{pmatrix} 1 & w_0 & w_1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ **on Γ,**

(3) $Y(z) = (I_3 + O(1/z)) \begin{pmatrix} z^n & 0 & 0 \\ 0 & z^{-n/2} & 0 \\ 0 & 0 & z^{-n/2} \end{pmatrix}$ **as $z \to \infty$.**

(assume n is even)



- **RH problem is ideal tool for asymptotic analysis...**

Bleher-Its (1999)
Deift-Kriecherbauer-McLaughlin-Venakides-Zhou (1999)

5. Asymptotic questions

Q0: **Can we choose Hermitian matrix $(C_{j,k})$ in such a way that we can do large n asymptotics on the RH problem with n-dependent weights**

$$\begin{cases} w_0(z) = e^{\frac{nt_3}{3t_0}z^3} \displaystyle\sum_{k=0}^{2} C_{j,k} \int_{\overline{\Gamma}_k} e^{-\frac{n}{t_0}\left(zs - \frac{t_3}{3}s^3\right)} ds \\ w_1(z) = e^{\frac{nt_3}{3t_0}z^3} \displaystyle\sum_{k=0}^{2} C_{j,k} \int_{\overline{\Gamma}_k} s e^{-\frac{n}{t_0}\left(zs - \frac{t_3}{3}s^3\right)} ds \end{cases} \qquad z \in \Gamma_j,$$

Q1: **Can we find the limiting behavior of zeros of P_n as $n \to \infty$?**

Q2: **Can we find the connection with Laplacian growth ?**

Q3: **What happens in the critical case ?**

Existence of OP

Theorem (Bleher-K, 2012)

With the choice

$$C = (C_{j,k}) = \frac{1}{2\pi i}\begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}$$

the following hold. Assume $0 < t_0 < t_{0,crit} = \frac{1}{8t_3^2}$

Existence of OP

Theorem (Bleher-K, 2012)

With the choice

$$C = (C_{j,k}) = \frac{1}{2\pi i}\begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}$$

the following hold. Assume $0 < t_0 < t_{0,crit} = \frac{1}{8t_3^2}$

(a) **The orthogonal polynomials** P_n **for the Hermitian form exist if** n **is sufficiently large.**

(b) **The zeros of** P_n **accumulate as** $n \to \infty$ **on the set**

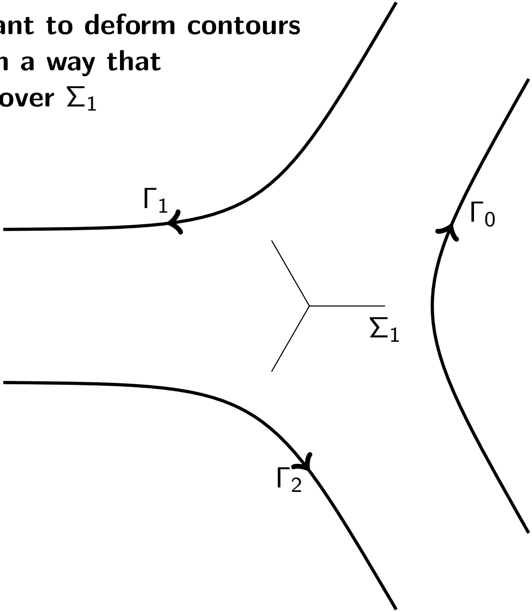
$$\Sigma_1 = [0, x^*] \cup [0, \omega x^*] \cup [0, \omega^2 x^*], \qquad \omega = e^{2\pi i/3},$$

$$x^* = \frac{3}{4t_3}\left(1 - \sqrt{1 - 8t_0 t_3^2}\right)^{2/3}$$

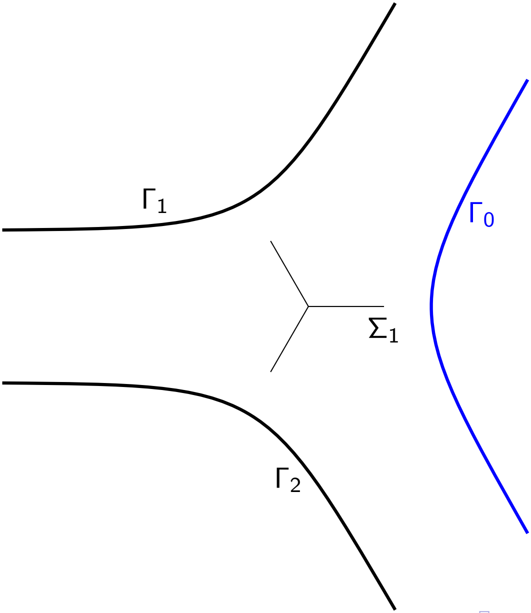
Theorem to be continued...

Why this choice for C ?

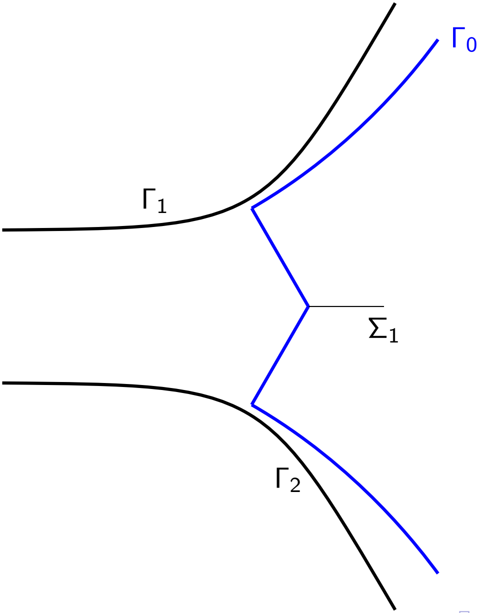
- **We want to deform contours in such a way that they cover Σ_1**



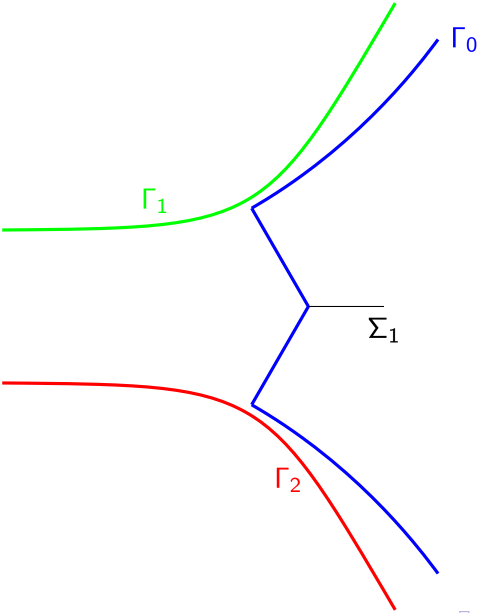
Deformation of contours



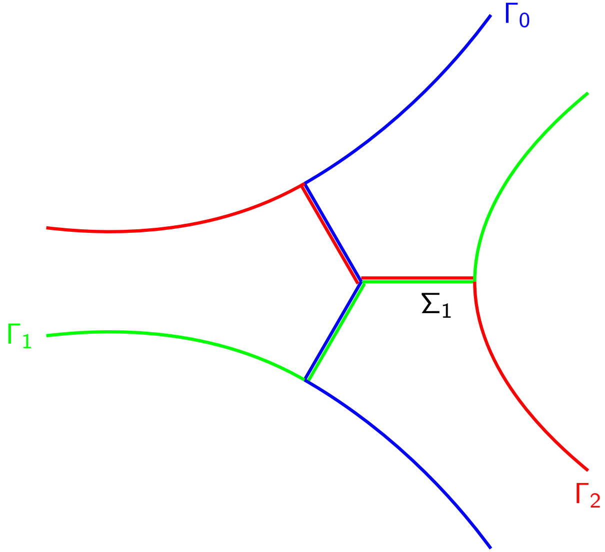
Deformation of contours



Deformation of contours



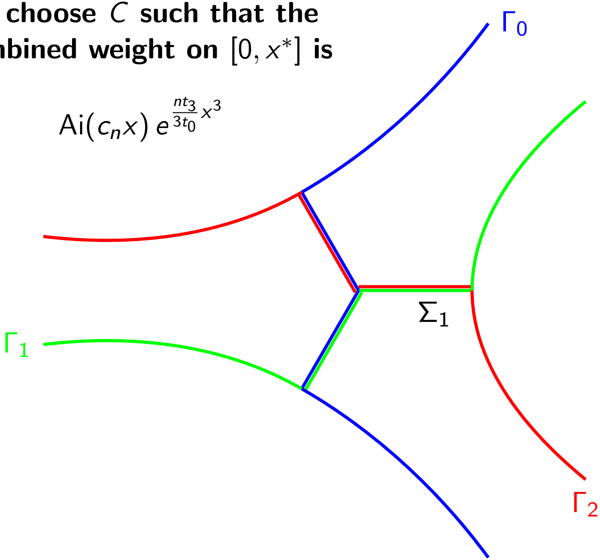
Choice for C



Choice for C

- **We choose C such that the combined weight on $[0, x^*]$ is**

$$\mathrm{Ai}(c_n x)\, e^{\frac{nt_3}{3t_0}x^3}$$



Multiple orthogonality with Airy weights

- **After deformation of contours the MOP conditions are**

$$\int_\Gamma P_n(z) z^k w_{0,n}(z) dz = 0, \qquad k = 0, \ldots, \tfrac{n}{2} - 1,$$
$$\int_\Gamma P_n(z) z^k w_{1,n}(z) dz = 0, \qquad k = 0, \ldots, \tfrac{n}{2} - 1,$$

- **On Σ_1 the new combined weights are**

$$w_{0,n}(z) = \omega^{2j} \operatorname{Ai}(c_n|z|)\, e^{\frac{nt_3}{3t_0} z^3}, \qquad z \in [0, \omega^j x^*],$$
$$w_{1,n}(z) = \omega^{j} \operatorname{Ai}'(c_n|z|)\, e^{\frac{nt_3}{3t_0} z^3}, \qquad c_n = \frac{n^{2/3}}{t_0^{2/3} t_3^{1/3}}.$$

Multiple orthogonality with Airy weights

- **After deformation of contours the MOP conditions are**

$$\int_\Gamma P_n(z) z^k w_{0,n}(z) dz = 0, \quad k = 0, \dots, \tfrac{n}{2} - 1,$$
$$\int_\Gamma P_n(z) z^k w_{1,n}(z) dz = 0, \quad k = 0, \dots, \tfrac{n}{2} - 1,$$

- **On Σ_1 the new combined weights are**

$$w_{0,n}(z) = \omega^{2j} \operatorname{Ai}(c_n|z|)\, e^{\frac{nt_3}{3t_0} z^3}, \qquad z \in [0, \omega^j x^*],$$
$$w_{1,n}(z) = \omega^{j} \operatorname{Ai}'(c_n|z|)\, e^{\frac{nt_3}{3t_0} z^3}, \qquad c_n = \frac{n^{2/3}}{t_0^{2/3} t_3^{1/3}}.$$

- **Large n behavior of the two weights for $z \in \Sigma_1 \setminus \{0\}$,**

$$w_{k,n}(z) \sim \exp(-nQ(z)), \qquad Q(z) = \frac{1}{t_0}\left(\frac{2}{3\sqrt{t_3}}|z|^{3/2} - \frac{t_3}{3} z^3\right).$$

Theorem (continued)

(c) **The OPs (P_n) have a limiting zero distribution μ_1^* on Σ_1.**

Limiting zero distribution

Theorem (continued)

(c) **The OPs (P_n) have a limiting zero distribution μ_1^* on Σ_1.**

(d) μ_1^* **is part of the minimizer (μ_1^*, μ_2^*) of a vector equilibrium problem that asks to minimize**

$$I(\mu_1) - I(\mu_1, \mu_2) + I(\mu_2) + \int Q d\mu_1$$

over (μ_1, μ_2) such that

- μ_1 **is a measure on Σ_1 with** $\mu_1(\Sigma_1) = 1$
- μ_2 **is a measure on Σ_2 with** $\mu_2(\Sigma_2) = \frac{1}{2}$

- **Logarithmic energy**

$$I(\mu, \nu) = \iint \log \frac{1}{|x-y|} d\mu(x) d\nu(y), \qquad I(\mu) = I(\mu, \mu),$$

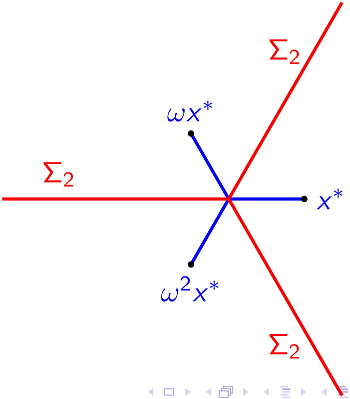
- **Minimize**

$$I(\mu_1) - I(\mu_1, \mu_2) + I(\mu_2) + \int Q d\mu_1,$$

$$Q(z) = \frac{1}{t_0}\left(\frac{2}{3\sqrt{t_3}}|z|^{3/2} - \frac{t_3}{3}z^3\right)$$

over (μ_1, μ_2) **such that**

$$\mathrm{supp}(\mu_1) \subset \Sigma_1,$$
$$\mathrm{supp}(\mu_2) \subset \Sigma_2,$$
$$\mu_1(\Sigma_1) = 1,$$
$$\mu_2(\Sigma_2) = 1/2.$$



- **Nikishin-type of interaction of measures on two plates.**

Structure of the minimizer

- **There is a unique minimizer (μ_1^*, μ_2^*) of the vector equilibrium problem.**
- **The minimizers induce an algebraic-geometric structure.**

Structure of the minimizer

- **There is a unique minimizer (μ_1^*, μ_2^*) of the vector equilibrium problem.**
- **The minimizers induce an algebraic-geometric structure.**

Definition

Define Cauchy transforms

$$F_k(z) = \int \frac{d\mu_k^*(s)}{z - s}, \qquad z \in \mathbb{C} \setminus \Sigma_k, \ \ k = 1, 2,$$

and the ξ-function on the first sheet

$$\xi_1(z) = t_3 z^2 + t_0 F_1(z), \qquad z \in \mathbb{C} \setminus \Sigma_1 = \mathcal{R}_1$$

Riemann surface

Theorem (continued)

(e) **The function ξ_1 has an analytic continuation to a three-sheeted Riemann surface**

(f) **ξ_1 is one of the solutions of the algebraic equation (spectral curve)**

$$\xi^3 - t_3 z^2 \xi^2 - \left(t_0 t_3 + \frac{1}{t_3}\right) + z^3 + A = 0$$

$$A = \frac{1 + 20 t_0 t_3^2 - 8 t_0^2 t_3^4 - (1 - 8 t_0 t_3^2)^{3/2}}{32 t_3^3}$$

Laplacian growth

Theorem (continued)

(g) **The equation** $\boxed{\xi_1(z) = \bar{z}}$ **defines a simple closed curve** $\partial\Omega$ **that is the boundary of a domain** Ω **containing** Σ_1 **in its interior.**

Laplacian growth

Theorem (continued)

(g) **The equation** $\boxed{\xi_1(z) = \bar{z}}$ **defines a simple closed curve** $\partial\Omega$ **that is the boundary of a domain** Ω **containing** Σ_1 **in its interior.**

(h) Ω **has exterior harmonic moments** $(0, 0, t_3, 0, 0, \ldots)$ **and**

$$\text{area}(\Omega) = \pi t_0$$

Theorem (continued)

(g) **The equation** $\boxed{\xi_1(z) = \bar{z}}$ **defines a simple closed curve** $\partial\Omega$ **that is the boundary of a domain** Ω **containing** Σ_1 **in its interior.**

(h) Ω **has exterior harmonic moments** $(0, 0, t_3, 0, 0, \ldots)$ **and**

$$\text{area}(\Omega) = \pi t_0$$

(i) **Also**

$$\int \frac{d\mu_1^*(\zeta)}{z - \zeta} = \frac{1}{\pi t_0} \iint_\Omega \frac{dA(\zeta)}{z - \zeta}. \qquad z \in \mathbb{C} \setminus \overline{\Omega}$$

Steepest descent analysis

- **The asymptotic formulas for P_n follow from a steepest descent analysis of the RH problem of size 3×3**
- **Sequence of explicit transformations**

$$Y \mapsto X \mapsto V \mapsto U \mapsto T \mapsto S \mapsto R$$

leading to a simple RH problem for R, that can be solved by Neumann series.

Steepest descent analysis

- **The asymptotic formulas for P_n follow from a steepest descent analysis of the RH problem of size 3×3**
- **Sequence of explicit transformations**

$$Y \mapsto X \mapsto V \mapsto U \mapsto T \mapsto S \mapsto R$$

  leading to a simple RH problem for R, that can be solved by Neumann series.
- **Major roles are played by the solution of the vector equilibrium problem and by the ξ-functions coming from the Riemann surface.**

- **The asymptotic formulas for P_n follow from a steepest descent analysis of the RH problem of size 3×3**
- **Sequence of explicit transformations**

$$Y \mapsto X \mapsto V \mapsto U \mapsto T \mapsto S \mapsto R$$

leading to a simple RH problem for R, that can be solved by Neumann series.

- **Major roles are played by the solution of the vector equilibrium problem and by the ξ-functions coming from the Riemann surface.**
- **There is some similarity with the steepest descent analysis of the RH problem for biorthogonal polynomials from the two-matrix model with quartic potential.**

Duits-K (2009), Duits-K-Mo (2012)

6. Outlook

- For $t_0 < t_{0,crit}$, the spectral curve has three branch points

$$x^*, \quad e^{2\pi i/3} x^*, \quad e^{-2\pi i/3} x^*$$

and three nodes

$$\widehat{x} > x^*, \quad e^{2\pi i/3} \widehat{x}, \quad e^{-2\pi i/3} \widehat{x}$$

- At the critical value $t_{0,crit}$ the nodes coalesce with the branch points.
- Local behavior can then be described by functions that are associated with the Painlevé I equation (on to do list).
- What happens beyond the critical value ??