Progress in the method of Ghosts and Shadows for Beta Ensembles

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Wishart Matrices (arbitrary covariance)

- G=mxn matrix of Gaussians
- Σ=mxn semidefinite matrix
- G'G Σ is similar to A=Σ^{½}G'GΣ$^{-½}$
- For β=1,2,4, the joint eigenvalue density of A has a formula:

$$c_{m,n,\Sigma,\beta} \prod_{i=1}^{n} \lambda_i^{\frac{m-n+1}{2}\beta - 1} \prod_{j<k} |\lambda_k - \lambda_j|^\beta \cdot {}_0F_0^{(\beta)}(-\frac{\beta}{2}\Lambda, \Sigma^{-1}),$$

where $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$.

- Known for β=2 in some circles as Harish-Chandra-Itzykson-Zuber

Main Purpose of this talk

$$c_{m,n,\Sigma,\beta} \prod_{i=1}^{n} \lambda_i^{\frac{m-n+1}{2}\beta-1} \prod_{j<k} |\lambda_k - \lambda_j|^{\beta} \cdot {}_0F_0^{(\beta)}(-\frac{\beta}{2}\Lambda, \Sigma^{-1}),$$

where $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$.

- Eigenvalue density of G'G Σ (similar to $A=\Sigma^{½}G'G\Sigma^{-½}$)
- Present an algorithm for sampling from this density
- Show how the method of Ghosts and Shadows can be used to derive this algorithm
- Further evidence that β=1,2,4 need not be special

Eigenvalues of GOE (β=1)

- Naïve Way:

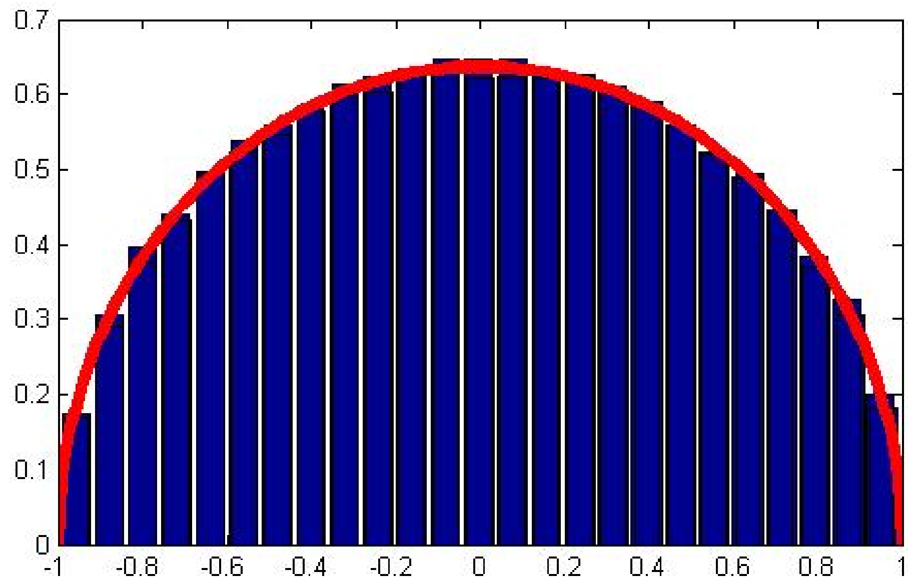
  MATLAB®: `A=randn(n); S=(A+A')/sqrt(2*n);eig(S)`

  R:

  `A=matrix(rnorm(n*n),ncol=n);S=(a+t(a))/sqrt(2*n);eigen(S,symmetric=T,only.values=T)$values;`

  Mathematica:

  `A=RandomArray[NormalDistribution[],{n,n}];S=(A+Transpose[A])/Sqrt[n];Eigenvalues[s]`



Eigenvalues of GOE/GUE/GSE and beyond…

- Real Sym. Tridiagonal:
- β=1:real, β=2:complex, β=4:quaternion

$$H_n^\beta = \frac{1}{\sqrt{\beta}} \begin{bmatrix} g_1 & \chi_{(n-1)\beta} & & & \\ \chi_{(n-1)\beta} & g_2 & \chi_{(n-2)\beta} & & \\ & \ddots & \ddots & \ddots & \\ & & \chi_{2\beta} & g_{n-1} & \chi_\beta \\ & & & \chi_\beta & g_n \end{bmatrix}$$

Diagonals N(0,2), Off-diagonals "chi" random-variables

- General β>0
- Computations significantly faster

Introduction to Ghosts

- G_1 is a standard normal $N(0,1)$
- G_2 is a complex normal $(G_1 + iG_1)$
- G_4 is a quaternion normal $(G_1 + iG_1 + jG_1 + kG_1)$
- G_β $(\beta > 0)$ seems to often work just fine

  → "Ghost Gaussian"

Chi-squared

- Defn: χ^2_β is the sum of β iid squares of standard normals if β=1,2,…
- Generalizes for non-integer β as the "gamma" function interpolates factorial
- χ_β is the sqrt of the sum of squares (which generalizes) (wikipedia chi-distriubtion)
- $|G_1|$ is χ_1, $|G_2|$ is χ_2, $|G_4|$ is χ_4
- So why not $|G_\beta|$ is χ_β ?
- I call χ_β the shadow of G_β

Linear Algebra Example

- Given a ghost Gaussian, G_β, $|G_\beta|$ is a real chi-beta variable χ_β variable.
- Gram-Schmidt:

$$\begin{bmatrix} G_\beta & G_\beta & G_\beta \\ G_\beta & G_\beta & G_\beta \\ G_\beta & G_\beta & G_\beta \end{bmatrix} \xrightarrow[H_3]{} \begin{bmatrix} \chi_{3\beta} & G_\beta & G_\beta \\ & G_\beta & G_\beta \\ & G_\beta & G_\beta \end{bmatrix} \xrightarrow[H_2]{} \begin{bmatrix} \chi_{3\beta} & G_\beta & G_\beta \\ & \chi_{2\beta} & G_\beta \\ & & G_\beta \end{bmatrix} \xrightarrow[H_1]{} \begin{bmatrix} \chi_{3\beta} & G_\beta & G_\beta \\ & \chi_{2\beta} & G_\beta \\ & & X_\beta \end{bmatrix}$$

or

$$\begin{bmatrix} G_\beta & G_\beta & G_\beta \\ G_\beta & G_\beta & G_\beta \\ G_\beta & G_\beta & G_\beta \end{bmatrix} = [Q] * \begin{bmatrix} \chi_{3\beta} & G_\beta & G_\beta \\ & \chi_{2\beta} & G_\beta \\ & & X_\beta \end{bmatrix}$$

Linear Algebra Example

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Q has β Haar Measure!
We have computed moments!

Tridiagonalizing Example

Symmetric Part of

$$\begin{matrix} G_\beta & G_\beta & G_\beta & \cdots & G_\beta \\ G_\beta & G_\beta & G_\beta & \cdots & G_\beta \\ G_\beta & G_\beta & G_\beta & \cdots & G_\beta \\ & & & & \\ G_\beta & G_\beta & G_\beta & G_\beta & G_\beta \end{matrix}$$

$$\rightarrow \quad H_n^\beta = \frac{1}{\sqrt{\beta}} \begin{bmatrix} g_1 & \chi_{(n-1)\beta} & & & \\ \chi_{(n-1)\beta} & g_2 & \chi_{(n-2)\beta} & & \\ & \ddots & \ddots & \ddots & \\ & & \chi_{2\beta} & g_{n-1} & \chi_\beta \\ & & & \chi_\beta & g_n \end{bmatrix}$$

(Silverstein, Trotter, etc)

Histogram without Histogramming: Sturm Sequences

- Count #eigs < 0.5: Count sign changes in

  Det((A-0.5*I)[1:k,1:k])

- Count #eigs in [x,x+h]

  Take difference in number of sign changes at x+h and x

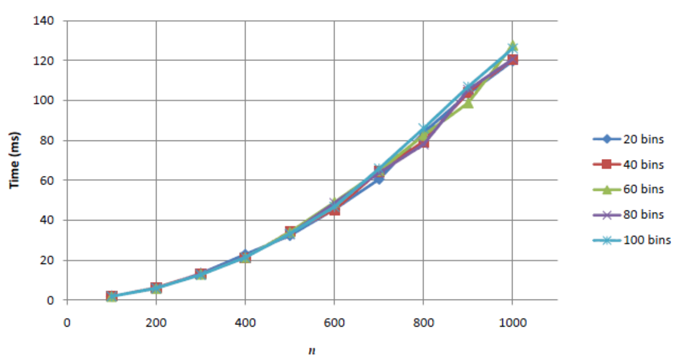


FIGURE 1. Performance of naive histogramming algorithm. This figure makes readily apparent the dominance of the eigenvalue computation (which takes $O(n^2)$ time): the number of histogram bins makes no significant difference in performance.

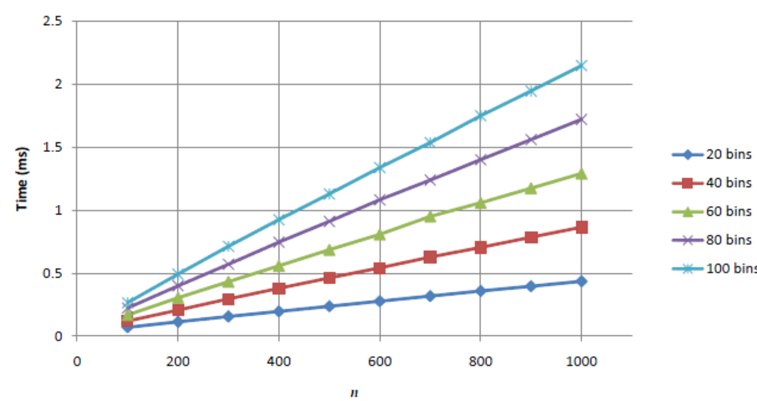


FIGURE 2. Performance of Sturm sequence based histogramming algorithm. This figure clearly demonstrates the bilinear dependency from the $O(mn)$ computation time.

Mentioned in Dumitriu and E 2006, Used theoretically in Albrecht, Chan, and E 2008

A good computational trick is a good theoretical trick!

Corollary 5.3. ("Eigenvalue distribution" or "Level density") *Let $f_{G_{-\lambda}}$ be the Gaussian density with mean $-\lambda$ and variance 1, and let $D_p(z)$ be the parabolic cylinder function. Define $z_i \equiv sign(s_{i-1})(s_i + \lambda + s_{i-1})$, and $p_i \equiv \frac{1}{2}\beta(i-1)$. If Λ is drawn from the β-Hermite eigenvalue density, then*

$$\cdots \lambda] = \frac{1}{n}\sum_{i=1}^{n}\int_{-\infty}^{0} f_{r_i}(s_i)\, ds_i,$$

where

$$f_{r_i}(s_i) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty} f_{r_1,r_2,\ldots} \cdots s_i)\, ds_1 ds_2 \ldots ds_{i-1},$$

$$f_{r_1,r_2,\ldots,r_n}(s_1,s_2,\ldots,s_n) = f_{G_{-\lambda}}(s_1)\prod_{i=2} f_{r_i|r_{i-1}} \cdots \text{ and}$$

$$f_{r_i|r_{i-1}}(s_i|s_{i-1}) = \frac{|s_{i-1}|^{p_i}}{\sqrt{2\pi}} e^{-\frac{1}{4}[2(s_i+\lambda)^2 - z_i^2]} D_{-p_i}(z_i).$$

Finite Semi-Circle Laws for Any Beta!

Corollary 5.4. (Largest eigenvalue distribution) *As in Corollary 5.3, let $f_{G_{-\lambda}}$ be the Gaussian density with mean $-\lambda$ and variance 1, $D_p(z)$ be the parabolic cylinder function, $z_i \equiv sign(s_{i-1}) \cdots \lambda + s_{i-1})$, and $p_i \equiv \frac{1}{2}\beta(i-1)$. If Λ_{max} is drawn from the β-Hermite largest eigenvalu… then*

$$\Pr[\Lambda_{max} < \lambda] = \int_{-\infty}^{0}\int_{-\infty}^{0}\cdots\int_{-\infty} \cdots (s_1,s_2,\ldots,s_n)\, ds_1 ds_2 \ldots ds_n,$$

where

$$f_{r_1,r_2,\ldots,r_n}(s_1,s_2,\ldots,s_n) = f_{G_{-\lambda}}(s_1)\prod_{i=2}^{n} f_{r_i|r_{i-1}}(s_i|s_{i-1}), \text{ and}$$

$$f_{r_i|r_{i-1}}(s_i|s_{i-1}) = \frac{|s_{i-1}|^{p_i}}{\sqrt{2\pi}} e^{-\frac{1}{4}[2(s_i+\lambda)^2 - z_i^2]} D_{-p_i}(z_i).$$

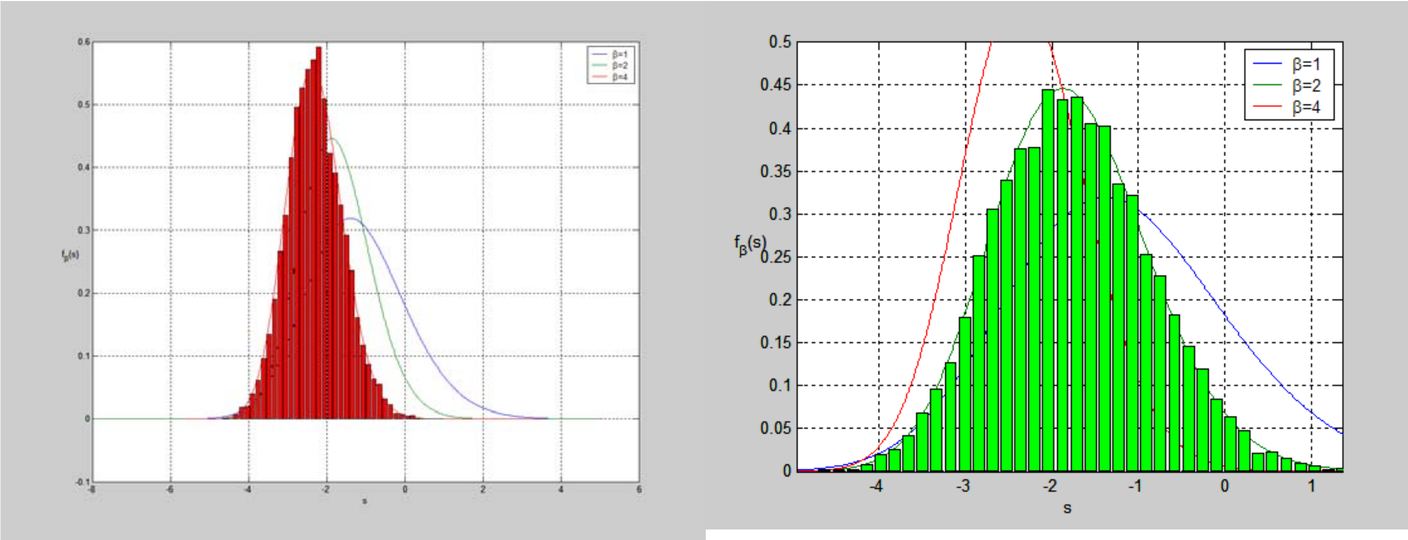
Finite Tracy-Widom Laws for Any Beta!

Stochastic Differential Eigen-Equations

- Tridiagonal Models Suggest SDE's with Brownian motion as infinite limit:
  - E: 2002
  - E and Sutton 2005, 2007
  - Brian Rider and (Ramirez, Cambronero, Virag, etc.)
    - (Lots of beautiful results!)

  - (Not today's talk)

"Tracy-Widom" Computations
Eigenvalues without the whole matrix!



Never construct the entire tridiagonal matrix!
Just say the upper $10n^{1/3}$ by $10n^{1/3}$
Compute largest eigenvalue of that, perhaps using Lanczos with shift and invert strategy!
Can compute for n amazingly large!!!!! And any beta.

E: 2003 Persson and E: 2005

Other Computational Results

- Infinite Random Matrix Theory
  - The Free Probability Calculator (Raj)
- Finite Random Matrix Theory
  - MOPS (Ioana Dumitriu) (β: orthogonal polynomials)
  - Hypergeometrics of Matrix Argument (Plamen Koev) (β: distribution functions for finite stats such as the finite Tracy-Widom laws for Laguerre)

- Good stuff, but not today.

Scary Ideas in Mathematics

- Zero
- Negative
- Radical
- Irrational
- Imaginary
- Ghosts: Something like a commutative algebra of random variables that generalizes random Reals, Complexes, and Quaternions and inspires theoretical results and numerical computation

Did you say "**commutative**"??

- Quaternions don't commute.
- Yes but random quaternions do!
- If x and y are G_4 then x*y and y*x are identically distributed!

RMT Densities

- Hermite:

  $c \prod |\lambda_i - \lambda_j|^\beta e^{-\Sigma \lambda_i^2/2}$ (Gaussian Ensemble)

- Laguerre:

  $c \prod |\lambda_i - \lambda_j|^\beta \prod \lambda_i^m e^{-\Sigma \lambda_i}$ (Wishart Matrices)

- Jacobi:

  $c \prod |\lambda_i - \lambda_j|^\beta \prod \lambda_i^{m_1} \prod (1-\lambda_i)^{m_2}$ (Manova Matrices)

- Fourier:

  $c \prod |\lambda_i - \lambda_j|^\beta$ (on the complex unit circle) "Jack Polynomials"

Traditional Story: Count the real parameters β=1,2,4 for real, complex, quaternion
Applications: β=1: All of Multivariate Statistics β=2:Tons,
β=4: Almost nobody cares
Dyson 1962 "Threefold Way" Three Associative Division Rings

β-Ghosts

- β=1 has one real Gaussian (G)
- β=2 has two real Gaussians (G+iG)
- β=4 has four real Gaussians (G+iG+jG+kG)
- $\beta \geq 1$ has one real part and (β-1) "Ghost" parts

Introductory Theory

- There is an advanced theory emerging
  - A β-ghost is a spherically symmetric random variable defined on R^β
  - A shadow is a derived real or complex quantity

Wishart Matrices (arbitrary covariance)

- G=mxn matrix of Gaussians
- Σ=mxn semidefinite matrix
- G'G Σ is similar to A=Σ½G'GΣ-½
- For β=1,2,4, the joint eigenvalue density of A has a formula:

$$c_{m,n,\Sigma,\beta} \prod_{i=1}^{n} \lambda_i^{\frac{m-n+1}{2}\beta - 1} \prod_{j<k} |\lambda_k - \lambda_j|^\beta \cdot {}_0F_0^{(\beta)}(-\frac{\beta}{2}\Lambda, \Sigma^{-1}),$$

where $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$.

Joint Eigenvalue density of G'G Σ

$$c_{m,n,\Sigma,\beta} \prod_{i=1}^{n} \lambda_i^{\frac{m-n+1}{2}\beta - 1} \prod_{j<k} |\lambda_k - \lambda_j|^\beta \cdot {}_0F_0^{(\beta)}(-\frac{\beta}{2}\Lambda, \Sigma^{-1}),$$

where $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$.

The "${}_0F_0$" function is a hypergeometric function of two matrix arguments that depends only on the eigenvalues of the matrices. Formulas and software exist.

Generalization of Laguerre

- Laguerre:

$$c_{m,n,\beta} \prod_{i=1}^{n} \lambda_i^{\frac{m-n+1}{2}\beta-1} \prod_{j<k} |\lambda_k - \lambda_j|^\beta \cdot \exp\left(-\frac{\beta}{2}\sum_{i=1}^{n} \lambda_i\right)$$

- Versus Wishart:

$$c_{m,n,\Sigma,\beta} \prod_{i=1}^{n} \lambda_i^{\frac{m-n+1}{2}\beta-1} \prod_{j<k} |\lambda_k - \lambda_j|^\beta \; {}_0F_0^{(\beta)}(-\frac{\beta}{2}\Lambda, \Sigma^{-1})$$

General β?

The joint density:

$$c_{m,n,\Sigma,\beta} \prod_{i=1}^{n} \lambda_i^{\frac{m-n+1}{2}\beta - 1} \prod_{j<k} |\lambda_k - \lambda_j|^{\beta} \cdot {}_0F_0^{(\beta)}(-\frac{\beta}{2}\Lambda, \Sigma^{-1}),$$

is a probability density for all $\beta>0$.

Goals:

- Algorithm for sampling from this density
- Get a feel for the density's "ghost" meaning

Main Result

- An algorithm derived from ghosts that samples eigenvalues
- A MATLAB implementation that is consistent with other beta-ized formulas
  - Largest Eigenvalue
  - Smallest Eigenvalue

Real quantity

Working with Ghosts

$$
\begin{bmatrix} g_{1,1} & g_{1,2} & g_{1,3} \\ g_{2,1} & g_{2,2} & g_{2,3} \\ g_{3,1} & g_{3,2} & g_{3,3} \end{bmatrix} \rightarrow \begin{bmatrix} \chi_{3\beta} & g_{1,2} & g_{1,3} \\ 0 & g_{2,2} & g_{2,3} \\ 0 & g_{3,2} & g_{3,3} \end{bmatrix} \rightarrow
$$

$$
\begin{bmatrix} \chi_{3\beta} & \chi_{2\beta} & 0 \\ 0 & g_{2,2} & g_{2,3} \\ 0 & g_{3,2} & g_{3,3} \end{bmatrix} \rightarrow \begin{bmatrix} \chi_{3\beta} & \chi_{2\beta} & 0 \\ 0 & \chi_{2\beta} & g_{2,3} \\ 0 & 0 & g_{3,3} \end{bmatrix} \rightarrow
$$

$$
\begin{bmatrix} \chi_{3\beta} & \chi_{2\beta} & 0 \\ 0 & \chi_{2\beta} & \chi_{\beta} \\ 0 & 0 & g_{3,3} \end{bmatrix} \rightarrow \begin{bmatrix} \chi_{3\beta} & \chi_{2\beta} & 0 \\ 0 & \chi_{2\beta} & \chi_{\beta} \\ 0 & 0 & \chi_{\beta} \end{bmatrix}
$$

More practice with Ghosts

$$\begin{bmatrix} \sigma_1 g_{1,1} & \sigma_2 g_{1,2} \\ \sigma_1 g_{2,1} & \sigma_2 g_{2,2} \\ \sigma_1 g_{3,1} & \sigma_2 g_{3,2} \end{bmatrix} \longrightarrow \begin{bmatrix} \sigma_1 \chi_{3\beta} & \sigma_2 g_{1,2} \\ 0 & \sigma_2 g_{2,2} \\ 0 & \sigma_2 g_{3,2} \end{bmatrix},$$

Bidiagonalizing Σ=I

$$\text{LAGUERRE: } Z = \begin{bmatrix} \chi_{m\beta} & \chi_{(n-1)\beta} & & & \\ & \chi_{(m-1)\beta} & \chi_{(n-2)\beta} & & \\ & & \ddots & \ddots & \\ & & & \chi_{2\beta} & \chi_{\beta} \\ & & & & \chi_{\beta} \end{bmatrix}$$

- Z'Z has the Σ=I density giving a special case of

$$c_{m,n,\Sigma,\beta} \prod_{i=1}^{n} \lambda_i^{\frac{m-n+1}{2}\beta-1} \prod_{j<k} |\lambda_k - \lambda_j|^{\beta} \cdot {}_0F_0^{(\beta)}(-\frac{\beta}{2}\Lambda, \Sigma^{-1}),$$

The Algorithm for Z=GΣ½

$$Z \equiv \begin{bmatrix} g_{1,1} & \cdots & g_{1,n} \\ \vdots & & \vdots \\ g_{m,1} & \cdots & g_{m,n} \end{bmatrix} \cdot \begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{bmatrix}$$

The Algorithm for Z=GΣ½

$$Z \equiv \begin{bmatrix} g_{1,1} & \cdots & g_{1,n} \\ \vdots & & \vdots \\ g_{m,1} & \cdots & g_{m,n} \end{bmatrix} \cdot \begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{bmatrix}$$

By recursion, we can assume the first $n-1$ columns of Z have a Ghost SVD:

$$Z = \begin{bmatrix} & d_n g_{1,n} \\ UTV^t & \vdots \\ & d_n g_{m,n} \end{bmatrix}$$

The real diagonal matrix T is assumed ordered.

Removing U and V

$$Z = \begin{bmatrix} & & d_n g_{1,n} \\ & UTV^t & \vdots \\ & & d_n g_{m,n} \end{bmatrix}$$

We can use the assumptions to get rid of U and V without affecting the Ghost Spectrum of Z:

$$Z = \begin{bmatrix} t_1 & & & d_n g_{1,n} \\ & \ddots & & \vdots \\ & & t_{n-1} & d_n g_{n-1,n} \\ & & & \vdots \\ & & & d_n g_{m,n} \end{bmatrix}$$

Algorithm cont.

We can multiply on the left by a Ghost Orthogonal matrix of "Ghost Signs," $\mathrm{diag}(\phi_1, \ldots, \phi_m)$, to create real numbers at the expensive of making the t_i's Ghost:

$$Z = \begin{bmatrix} t_1\phi_1 & & & d_n\chi_\beta \\ & \ddots & & \vdots \\ & & t_{n-1}\phi_{n-1} & d_n\chi_\beta \\ & & & \vdots \\ & & & d_n\chi_\beta \end{bmatrix}$$

Completion of Recursion

We can multiply on the right by more Ghost Signs to make the whole matrix real:

$$Z = \begin{bmatrix} t_1 & & & d_m \chi_\beta \\ & \ddots & & \vdots \\ & & t_{m-1} & d_m \chi_\beta \\ & & & \vdots \\ & & & d_m \chi_\beta \end{bmatrix}$$

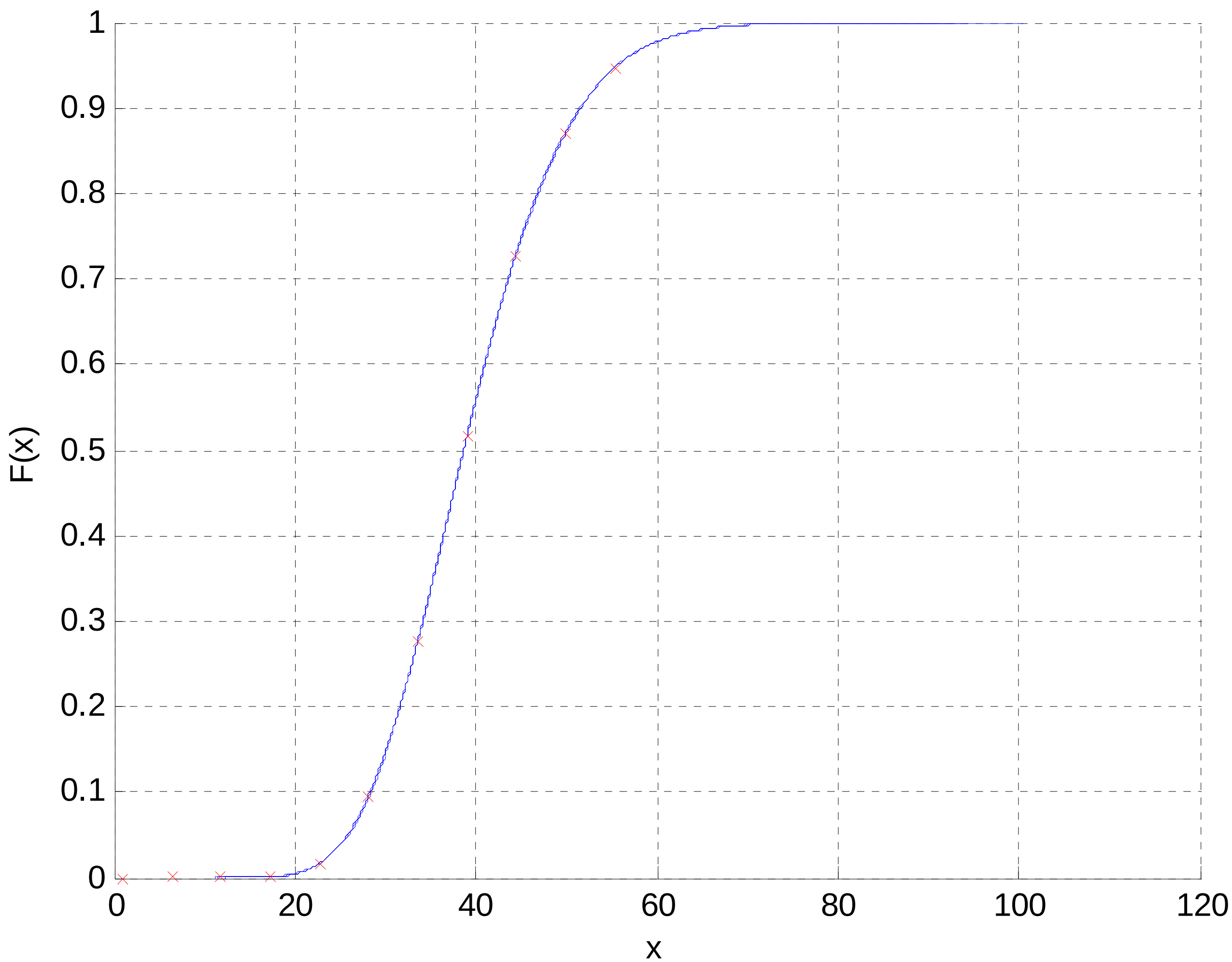
We can take the real SVD of this matrix, completing the recursion.

Numerical Experiments – Largest Eigenvalue

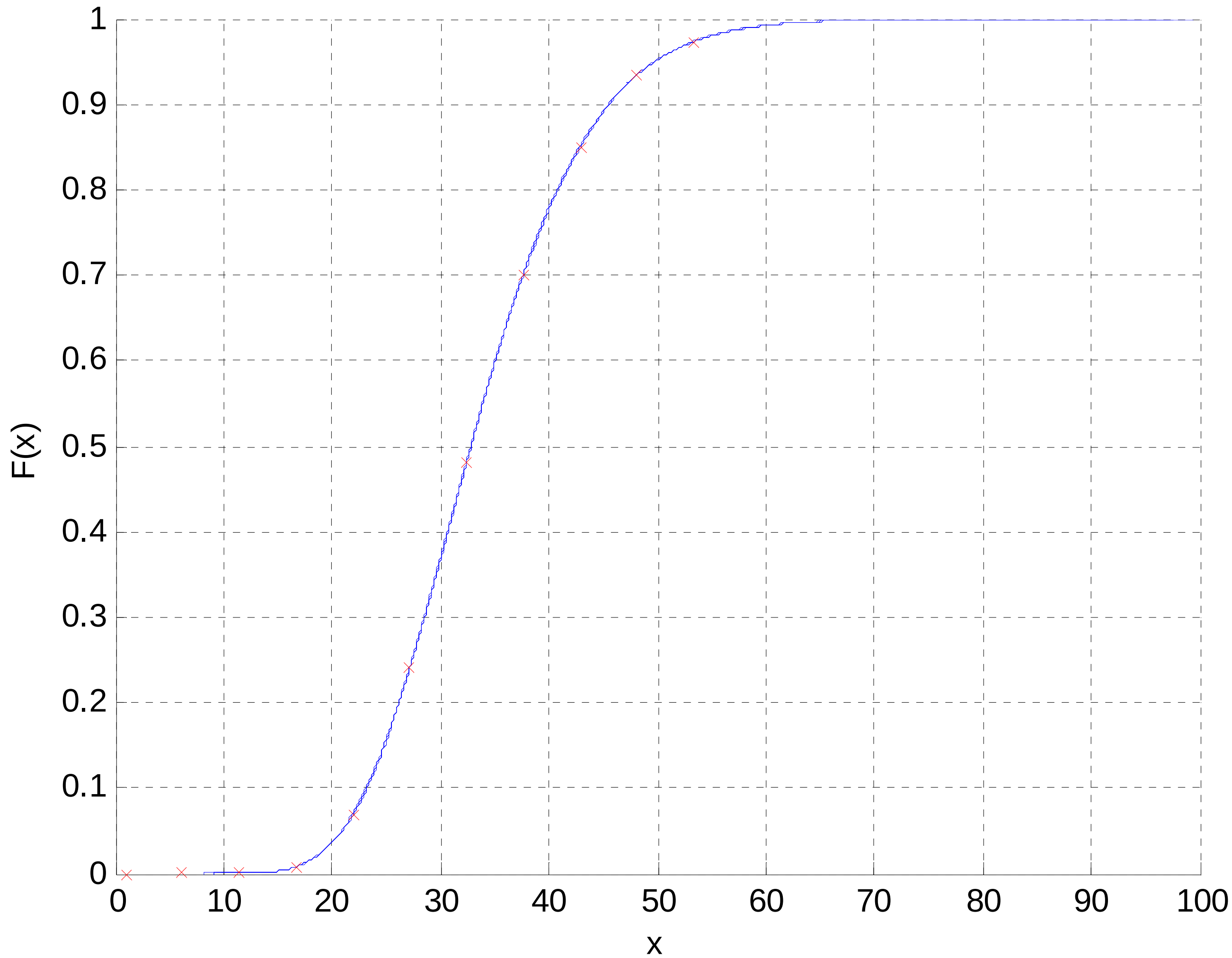
- Analytic Formula for largest eigenvalue dist

$$c_{m,n,\beta} \cdot \det\left(\frac{x}{2}\beta\Sigma^{-1}\right)^{\frac{m}{2}\beta} \cdot {}_1F_1^{(\beta)}\left(\frac{m}{2}\beta; \frac{m+n-1}{2}\beta + 1; -\frac{x}{2}\beta\Sigma^{-1}\right).$$

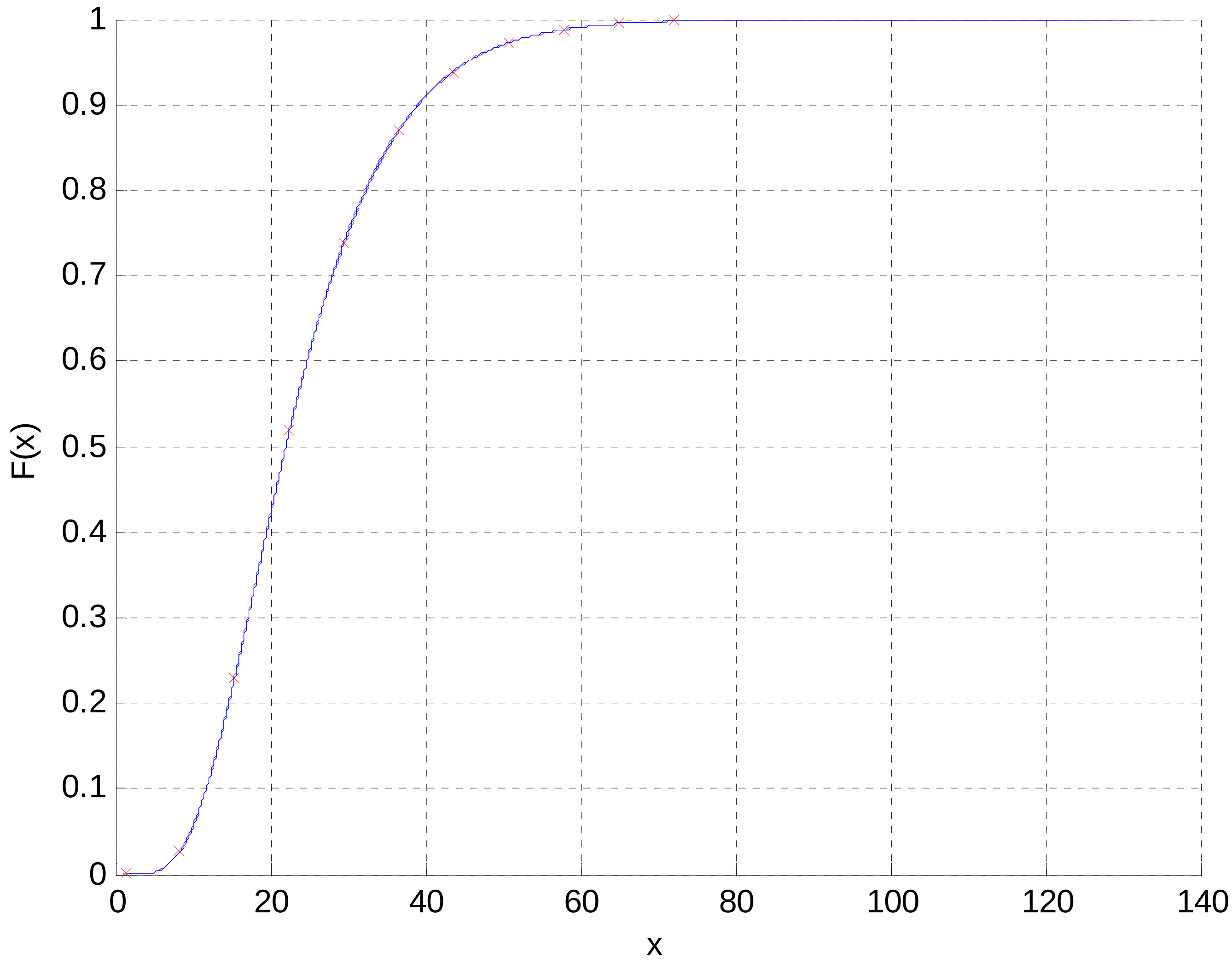
- E and Koev: software to compute



$$m = 3,\ n = 3,\ \beta = 5,\ \Sigma = \mathrm{diag}(1.1, 1.2, 1.4)$$



$$m = 4,\ n = 4,\ \beta = 2.5,\ \Sigma = \mathrm{diag}(1.1, 1.2, 1.4, 1.8)$$



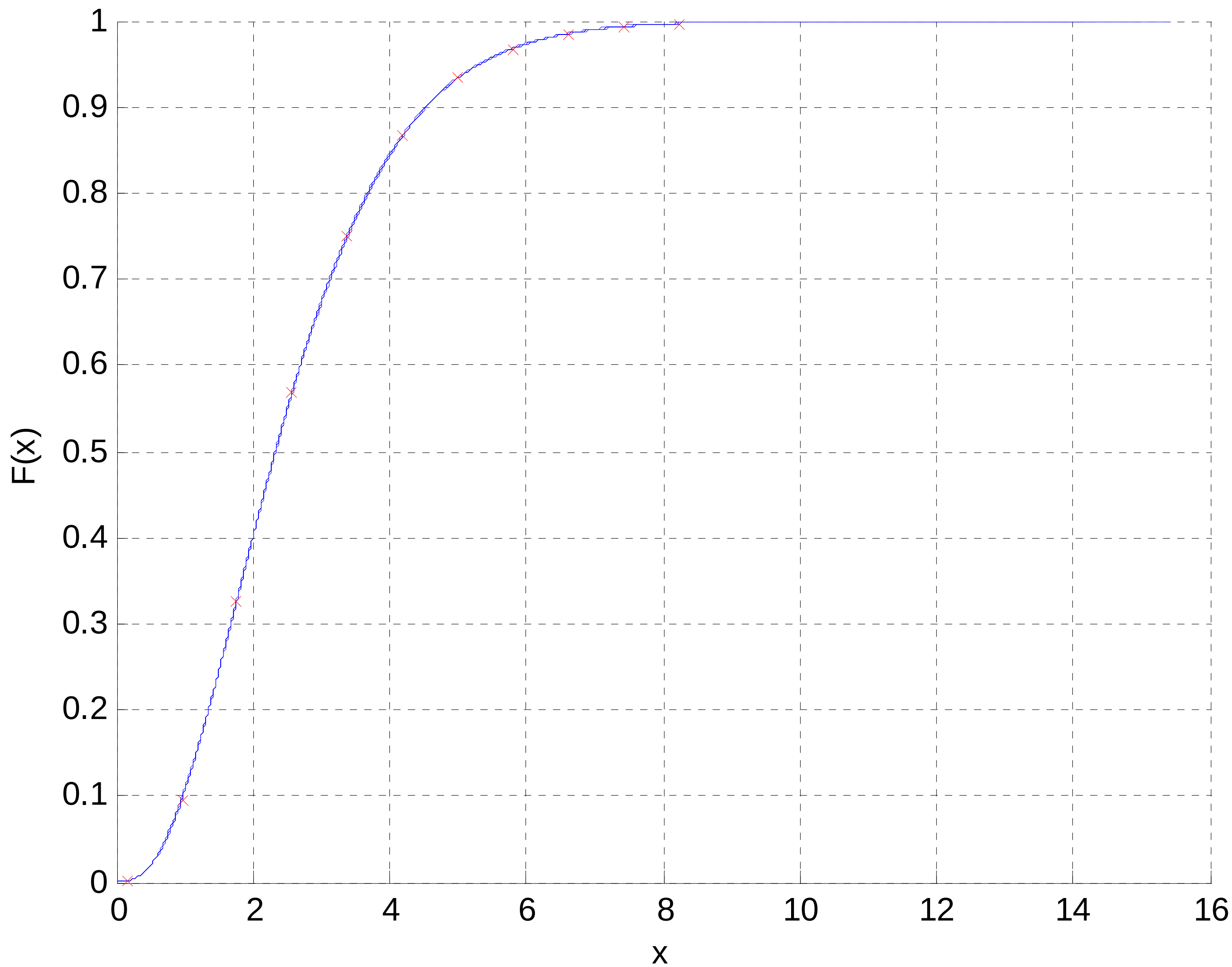
$$m = 5,\ n = 4,\ \beta = 0.75,\ \Sigma = \mathrm{diag}(1, 2, 3, 4)$$

Smallest Eigenvalue as Well

The cdf of the smallest eigenvalue,

for integer values of $r = (n - m + 1)\beta/2 - 1$, is

$$P(x) = 1 - \exp\left(\mathrm{tr}\left(\frac{x}{2}\Sigma^{-1}\right)\right)\sum_{k=0}^{mp}\widehat{\sum_{\kappa}}\frac{C_{\kappa}^{(\beta)}\left(\frac{x}{2}\Sigma^{-1}\right)}{k!}.$$



$m = 5,\ n = 4,\ \beta = 3,\ \Sigma = \mathrm{diag}(1.1, 1.2, 1.4, 1.8)$

Goals

- Continuum of Haar Measures generalizing orthogonal, unitary, symplectic
- New Definition of Jack Polynomials generalizing the zonals
- Computations! E.g. Moments of the Haar Measures
- Place finite random matrix theory "β"into same framework as infinite random matrix theory: specifically β as a knob to turn down the randomness, e.g. Airy Kernel

  $-d^2/dx^2+x+(2/\beta^{½})dW$ ←White Noise

Formally

- Let $S_n=2\pi/\Gamma(n/2)$="surface area of sphere"
  - Defined at any n= $\beta>0$.
- A β-ghost x is formally defined by a function $f_x(r)$ such that $\int_{r=0}^{\infty} f_x(r)\, r^{\beta-1} S_{\beta-1} dr=1$.
- Note: For β integer, the x can be realized as a random spherically symmetric variable in β dimensions
- Example: A β-normal ghost is defined by
  - $f(r)=(2\pi)^{-\beta/2}e^{-r^2/2}$
- Example: Zero is defined with constant*$\delta(r)$.
- Can we do algebra? Can we do linear algebra?
- Can we add? Can we multiply?

A few more operations

- $||x||$ is a real random variable whose density is given by $f_x(r)$
- $(x+x')/2$ is real random variable given by multiplying $||x||$ by a beta distributed random variable representing a coordinate on the sphere

Addition of Independent Ghosts:

- Addition returns a spherically symmetric object
- Have an integral formula
- Prefer: Add the real part, imaginary part completed to keep spherical symmetry

Multiplication of Independent Ghosts

- Just multiply IIzII's and plug in spherical symmetry
- Multiplication is commutative
  - (Important Example: Quaternions don't commute, but spherically symmetric random variables do!)

Understanding $\prod|\lambda_i-\lambda_j|^\beta$

- Define volume element (dx)^ by

  $(r\,dx)^\wedge = r^\beta (dx)^\wedge$ (β-dim volume, like fractals, but don't really see any fractal theory here)

- Jacobians: $A=Q\Lambda Q'$ (Sym Eigendecomposition)

  $Q'dAQ = d\Lambda + (Q'dQ)\Lambda - \Lambda(Q'dQ)$

  $(dA)^\wedge = (Q'dAQ)^\wedge =$ diagonal ^ strictly-upper

  diagonal $= \prod d\lambda_i = (d\Lambda)^\wedge$

  off-diag $= \prod\Big((Q'dQ)_{ij}(\lambda_i-\lambda_j)\Big)^\wedge = (Q'dQ)^\wedge \prod|\lambda_i-\lambda_j|^\beta$

Haar Measure

- $E_Q(\text{trace}(AQBQ')^k)=\sum C_\kappa(A)C_\kappa(B)/C_\kappa(I)$

  …hod: Suppose you know C_κ's a-priori.

  …rminants (Think

  C…

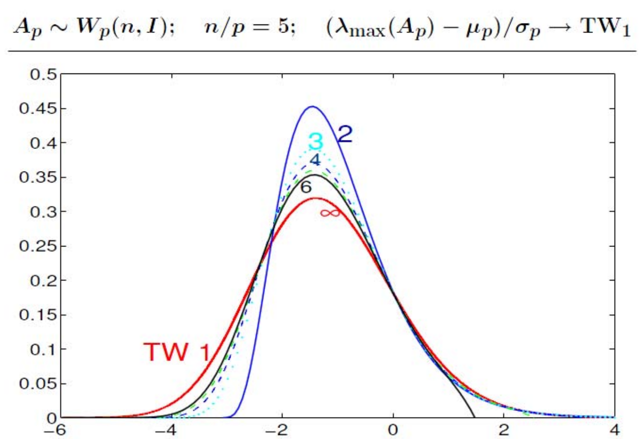
$$
\begin{aligned}
E(|q_{11}|^4) &= \frac{1+\alpha}{(n+\alpha)n} \\
E(|q_{11}|^2|q_{12}|^2) &= \frac{1}{(n+\alpha)n} \\
E(|q_{11}|^2|q_{22}|^2) &= \frac{n+\alpha-1}{n(n-1)(n+\alpha)} \\
E(|q_{11}|^6) &= \frac{(1+\alpha)(1+2\alpha)}{n(n+\alpha)(n+2\alpha)}
\end{aligned}
$$

$$
\begin{aligned}
E(|q_{11}|^4|q_{12}|^2) &= \frac{1+\alpha}{n(n+\alpha)(n+2\alpha)} \\
E(|q_{11}|^2|q_{12}|^2|q_{13}|^2) &= \frac{1}{n(n+\alpha)(n+2\alpha)} \\
E(|q_{11}|^2|q_{12}|^2|q_{23}|^2) &= \frac{n+2\alpha-1}{(n-1)n(n+\alpha)(n+2\alpha)} \\
E(|q_{11}|^4|q_{22}|^2) &= \frac{(1+\alpha)(n+2\alpha-1)}{(n-1)n(n+\alpha)(n+2\alpha)}
\end{aligned}
$$

- Then can fo…
- Example: $E(|q_{11}|^2|q_{22}$…
- $\alpha:=2/\beta$
- Can Gram-Schmidt the ghosts. Same a… coming up!

Further Uses of Ghosts

- Multivariate Orthogonal Polynomials
- Tracy-Widom Laws
- Largest Eigenvalues/Smallest Eigenvalues



- Expect lots of uses to be discovered…